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A conditional stability estimate of continuation problem for the Helmholtz equation

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Abstract. In this paper, we consider the continuation problem for the Helmholtz equation. The main result is a conditional stability estimate for a solution to the considered problem. The estimate shows that the closer solution to the surface is more stable. The results of the numerical experiments confirm this conclusion.

Keywords: Helmholtz equation, Stability estimate, Inverse problem
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PROBLEM STATEMENT

In many inverse problems required heterogeneities are located at some depth under the layer, the parameters of which are known. In this case, an important means for practitioners are continuation problems of geophysical fields from the earth's surface in the direction of occurrence of heterogeneities. Continuation problems for equations of mathematical physics from a part of the boundary in many cases are strongly ill-posed. Solving these problems by different numerical methods an analysis of a stability and error estimates for approximate solutions was deficiently researched.

Conditional stability of the continuation problem for the Helmholtz equation is studied. The numerical researches in this paper are actual because recently the continuation problems for the Helmholtz equation has received new practical applications [1].

Stability question of a solution to a Cauchy problem for the Helmholtz equation was studied by many authors. For example, N. H. Tuan and P. H. Quan [2] considered the case $0 < k < 1$ and proposed a regularization technique which allows one to obtain a stable solution in a two-dimensional domain. T. Reginska and K. Reginski [3] showed that if k satisfies a certain condition, then the Cauchy problem for the Helmholtz equation has a stable solution in a three-dimensional domain. V. Isakov and S. Kindermann [4] used the singular value decomposition to prove that in a simple domain the considered problem becomes more stable with increasing k.

For the first time the continuation problem for the Laplace equation was regularized by an iterative method in 1991 by V. A. Kozlov, V. G. Maz'ya, A. V. Fomin [5].

Hence, we consider the following initial boundary value problem for the Helmholtz equation in the domain $\Omega = (0,l) \times (0,\pi)$

$$
\begin{aligned}
u_{xx} + u_{yy} + k^2 u &= 0, \ (x,y) \in \Omega, && (1)\\
u_x(0,y) &= 0, \ y \in [0,\pi], && (2)\\
u(0,y) &= f(y), \ y \in [0,\pi], && (3)\\
u_y(x,0) &= u_y(x,\pi) = 0, \ x \in [0,l], && (4)
\end{aligned}
$$

where k, l are given constants. It is required to find a function $u(x,y)$ in Ω from $f(y)$.

Investigated problem (1)–(4) is considered as an inverse problem with respect to the next direct (well-posed) problem

$$
\begin{aligned}
u_{xx} + u_{yy} + k^2 u &= 0, \ (x,y) \in \Omega, && (5)\\
u_x(0,y) &= 0, \ y \in [0,\pi], && (6)\\
u(l,y) &= q(y), \ y \in [0,\pi], && (7)\\
u_y(x,0) &= u_y(x,\pi) = 0, \ x \in [0,l]. && (8)
\end{aligned}
$$

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Then the inverse problem is formulated as follows: find $q(y)=u(l,y)$ from the relations (5), (6), (8) and the additional information

$$u(0,y)=f(y),\ y\in[0,\pi]. \tag{9}$$

A solution of the problem will be found in our specially defined class [6].

Definition 1. *A function $u\in L_2(\Omega)$ is called a generalized solution of the direct problem (5)–(8) if for any $\omega\in H^2(\Omega)$ such that*

$$\begin{aligned}\omega_x(0,y) &= 0,\ y\in[0,\pi], &(10)\\ \omega(l,y) &= 0,\ y\in[0,\pi], &(11)\\ \omega_y(x,0) &= \omega_y(x,\pi)=0,\ x\in[0,l], &(12)\end{aligned}$$

the following equality holds:

$$\int_0^l\int_0^\pi u\big(\omega_{xx}+\omega_{yy}+k^2\omega\big)dydx-\int_0^\pi q(y)\omega_x(l,y)dy=0. \tag{13}$$

Theorem 1 (Existence of a generalized solution of the direct problem)**.** *[7] If $q\in L_2(0,\pi)$ and $k^2l^2<1$, then the direct problem (5)–(8) has a unique generalized solution $u\in L_2(\Omega)$ and the following estimate is valid:*

$$\|u\|_{L_2(\Omega)}\le\|q\|_{L_2(0,\pi)}\frac{\sqrt{l}}{1-k^2l^2}. \tag{14}$$

The iterative method applied by us for solving the inverse problem (5), (6), (8), (9) involves solving the conjugate problem, which is as follows:

$$\begin{aligned}\psi_{xx}+\psi_{yy}+k^2\psi &= 0,\ (x,y)\in\Omega, &(15)\\ \psi(l,y) &= 0,\ y\in[0,\pi], &(16)\\ \psi_x(0,y) &= \mu(y),\ y\in[0,\pi], &(17)\\ \psi_y(x,\pi) &= \psi_y(x,0)=0,\ x\in[0,l], &(18)\end{aligned}$$

where a given function $\mu(y)$ is required to define the function $\psi(x,y)$.

Similarly we introduce a class of solutions to the conjugate problem.

Definition 2. *A function $\psi\in L_2(\Omega)$ is called a generalized solution of the conjugate problem (15)-(18) if for any $v\in H^2(\Omega)$ such that*

$$\begin{aligned}v_x(0,y) &= 0,\ y\in[0,\pi], &(19)\\ v(l,y) &= 0,\ y\in[0,\pi], &(20)\\ v_y(x,0) &= v_y(x,\pi)=0,\ x\in[0,l], &(21)\end{aligned}$$

the following equality holds:

$$\int_0^l\int_0^\pi \psi\big(v_{xx}+v_{yy}+k^2v\big)dydx-\int_0^\pi \mu(y)v(0,y)dy=0. \tag{22}$$

The main theoretical result of this work is as follows

Theorem 2. *Assume that for $f\in L_2(0,\pi)$ there exists a solution $u\in L_2(\Omega)$ to the problem (1)-(4), then the following conditional stability estimate holds [8]:*

$$\begin{aligned}\|u\|^2(x)\le&\Big(\|q\|^2+\big|\|f_y\|^2-\frac{k^2}{2}\|f\|^2\big|\Big)^{\frac{x}{l}}\Big(\|f\|^2+\big|\|f_y\|^2-\frac{k^2}{2}\|f\|^2\big|\Big)^{\frac{l-x}{l}}e^{2x(l-x)}\\ &-\big|\|f_y\|^2-\frac{k^2}{2}\|f\|^2\big|,\end{aligned} \tag{23}$$

where $\|u\|^2(x)=\int_0^\pi u^2(x,y)dy$.

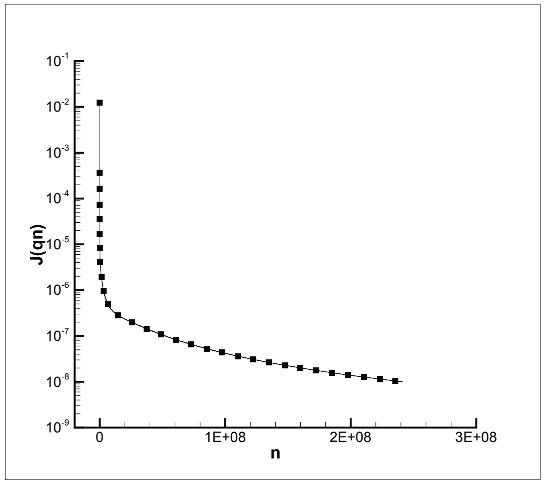

FIGURE 1. Graphic of the functional $J(q_n)$

ALGORITHMS FOR SOLVING THE DIRECT AND INVERSE PROBLEMS

The solution of direct problems (5)-(8) and the conjugate problem (15)-(18) will be constructed using the Fourier series:

$$u(x,y) = \sum_{m=1}^{M} a_m(x)\cos my, \tag{24}$$

$$\psi(x,y) = \sum_{m=1}^{M} b_m(x)\cos my. \tag{25}$$

The inverse problem will be solved by the conjugate gradient method. As minimizing functional is taken next

$$J(q) = \frac{\pi}{2}\sum_{m=1}^{M}\left(a_m(0;q) - f_m\right)^2, \tag{26}$$

where f_m are the corresponding Fourier coefficients of decomposition of the function f.

The derivative of the functional J defined by the formula

$$J'(q) = \psi_x(l,y), \tag{27}$$

where $\psi(x,y)$ – a solution to the conjugate problem (15)-(18) with $\mu(y) = 2[u(0,y;q) - f(y)]$.

NUMERICAL RESULTS

We assume that $M = 7$, where M is the number of Fourier coefficients, and solve our problem in the following domain $\Omega = (0,1)\times(0,\pi)$ with the parameter $k = 0.9$ and boundary condition

$$q(y) = y(\pi - y).$$

As a stopping criterion of the iterative process was considered a condition $J(q_n) \leq 10^{-8}$ with the initial approximation

$$q_m^0 = 0.1,\ m = \overline{1,M}.$$

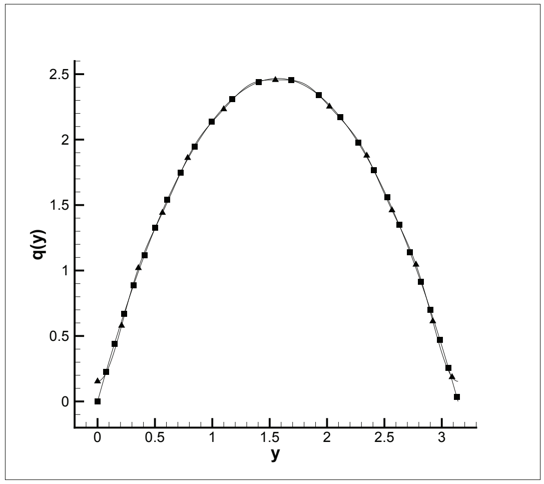

FIGURE 2. Graphics of exact and approximate solutions of $q(y)$

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