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Solvability one of stationary problem of magnetohydrodynamics

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Abstract. In this work, we prove unique solvability of a boundary-value problem of magnetohydrodynamics in Sobolev and Hölder spases.

Keywords: Magnetohydrodynamics, Generalized solution, Statinary problem
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INTRODUCTION

In this work, we study the problem on a viscous incompressible fluid fills a bounded vessel $\Omega \subset \mathbb{R}^3$ with a perfectly conducting boundary S. In Ω, an exterior force $\vec{f}(x)$ acting on the fluid and electric current of density $\vec{j}(x)$ are defined. One must find the velocity vector field $\vec{\upsilon}(x)$ and the pressure function $p(x)$, as well as the magnetic and electric fields, $\vec{H}(x)$ and $\vec{E}(x)$.

The steady motion of a viscous incompressible electrically conducting fluids is described by the following system of equations of magnetohydrodynamics consisting of Navier-Stokes equations

$$-\nu\Delta\vec{\upsilon}(x) + \sum_{k=1}^{3}\left(\upsilon_k \vec{\upsilon}_{x_k} - \frac{\mu}{\rho}H_k\vec{H}_{x_k}\right) + \frac{1}{\rho}\nabla\left(p(x) + \frac{\mu}{2}\left|\vec{H}(x)\right|^2\right) = \vec{f}(x), \quad x \in \Omega, \tag{1}$$

$$div\vec{\upsilon}(x) = 0, \quad x \in \Omega, \tag{2}$$

and Maxwell's equations without displacement current

$$rot\vec{H}(x) - \sigma\left(\vec{E}(x) + \mu\left[\vec{\upsilon}\times\vec{H}\right]\right) = \vec{j}(x), \quad x \in \Omega, \tag{3}$$

$$div\mu\vec{H}(x) = 0, \quad x \in \Omega, \tag{4}$$

$$rot\vec{E}(x) = 0, \quad x \in \Omega \tag{5}$$

with given $\vec{f}(x)$ and $\vec{j}(x)$. Here, the magnetic permeability of a fluid μ, the conductivity σ, the kinematic coefficient of viscosity ν and the density ρ are given positive constants. The studying problem consists to find solutions of (1)-(4) in Ω, satisfying the following boundary conditions on S:

$$\vec{\upsilon}(x)|_S = 0\,, \quad \vec{E}_\tau(x)|_S = 0\,, \quad \vec{H}\cdot\vec{n}|_S = 0\,, \tag{6}$$

where $\vec{n}$ is the unit outward normal to S, and $\vec{E}_\tau = \vec{E} - \vec{n}\left(\vec{n}\cdot\vec{E}\right)$.

Authors [1] have studied problem (1)-(6) passing to the generalized statement, i.e. boundary conditions replaced by the requirement of belonging of unknown vector functions in some functional Hilbert spaces and other equations by integral identities. The theorem of existence and uniqueness of generalized solutions of problem (1) - (6) in the space $L_2(\Omega)$ was proved in [2]. The existence of weak solution to the problem was studied in [2]. We extended these results as $p \geq \frac{3}{2}$ in a bounded domain up to the boundary.

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AUXILIARY PROPOSITIONS AND MAIN RESULTS

We note that the passing to a generalized statement requires a number of auxiliary propositions proved in [1], which we refer in the right places.

Definition 1. *Let $\mathbf{H}(\Omega)$ be the space of solenoidal vector functions of $\vec{\varphi} \in \overset{\circ}{W}{}^1_2(\Omega)$, i. e.,*

$$\mathbf{H}(\Omega) := \left\{ \vec{\varphi} \in \overset{\circ}{W}{}^1_2(\Omega) : \ div\vec{\varphi} = 0 \ in\ \Omega, \ and\ \vec{\varphi} = 0\ on\ S \right\}.$$

Definition 2. *Let $\mathbf{H}_1(\Omega)$ be the space of solenoidal vector functions of $\vec{\varphi} \in W^1_2(\Omega)$, i. e.,*

$$\mathbf{H}_1(\Omega) := \left\{ \vec{\varphi} \in W^1_2(\Omega) : \ div\vec{\varphi} = 0 \ \ in\ \ \Omega, \ \ and\ \vec{\varphi}_n \equiv \vec{\varphi} \cdot \vec{n} = 0 \ \ on\ \ S \right\}.$$

Definition 3. *A pair of functions $\left(\vec{\upsilon}, \vec{H}\right)$ is called a generalized solution of problem* (1)-(6)*, if $\vec{\upsilon}(x) \in \mathbf{H}(\Omega)$ and $\vec{H}(x) \in \mathbf{H}_1(\Omega)$ and for any $\vec{\varphi} \in \mathbf{H}(\Omega)$ and $\vec{\psi} \in \mathbf{H}_1(\Omega)$ hold the following integral identities*

$$\nu \int\limits_{\Omega} \nabla \vec{\upsilon} \nabla \vec{\varphi} dx - \int\limits_{\Omega} \vec{\upsilon} \left(\vec{\upsilon} \cdot \nabla\right) \vec{\varphi} dx + \frac{\mu}{\rho} \int\limits_{\Omega} \vec{H} \left(\vec{H} \cdot \nabla\right) \vec{\varphi} dx = \int\limits_{\Omega} \vec{f} \vec{\varphi} dx, \tag{7}$$

$$\frac{1}{\sigma} \int\limits_{\Omega} rot\vec{H} rot\vec{\psi} dx - \int\limits_{\Omega} \mu \left[\vec{\upsilon} \times \vec{H}\right] rot\vec{\psi} dx = \frac{1}{\sigma} \int\limits_{\Omega} \vec{j} rot\vec{\psi} dx. \tag{8}$$

For problem (1)-(6), the following theorem holds.

Theorem 1. *Let $S \in C^3$. If $\vec{f}(x) \in L_p(\Omega)$, $\vec{j}(x) \in W^1_p(\Omega)$ with $p \geq \frac{6}{5}$, then the generalized solution of problem* (1)-(6) *belongs to $W^2_p(\Omega) \times W^2_p(\Omega)$, and there exist $\nabla p(x) \in L_p(\Omega)$ and $\vec{E}(x) \in W^1_p(\Omega)$ such that equations* (1)-(2) *are satisfied, and the following estimate holds*

$$\begin{aligned} &\|\vec{\upsilon}\|_{W^2_p(\Omega)} + \left\|\vec{H}\right\|_{W^2_p(\Omega)} + \|\nabla p\|_{L_p(\Omega)} + \left\|\vec{E}\right\|_{W^2_p(\Omega)} \\ &\leq C \left(\left\|\vec{f}\right\|_{L_p(\Omega)} + \left\|\vec{j}\right\|_{W^1_p(\Omega)} + \left\|\vec{f}\right\|^8_{L_p(\Omega)} + \left\|\vec{j}\right\|^8_{W^1_p(\Omega)} \right). \end{aligned} \tag{9}$$

Theorem 2. *Let $S \in C^{2+\alpha}$. If $\vec{f}(x) \in C^{\alpha}(\Omega)$, $\vec{j}(x) \in C^{1+\alpha}(\Omega)$, $\alpha \in (0,1)$, then $\vec{\upsilon}(x) \in C^{2+\alpha}(\Omega)$, $\vec{H}(x) \in C^{2+\alpha}(\Omega)$, $\nabla p(x) \in C^{\alpha}(\Omega)$, $\vec{E}(x) \in C^{1+\alpha}(\Omega)$ and the following estimate is valid*

$$\begin{aligned} &\|\vec{\upsilon}\|_{C^{2+\alpha}(\Omega)} + \left\|\vec{H}\right\|_{C^{2+\alpha}(\Omega)} + \|\nabla p\|_{C^{\alpha}(\Omega)} + \left\|\vec{E}\right\|_{C^{1+\alpha}(\Omega)} \\ &\leq C \left(\left\|\vec{f}\right\|_{C^{\alpha}(\Omega)} + \left\|\vec{j}\right\|_{C^{1+\alpha}(\Omega)} + \left\|\vec{f}\right\|^{16}_{C^{\alpha}(\Omega)} + \left\|\vec{j}\right\|^{16}_{C^{1+\alpha}(\Omega)} \right). \end{aligned} \tag{10}$$

Proof. The proofs of these theorems are based on the results of propositions given below and the smoothness of the generalized solution to Stokes problem [1]. We also use the following new estimate for nonlinear terms in equations of magnetohydrodynamics. For $\vec{\upsilon}$, $\vec{H} \in W^1_2(\Omega) \subset L_6(\Omega)$, we have

$$\|(\vec{\upsilon} \cdot \nabla)\vec{\upsilon}\|_{L_{3/2}(\Omega)} \leq \|\vec{\upsilon}\|_{L_6(\Omega)} \|\nabla\vec{\upsilon}\|_{L_2(\Omega)} \leq c \|\vec{\upsilon}\|^2_{W^1_2(\Omega)}.$$

$$\left\|\left(\vec{H} \cdot \nabla\right)\vec{H}\right\|_{L_{3/2}(\Omega)} + \left\|\nabla\vec{H}\right\|_{L_{3/2}(\Omega)} \leq \left\|\vec{H}\right\|_{L_6(\Omega)} \left\|\nabla\vec{H}\right\|_{L_2(\Omega)} \leq C \left\|\vec{H}\right\|^2_{W^1_2(\Omega)}.$$

We begin the proof of Theorem 1 from the transfers relations (7), (8) as follows: First, we rewrite (7) in the form

$$\nu \int_{\Omega} \nabla\vec{\upsilon} \nabla\vec{\varphi} dx = \int_{\Omega} \vec{F} \vec{\varphi} dx, \ \ \forall \vec{\varphi} \in H(\Omega), \tag{11}$$

where $\vec{F} \equiv \vec{f} - (\vec{\upsilon} \cdot \nabla)\, \vec{\upsilon} + \frac{\mu}{\rho} \left(\vec{H}\nabla\right) \vec{H}$.

Before we transform (8), we recall Weyl's orthogonal decomposition of the space $L_2(\Omega)$ of square summable vector fields [3], [4]:

$$L_2(\Omega) = \overset{0}{G}(\Omega) \oplus J(\Omega), \tag{12}$$

where $\overset{\circ}{G}(\Omega) = \left\{u = \nabla\phi,\ \phi \in \overset{\circ}{W}{}_2^1(\Omega)\right\}$, and $J(\Omega)$ is the solenoidal vector fields from $L_2(\Omega)$ (i.e. orthogonal to any $\nabla\phi$, $\phi \in \overset{\circ}{W}{}_2^1(\Omega)$).

Let P_J be orthogonal projection to the space $J(\Omega)$. It is clearly, we have that $\|P_J\|_{L_2 \to L_2} \leq 1$. Moreover, following propositions [1] hold:

Proposition 3. *If $\vec{F} \in L_q(\Omega)$, $q \geq \frac{6}{5}$ in (14), then $\vec{\upsilon} \in W_q^2(\Omega)$ and satisfies the following inequality:*

$$\|\vec{\upsilon}\|_{W_q^2(\Omega)} \leq c \left\|\vec{F}\right\|_{L_q(\Omega)}.$$

Moreover, there is exists a function $\nabla p \in L_q(\Omega)$ such that

$$-\nu\Delta\vec{\upsilon} + \frac{1}{\rho}\nabla\left(p + \frac{\mu\vec{H}^2}{2}\right) = \vec{f}(x)$$

and

$$\|\nabla p\|_{L_q(\Omega)} \leq c \left\|\vec{F}\right\|_{L_q(\Omega)}.$$

Proposition 4. *If $\vec{u}(x) \in W_q^k(\Omega)$, $k = 0, 1$, then $P_J\vec{u}(x) \in W_q^k(\Omega)$ and*

$$\|P_J\vec{u}\|_{W_q^k(\Omega)} \leq c\,\|\vec{u}\|_{W_q^k(\Omega)},\ \ k = 0,\, 1. \tag{13}$$

Proposition 5. *For any vector function $\vec{\xi}(x) \in J(\Omega) \bigcap W_p^1(\Omega)$ the problem*

$$rot\vec{H}(x) = \vec{\xi}(x),\ \ div\vec{H}(x) = 0,\ x \in \Omega,\ \ \vec{H} \cdot \vec{n}\,|_S = 0 \tag{14}$$

has a unique solution $\vec{H}(x) \in W_p^2(\Omega)$ and the following estimate holds

$$\left\|\vec{H}\right\|_{W_p^2(\Omega)} \leq c \left\|\vec{\xi}\right\|_{W_p^1(\Omega)}. \tag{15}$$

By identity (8), the expression $\frac{1}{\sigma} rot\vec{H} - \mu\left[\vec{\upsilon} \times \vec{H}\right] - \frac{1}{\sigma}\vec{j}$ is orthogonal to all functions $rot\vec{\psi}$, $\vec{\psi} \in \mathbf{H}_1(\Omega)$ i.e. all functions $\vec{\xi}(x) \in J(\Omega)$. Hence,

$$\frac{1}{\sigma} rot\vec{H} - \mu\left[\vec{\upsilon} \times \vec{H}\right] - \frac{1}{\sigma}\vec{j} = \nabla\phi,\ \ \phi \in \overset{0}{W}{}_2^1(\Omega). \tag{16}$$

Formula (16) can be treated as the orthogonal decomposition (12) for the vector function $\vec{g}(x) = \frac{1}{\sigma}\vec{j} + \mu\left[\vec{\upsilon} \times \vec{H}\right]$.

Remark 1. *We can take $\vec{E}(x) = \nabla\phi$ in (16).*

It follows from (16) that

$$\vec{g}(x) = \frac{1}{\sigma} rot\vec{H} - \nabla\phi,\ \ -\nabla\phi \in \overset{\circ}{G}(\Omega),\ \frac{1}{\sigma} rot\vec{H} \in \overset{\circ}{J}(\Omega). \tag{17}$$

If $\vec{g}(x) \in W_q^1(\Omega)$, then by Propositions 4 and 5 we have the following estimate

$$\left\|\vec{H}\right\|_{W_q^2(\Omega)} \leq c \left\|rot\vec{H}\right\|_{W_q^1(\Omega)} \leq c\,\|\vec{g}\|_{W_q^1(\Omega)}.$$

In this case, we can take the vector $\vec{E}(x)$ in the form $\vec{E}(x) = \nabla\phi = P_{\overset{\circ}{G}}\vec{g}(x)$ and it satisfies the inequality [5]

$$\left\|\vec{E}(x)\right\|_{W_q^1(\Omega)} \le c\,\|\vec{g}\|_{W_q^1(\Omega)}. \tag{18}$$

Now, we prove inequality (9). Using the Hölder inequality

$$\left|\int_\Omega \nu\upsilon dx\right| \le \|\nu\|_{L_s(\Omega)}\,\|\upsilon\|_{L_{s'}(\Omega)}, \quad \frac{1}{s}+\frac{1}{s'}=1$$

we estimate nonlinear terms in (7) and (8). By the fact that $\vec{\upsilon},\vec{H} \in L_6(\Omega)$, we have

$$\|(\vec{\upsilon}\nabla)\vec{\upsilon}\|_{L_{\frac{3}{2}}(\Omega)} \le \left(\int_\Omega |\vec{\upsilon}|^{\frac{3}{2}}|\nabla\vec{\upsilon}|^{\frac{3}{2}}dx\right)^{\frac{2}{3}} \le \left(\int_\Omega |\nabla\vec{\upsilon}|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\vec{\upsilon}|^6 dx\right)^{\frac{1}{6}} \le c\,\|\vec{\upsilon}\|^2_{W_2^1(\Omega)}.$$

Similarly, we get

$$\left\|\left(\vec{H}\nabla\right)\vec{H}\right\|_{L_{\frac{3}{2}}(\Omega)} \le c\left\|\vec{H}\right\|^2_{W_2^1(\Omega)},$$

$$\left\|\left(\vec{\upsilon}\times\vec{H}\right)\right\|_{W_{\frac{3}{2}}(\Omega)} \le c\,\|\vec{\upsilon}\|^2_{W_2^1(\Omega)} + \left\|\vec{H}\right\|^2_{W_2^1(\Omega)}.$$

Consequently,

$$\vec{F}(x) = \vec{f} - (\vec{\upsilon}\nabla)\vec{\upsilon} + \frac{\mu}{\rho}\left(\vec{H}\nabla\right)\vec{H} \in L_{p_1}(\Omega),\ \text{for } p_1 = \min\left(p, \frac{3}{2}\right)$$

and

$$\vec{g}(x) = \frac{1}{\sigma}\vec{j}(x) + \mu\left[\vec{\upsilon}\times\vec{H}\right] \in W^1_{p_1}(\Omega),\ \text{for } p_1 = \min\left(p, \frac{3}{2}\right).$$

If $p \le \frac{3}{2}$, then estimate (9) follows from the estimates (12), (13), and (15).

Let $p > \frac{3}{2}$. Then, $p_1 = \frac{3}{2}$ and by (12), (13), and (15), we have

$$\|\vec{\upsilon}\|_{W^2_{3/2}(\Omega)} \le c\left\|\vec{F}\right\|_{L_{3/2}(\Omega)}$$

and

$$\left\|\vec{H}\right\|_{W^2_{3/2}(\Omega)} \le c\left\|rot\vec{H}\right\|_{W^1_{3/2}(\Omega)} \le c\,\|\vec{g}\|_{W^2_{3/2}(\Omega)}.$$

Using this information, we estimate the nonlinear terms. By the embedding theorems, $\vec{\upsilon},\vec{H} \in W_3^1(\Omega)$ and $\vec{\upsilon},\vec{H} \in L_Q(\Omega)$ with an arbitrary large Q. We also have for any small $\theta > 0$

$$\|(\vec{\upsilon}\nabla)\vec{\upsilon}\|_{L_{3-\theta}} \le \left(\int\limits_\Omega |\vec{\upsilon}|^{3-\theta}|\nabla\vec{\upsilon}|^{3-\theta}dx\right)^{\frac{1}{3-\theta}} \le \left(\int\limits_\Omega |\nabla\vec{\upsilon}|^3 dx\right)^{\frac{1}{3}}\left(\int\limits_\Omega |\vec{\upsilon}|^\theta dx\right)^{\frac{1}{\theta}} \le c\,\|\vec{\upsilon}\|^2_{W^2_{3/2}(\Omega)},$$

$$Q = \frac{(3-\theta)3}{\theta} = \frac{9}{\theta} - 3.$$

Likewise, we obtain inequalities

$$\left\|\left(\vec{H}\nabla\right)\vec{H}\right\|_{L_{3-\theta}(\Omega)} \le c\left\|\vec{H}\right\|^2_{W^2_{3/2}(\Omega)},$$

$$\left\|\left(\vec{\upsilon}\times\vec{H}\right)\right\|_{W^1_{3-\theta}(\Omega)} \le c\left(\|\vec{\upsilon}\|^2_{W^1_{3/2}(\Omega)} + \left\|\vec{H}\right\|^2_{W^1_{3/2}(\Omega)}\right).$$

Consequently, $\vec{F}(x) \in L_{p_2}(\Omega)$, $\vec{g}(x) \in W^1_{p_2}(\Omega)$ with $p_2 = \min(p, 3-\theta)$. In the case $p \leq 3-\theta$, it is sufficiently to applied the estimates (5), (7), and (8). In the case $p > 3-\theta$ we have $p_2 = 3-\theta$, and $\vec{v}$, $\vec{H} \in W^2_{3-\theta}(\Omega)$. Therefore $\vec{v}, \vec{H} \in W^2_{Q_1}(\Omega)$ ñ $\frac{1}{Q_1} = \frac{1}{1-\frac{\theta}{3}} - 1 = \frac{\theta}{3-\theta}$, i.e. $Q_1 = \frac{3-\theta}{\theta}$. It is obviously that $Q_1 >> 1$ for small θ. Moreover, $\vec{v}, \vec{H} \in L_\infty(\Omega)$.

Estimating the nonlinear terms as above in the case $\vec{v}, \vec{H} \in W^1_{Q_1}(\Omega)$, $\vec{v}, \vec{H} \in L_\infty(\Omega)$, we see that they are summarable with arbitrary large exponents. Therefore, $\vec{F}(x) \in L_p(\Omega)$, $\vec{g}(x) \in W^1_p(\Omega)$ and it means that $\vec{v}, \vec{H} \in W^2_p(\Omega)$. Thus the proof of Theorem 1 is completed.

Theorem 2 can be proved in the analogical way as above. The rest of the proof proceeds as in the proof of Theorem 1, and we omit it. □

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