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Citation: AIP Conference Proceedings **1759**, 020034 (2016); doi: 10.1063/1.4959648
View online: http://dx.doi.org/10.1063/1.4959648
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1759?ver=pdfcov
Published by the AIP Publishing

The first regularized trace of integro-differential Sturm-Liouville operator on the segment with punctured points at integral perturbation of transmission conditions

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Abstract. The paper is devoted to calculation of a first regularized trace of one integro-differential operator with the main part of the Sturm-Liouville type on a segment with punctured points at integral perturbation of "transmission" conditions. The Sturm-Liouville operator $-y''(x)+q(x)y(x)+\gamma\int_0^\pi y(t)dt=\lambda y(x)$ given on the segments $\frac{\pi}{n}(k-1)<x<\frac{\pi}{n}k$, $k=\overline{1,n}$; $n\geq 2$ is considered. Boundary conditions of the Dirichlet type: $y(0)=0$, $y(\pi)=0$ are given on the left-hand and right-hand ends of the segment $[0,\pi]$. The functions are continuous on $[0,\pi]$, the first derivatives of which have jumps at the points $x=\frac{\pi}{n}k$, are solutions. The value of jumps is expressed by the formula $y'(\frac{\pi k}{n}-0)=y'(\frac{\pi k}{n}+0)-\beta_k\int_0^\pi y(t)dt, k=\overline{1,n-1}$. The basic result of the paper is the exact formula of the first regularized trace of the considered differential operator.

Keywords: Spectral problem, First regularized trace, Integro-differential operator, Inner-boundary condition
PACS: 02.30.Jr, 02.30.Tb

INTRODUCTION

At the present stage of its development, the spectral theory of the ordinary differential operators is one of the important parts of the general spectral theory and has been actively developing by different mathematical schools. One of the spectral theory directions is the trace theory of differential operators. Primarily, the trace theory has been developed by the Moscow school under the head of academician V. A. Sadovnichij [1, 2]. In recent years they have presented the formula of the first regularized trace of discrete operators, regularized traces of singular operators, regularized traces of self-adjoint discrete operators with a non-nuclear resolvent, trace formula of M. G. Krein in case of perturbations such as Hilbert-Schmidt, regularized trace of the operator Sturm-Liouville equation on a finite interval, the trace formula for potential containing the delta-function, formulas for regularized traces of operators with relatively compact perturbations, and so on.

However, a number of essential problems of spectral theory is still not resolved, finding regularized traces of differential operators in areas with deleted points is regarded to these problems. This direction is closely related to the research of operators with potentials, containing a delta function, however this direction also has its own characteristics. Nowadays, this direction is in the process of accumulating the primary knowledge. In order to accumulate the primary knowledge we have to obtain formulas of explicit form of calculating the regularized traces of different specific differential operators in deleted points areas.

THE CONCEPT OF A REGULARIZED TRACE

The theory of regularized traces of linear operators originates from the fundamental fact of finite-dimensional theory is an invariance of the matrix trace of a linear operator and its coincidence with the spectral trace, and explores the question about spreading the concept of trace invariance on the unbounded operators.

Researching regularized traces of operators with discrete spectrum was started in the famous work of I. M. Gel'fand and B. M. Levitan [3], where the authors obtained a trace of the operator in Sturm-Liouville problem

International Conference on Analysis and Applied Mathematics (ICAAM 2016)
AIP Conf. Proc. 1759, 020034-1–020034-5; doi: 10.1063/1.4959648
Published by AIP Publishing. 978-0-7354-1417-4/$30.00

$$-y''(x)+q(x)y(x)=\lambda y(x),\ y'(0)=0,\ y'(\pi)=0.$$

For $q(x)\in C^1[0,\pi]$ under fulfillment of condition $\int_0^\pi q(x)dx=0$ the following formula

$$\sum_{n=0}^{\infty}(\lambda_n-\mu_n)=\frac{1}{4}\big(q(0)+q(\pi)\big)$$

has been obtained, where λ_n are eigenvalues of the problem and $\mu_n=n^2$ are eigenvalues of the same problem with $q(x)=0$. The sum of a series is called the first regularized trace of the operator. Formulas of this type are called the first regularized trace. One of the advantages of such formulas is that although the eigenvalues when $q(x)\neq 0$ can not be calculated in explicit form, the sum of the regularized trace is accurately determined and can always be calculated.

In the same year L. A. Dikiy came to similar results, he used a little bit different methods. The works of I. M. Gel'fand, M. G. Gasymova, R. F. Shevchenko, A. G. Kostyuchenko, V. A. Sadovnichii and many others were devoted to the obtained formulas for regularized traces of ordinary differential operators. The most common results for ordinary differential operators were obtained by V. B. Lidskii and V. A. Sadovnichii [4]. They found that the derivation of this type for a wide class of boundary value problems, generated by ordinary differential expressions on a finite interval with a complex spectral parameter, reduces to the study of regularized sums of roots of entire functions with a certain structure of the asymptotic.

OVERVIEW OF SIMILAR CATEGORY OF WORKS OF KAZAKH AUTHORS

Among the large number of published works on the formulas of regularized traces of differential operators it is necessary to note the works of Kazakh mathematicians. This is primarily the works of B. E. Kanguzhin and his disciples and followers.

In [5] a mathematical model of forced oscillations package of flat plates with point elastic links was proposed. In [6] a complete description of the correct solvability of boundary value problems for the biharmonic operator in a circle and ball was given, their correctly solvable finite perturbations arising in considering the problem in the deleted area were written. In [7] identities for eigenvalues of operator generated by ordinary differential expressions with internal boundary conditions were found. In [8] a complete description of correctly solvable boundary value problems for the Laplace operator in the circle and in the deleted circle was given, the formulas of resolvents of correct problems were shown. In [9] the Laplace operator in the pierced area was studied in the Hilbert space , an analogue of Green's formula was obtained and the class of self-adjoint extensions was described. In [10] the class of correct problems for the m-Laplace operator in the deleted area was considered and formulas of the regularized trace were obtained.

[12] is the closest to the theme of this paper. In this paper the formula of the first regularized trace of a differential Sturm-Liouville type operator on a segment with deleted points under an integral perturbation of "jump" conditions of the first derivatives in the deleted points was obtained. We will give this result in more detail.

The problem of eigenvalues

$$-y''(x)+q(x)y(x)=\lambda y(x),\ \frac{\pi}{n}(k-1)<x<\frac{\pi}{n}k,\ k=\overline{1,n};\ n\geq 2; \tag{1}$$

$$y(0)=0,\ \ y(\pi)=0, \tag{2}$$

$$y\left(\frac{\pi k}{n}-0\right)=y\left(\frac{\pi k}{n}+0\right),\ \ y'\left(\frac{\pi k}{n}-0\right)=y'\left(\frac{\pi k}{n}+0\right)-\beta_k\int_0^\pi y(t)dt,\ \ k=\overline{1,n-1} \tag{3}$$

has been considered, where $q(x)$ are enough times differentiable real-valued function; β_k is a real constant, λ is a spectral parameter.

Note that in [11], [2, p. 112] formulas for the first regularized trace of the problem when $\beta_k=0$, $k=\overline{1,n-1}$ for equation (1) with additional summands of the form $\sum_{k=1}^{n-1}\alpha_k y(\frac{\pi k}{n})$ are written . These works as well as [12] are the most related to the theme of our problem under consideration.

The main result of [12] is given in the following theorem.

Theorem 1. *For the first regularized trace of problem (1) - (3) the following formula is true :*

$$\sum_{m=0}^{\infty}\sum_{j=1}^{2n}\left(\lambda_{m,j}-(2nm+j)^2-\left(1+\frac{1}{2nm+j}\right)\frac{2}{\pi}\int_0^\pi q(t)dt\right)$$

$$= -\frac{1}{4}\Big(q(0)+q(\pi)\Big)+\frac{1}{2\pi}\int_0^\pi q(t)dt, \tag{4}$$

where $\lambda_{m,j}$ are eigenvalues of problem (1)-(3). Herewith, eigenvalues have an asymptotic behavior $\lambda_{m,j}=s^2_{m,j}$, where

$$s_{m,j}=(2nm+j)+\frac{c_{1,j}}{2nm+j}+\frac{c_{2,j}}{(2nm+j)^2}+O\left(\frac{1}{(2nm+j)^3}\right), \tag{5}$$

$$c_{1,j}=\frac{1}{2\pi}\int_0^\pi q(t)dt, c_{2,j}=\frac{1-(-1)^j}{2}\frac{i}{\pi}\sum_{k=1}^{n-1}(\beta_k+\beta_{n-k})\,e^{\frac{i\pi jk}{n}}, \tag{6}$$

$$j=\overline{1,2n}, m=0,1,2,.$$

Throughout this note we mainly use techniques from works [12-17].

Statement of the problem and formulation of the main result

Consider a problem on eigenvalues for an integral-differential operator:

$$-y''(x)+q(x)y(x)+\gamma\int_0^\pi y(t)dt=\lambda y(x),\ \frac{\pi}{n}(k-1)<x<\frac{\pi}{n}k,\ k=\overline{1,n};\ n\geq 2 \tag{7}$$

with the boundary conditions (2) and the generalized conditions of "bonding" (3) in the deleted points of the interval.

Our goal is to build formulas for the first regularized trace, similar to formula (4), for the spectral problem under consideration (7), (2) and (3).

The main result of this work is given in the following theorem.

Theorem 2. *For the first regularized trace of problem (7), (2), (3) the following formula holds:*

$$\sum_{m=0}^{\infty}\sum_{j=1}^{2n}\left(\lambda_{m,j}-(2nm+j)^2-\left(1+\frac{1}{2nm+j}\right)\frac{2}{\pi}\int_0^\pi q(t)dt\right)$$

$$= -\frac{1}{4}\Big(q(0)+q(\pi)\Big)+\frac{1}{2\pi}\int_0^\pi q(t)dt-\gamma\pi, \tag{8}$$

where $\lambda_{m,j}$ are eigenvalues of problem (7), (2), (3). In this case the eigenvalues at large m have the asymptotic behavior $\lambda_{m,j}=s^2_{m,j}$, $j=\overline{1,2n}$, where the quantities $s_{m,j}$, $c_{1,j}$, $c_{2,j}$ are determined by formulas (5) and (6), respectively.

A SHORT PROOF OF THE THEOREM

By standard calculations on each interval $I_k: \frac{\pi}{n}(k-1)<x<\frac{\pi}{n}k$, we can write the asymptotic behavior (when $|s|\to\infty$) of two linearly independent solutions of equation (1):

$$y_{1,k}(x,s)\sim e^{isx}\sum_{\nu=0}^{\infty}\frac{a_{\nu,k}(x)}{s^\nu},\ y_{2,k}(x,s)\sim e^{-isx}\sum_{\nu=0}^{\infty}(-1)^\nu\frac{a_{\nu,k}(x)}{s^\nu},$$

where $a_{0,k}(x)\equiv 1$,

$$a_{\nu,k}(x)=\frac{i}{2}\left\{a'_{\nu-1,k}(x)-a'_{\nu-1,k}\left(\pi\frac{k-1}{n}\right)-\int_{\pi\frac{k-1}{n}}^{x}q(t)a_{\nu-1,k}(t)dt\right\}. \tag{9}$$

Here, as usual, it is assumed that the complex plane ($\lambda=s^2, s=\sqrt{\lambda}$) is divided into four sectors by lines $\arg s=0$ and $\arg s=\frac{\pi}{2}$ and this asymptotic is available in each of the four sectors.

From formula (7) we obtain

$$a_{1,k}(x)=-\frac{i}{2}\int_{\pi\frac{k-1}{n}}^{x}q(t)dt,\ a_{2,k}(x)=\frac{1}{4}\left\{q(x)-q\left(\pi\frac{k-1}{n}\right)-\frac{1}{2}\left(\int_{\pi\frac{k-1}{n}}^{x}q(t)dt\right)^2\right\},$$

$$a_{\nu,k}\left(\pi\frac{k-1}{n}\right)=0, \nu=1,2,;$$

$$y_{1,k}\left(\pi\frac{k-1}{n},s\right)=e^{i\frac{\pi(k-1)s}{n}},\ y_{2,k}\left(\pi\frac{k-1}{n},s\right)=e^{-i\frac{\pi(k-1)s}{n}}.$$

On each of the intervals I_k a general solution of the integro – differential equation (7) we build by the variation method of arbitrary constants. We are introducing the solution in the form

$$Y(x)=A_k(x)y_{1,k}(x,s)+B_k(x)y_{2,k}(x,s).$$

We get representation of the general solution of equation (7). This solution is a two-parameter family. We denote these constants by A_k^0, B_k^0.

Satisfying the boundary conditions (2) and the generalized conditions of "bonding" (3), we obtain with respect to the constant A_k^0, B_k^0, a linear system of $2n$ equations, determinant $\Delta(s)$ of which will be characteristic determinant of the spectral problem (7), (2), (3).

The function $\Delta(s)$ is defined by the following asymptotic expression:

$$\Delta(s)=e^{i\pi s}\left\{1+\frac{a_1}{s}+\frac{a_2-(\beta_1+\ldots+\beta_{n-1})+\gamma\pi}{s^2}+O\left(\frac{1}{s^3}\right)\right\}$$

$$+e^{-i\pi s}\left\{-1+\frac{a_1}{s}-\frac{a_2-(\beta_1+\ldots+\beta_{n-1})+\gamma\pi}{s^2}+O\left(\frac{1}{s^3}\right)\right\}$$

$$+\sum_{k=1}^{n-1}e^{i\frac{\pi k}{n}s}\left\{\frac{\beta_k+\beta_{n-k}}{s^2}+O\left(\frac{1}{s^3}\right)\right\}-\sum_{k=1}^{n-1}e^{-i\frac{\pi k}{n}s}\left\{\frac{\beta_k+\beta_{n-k}}{s^2}+O\left(\frac{1}{s^3}\right)\right\}+\left\{\frac{4i\gamma}{s^3}+O\left(\frac{1}{s^4}\right)\right\}.$$

Here

$$a_1=-\frac{i}{2}\int_0^\pi q(t)dt,\ a_2=\frac{1}{4}\left\{q(\pi)-q(0)-\frac{1}{2}\left(\int_0^\pi q(t)dt\right)^2\right\}.$$

Analyzing the equation $\Delta(s)=0$, we obtain that problem (7), (2) (3) has a $4n$ series of eigenvalues with asymptotics (5),(6). Taking into account that the function $\Delta(s)$ is odd, we get that alongside with the eigenvalues $s_{m,j}$ from (5) the numbers $-s_{m,j}$ are also the roots of a characteristic polynomial. We denoted them by $s_{m,j+2n}$, $j=\overline{1,2n}$. Hence we obtain the vanishing of the coefficients $c_{2,j}$ from (6) with the even numbers j. Thus, in terms of the spectral parameter λ we get the $2n$ series of eigenvalues.

The function $\Delta(s)$ belongs to the class K of entire functions of first order [4]. Therefore, for this function one can apply the method of calculating the regularized sum of roots of quasipolynomials based on the construction of the zeta function associated with the function $\Delta(s)$ and using the method of successive approximations of Horn (see. [4]).

Since this technique is well-established, due to cumbersome calculations, we shall not give them here and by this we complete the proof of the theorem.

Remark 1. *In the particular case for $\gamma=0$, problem (7), (2), (3) coincides with problem (1) - (3) and Eq. (8) of Theorem 2 coincides with the main result of Theorem 1 (formula (4)).*

Remark 2. *In the particular case for $\gamma=0$, $\beta_k=0, k=\overline{1,n-1}$, problem (7), (2), (3) coincides with the Dirichlet problem and the main result of Theorem 2 (the formula (8)) coincides with the classical result:*

$$\sum_{m=0}^{\infty}\left(\lambda_m-m^2-\frac{1}{\pi}\int_0^\pi q(t)dt\right)=-\frac{1}{4}\Big(q(0)+q(\pi)\Big)+\frac{1}{2\pi}\int_0^\pi q(t)dt.$$

ACKNOWLEDGMENTS

The authors express their gratitude to T. Sh. Kalmenov, B. E. Kanguzhin and M. A. Sadybekov for their immense knowledge and guidance. This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan under the grant number 0825/GF4. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

REFERENCES

1. V. A. Sadovnichij, *Teorija Operatorov*, Drofa, Moskow, 2004, (in Russian)
2. S. I. Mitrohin, *Spektral'naja Teorija Operatorov: Gladkie, Razryvnye, Summiruemye Kojefficienty*, Internet-Universitet Informacionnyh Tehnologij, Moskow, 2009, (in Russian).
3. I. M. Gel'fand, and B. M. Levitan, *Dokl. AN SSSR* **88**, 593–596 (1953).
4. V. B. Lidskij, and V. A. Sadovnichij, *Funkcional'nyj Analiz i Ego Prilozhenija* **1**. 52–59 (1967).
5. V. V. Karachik, M. A. Sadybekov, and B. T. Torebek, *Electronic Journal of Differential Equations* **2015**, 1–9 (2015), (Article ID 244).
6. T. Sh. Kal'menov, and D. Suragan, *Differential Equations* **48**, 441–445 (2012).
7. B. E. Kanguzhin, D. B. Nurakhmetov, and N. E. Tokmagambetov, *Ufimsk. Mat. Zh.* **3**, 80–92 (2010).
8. B. E. Kanguzhin, and A. A. Aniyarov, *Mathematical Notes* **89**, 819–829 (2011)
9. B. E. Kanguzhin, D. B. Nurakhmetov, and N. E. Tokmagambetov, *Izv. Vyssh. Uchebn. Zaved. Mat.* **2**, 9-16 (2014).
10. B. E. Kanguzhin, and N. E. Tokmagambetov, *Doklady Mathematics* **91**, 1–4 (2015).
11. M. Martinovich, *Differenc. Uravnenija* **18**, 537–540 (1982).
12. N. S. Imanbaev, and M. A. Sadybekov, *Vestnik KazNU, Ser. Mat., Meh., Inf.* **2**, 66–71 (2014).
13. T. Sh. Kalmenov, and A. S. Shaldanbaev, *Journal of Inverse and Ill-Posed Problems* **18**, 471–492 (2010).
14. M. B. Muratbekov, T. Sh. Kalmenov, and M. M. Muratbekov, *Complex Variables and Elliptic Equations* **60**, 1752–1763 (2015).
15. T. Sh. Kal'menov, M. A. Sadybekov, and A. M. Sarsenbi, *Differential Equations* **47**, 144–148 (2011).
16. M. A. Sadybekov, and A. M. Sarsenbi, *Differential Equations* **44**, 685–691 (2008).
17. M. A. Sadybekov, and N. S. Imanbaev, *Differential Equations* **48**, 896–900 (2012)