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Inverse Problem with Integral Overdetermination for System of Equations of Kelvin-Voight Fluids

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Abstract. In the paper we consider an inverse problem for the three-dimensional nonlinear pseudoparabolic equations describing the Kelvin-Voight motion. The inverse problem consists of finding a velocity field and pressure which is gradient and also a right-hand said of the equation. Additional condition about the solution to the inverse problem is given in the form of integral overdetermination condition. The existence and uniqueness of weak generalized solution of this inverse problem in the sobelev space is proved.

Introduction

The inverse problems for the partial differential equations arise in many areas of the science when attempts to describe the internal characteristics of the medium in which the flowing of physical and chemical processes by the results of observations of these processes in available areas for measurement. In last years the theory of the inverse problems for the different of non-classical equations of mathematical physics is actively developed in [1-5]. Inverse problems of determining the right-hand side of linear and nonlinear Navier-Stokes equation under a final also an integral overdetermination conditions were studied in papers [1-3]. The weak generalized solutions of the inverse problem for parabolic equations were considered in [4,5].

The solvability of initial boundary direct problems for equations describing the Kelvin-Voight motion and which are some modifications was investigated by A.P. Oskolkov [6-8]. He proved the unique existence of global weak and also the strong solutions of nonlinear initial-boundary value problems for linear viscoelastic Kelvin-Voight equations in threedimensional case.

In this work we discuss the unique solvability of the inverse problem for the three-dimensional nonlinear equations of Kelvin-Voigt fluids [6-7], in the case when additional information on solving the direct problem is given in the integral form.

By the successive approach method the existence and uniqueness of global in time weak generalized solution $(\vec{\upsilon}, f) \in L_\infty\left(0,T;\overset{\circ}{J}{}^1(\Omega)\right) \cup L_2\left(0,T;\overset{\circ}{J}{}^1(\Omega)\right) \times L^2(0,T)$ of inverse problem is proved.

Statement of the problem

Let Ω is a bounded domain in $R^m, m = 2,3$ with smooth boundary $\partial\Omega \in C^2$. In the cylinder $Q_T = \Omega \times (0,T)$, $T > 0$ with the lateral surface $\Sigma = \partial\Omega \times (0,T)$ we consider the following inverse problem consists of finding a set $\{\vec{\upsilon}(x,t), \nabla p(x,t), f(t)\}$ of functions satisfying the system of equations:

$$\vec{\upsilon}_t - \nu\Delta\vec{\upsilon} + (\vec{\upsilon}\nabla)\vec{\upsilon} - \chi\Delta\vec{\upsilon}_t + \nabla p = f(t)\vec{g}(x,t), \qquad (x,t) \in Q_T, \tag{1}$$

$$div\vec{\upsilon} = 0, \ (x,t) \in Q_T, \tag{2}$$

the initial condition

$$\vec{\upsilon}(x,0) = \vec{\upsilon}_0(x), \quad x \in \Omega, \tag{3}$$

the boundary condition

$$\vec{\upsilon}(x,t)=0,\ (x,t)\in\Sigma \tag{4}$$

and the integral overdetermination condition

$$\int\limits_{\Omega}(\vec{\upsilon}(x,t)\vec{u}(x,t)+\chi\nabla\vec{\upsilon}(x,t)\nabla\vec{u}(x,t))dx=e(t),\quad t\in[0,T], \tag{5}$$

where $(\vec{v},\nabla)\vec{v}=\sum_{i=1}^{m}v_i\frac{\partial}{\partial x_i}\vec{v}(x,t)$, v_i are components of the vector-valued function $\vec{v}(x,t)$. The functions $\vec{\upsilon}_0(x)$, $\vec{u}(x,t)$, $e(t)$, $\vec{g}(x,t)$ and the positive constants ν, χ are given while $\{\vec{\upsilon},\nabla p,f\}$ is unknown.

The global existence and uniqueness of the weak and also the strong solutions of the nonlinear initial-boundary direct problem (1)-(4) with the right-hand side $f_1(x,t)=f(t)\vec{g}(x,t)\in L_2(Q_T)$ was proved in [6]. We using results of [6-7] for the direct problem (1)-(4) and also the methodic of U.U. Abylkayrov in [4], we proof the theorem of existence and uniqueness of global in time weak generalized solution of the three dimensional nonlinear inverse problem (1)-(5).

We used the notations of functional spaces and they are norms in [6].

Definition. A pair of functions $(\vec{\upsilon},f)$ is called a generalized solution of the inverse problem (1)-(5), if $\vec{\upsilon}(x,t)$, $\vec{\upsilon}_x(x,t)\in L_2(\Omega)$ for all $t\in[0.T]$ and $\vec{\upsilon}_x(x,t)\in L_2(Q_T)$, $f(t)\in L^2(0,T)$ and satisfied the following integral identities:

$$\iint\limits_{Q_T}\{-\vec{\upsilon}\varphi_t+\nu\vec{\upsilon}_x\varphi_x-\vec{\upsilon}_k\vec{\upsilon}\varphi_{x_k}-\chi\vec{\upsilon}_x\varphi_{xt}\}dxdt=\int\limits_{\Omega}(\vec{\upsilon}_0\vec{\varphi}(x,0)+\chi\vec{\upsilon}_{ox}\vec{\varphi}_x(x,0))dx+\iint\limits_{Q_T}f(t)\vec{g}(x,t)\vec{\varphi}(x,t)dxdt \tag{6}$$

for any $\varphi(x,t)\in\overset{\circ}{W}{}_2^{1,1}(Q_T)\cap J(Q_T)$, such that $\varphi_{xt}\in L_2(Q_T)$, $\varphi(x,T)=\varphi_x(x,T)=0$,

$$e'(t)=\int\limits_{\Omega}(u_t\vec{\upsilon}-\nu\vec{\upsilon}_x\vec{u}_x+\chi\vec{\upsilon}_x\vec{u}_{xt}+\upsilon_k\vec{\upsilon}\vec{u}_{x_k})dx+f(t)\int\limits_{\Omega}\vec{u}\vec{g}(x,t)dx \tag{7}$$

where

$$\vec{u}\in C^1(0,T;J^1(\Omega)),\ e(t)\in W_2^1(0,T),\ \vec{g}(x,t)\in L_\infty(0,T;L_2(\Omega)),\ \int\limits_{\Omega}\vec{u}(x,t)\vec{g}(x,t)dx\neq 0\ \text{ for }\forall t\in[0,T]. \tag{8}$$

Lemma. The inverse problem (1)-(5) is equivalent to the statement of the problem (1)-(4), (7) for sufficiently smooth solution $(\vec{\upsilon},f)$, and for the joint date of the problem.

Remark. In the problem (1)-(6) under the condition (8) the function $f(t)$ can be expressed explicitly, i.e.

$$f(t)=(u,\ g)_{2,\Omega}^{-1}\left[e'(t)-\int\limits_{\Omega}(\vec{\upsilon}u_t-\nu\nabla\vec{\upsilon}\nabla\vec{u}+\chi\nabla\vec{\upsilon}\nabla\vec{u}_t+\vec{\upsilon}_k\vec{\upsilon}\vec{u}_{x_k})dx\right]. \tag{9}$$

The main result of this work is the following proposition.

Theorem. If supposition (8) is valid and $\vec{\upsilon}_0 \in \overset{\circ}{W}{}_2^1(\Omega) \cap J(\Omega)$, then exists unique weak generalized solution $(\vec{\upsilon}, f)$ of the inverse problem (1)-(5).

Proof. To prove this theorem we can use the successive approximations method [4]. Take as zero approximation $\vec{\upsilon}^0 = 0$ and define $(\vec{\upsilon}^m, f^m)$ through the relation:

$$f^m(t) = (u,\ g)_{2,\Omega}^{-1}\left[e'(t) - \int\limits_\Omega \vec{u}_t \vec{\upsilon}^{m-1} dx + \int\limits_\Omega (\nu\nabla\vec{u} - \chi\nabla\vec{u}_t)\vec{\upsilon}_x^{m-1} dx - \int \upsilon_k^{m-1} \vec{\upsilon}^{m-1} \vec{u}_{x_k} dx \right] \tag{10}$$

$$\iint\limits_{Q_T} \left\{ -\vec{\upsilon}^m \vec{\varphi}_t + \nu \vec{\upsilon}_x^m \vec{\varphi}_x^m - \chi \vec{\upsilon}_x^m \vec{\varphi}_{xt} - \upsilon_k^m \vec{\upsilon}^m \vec{\varphi}_{x_k} \right\} dxdt = \int\limits_\Omega \left(\vec{\upsilon}_0 + \chi\nabla\vec{\upsilon}_0 \right) \vec{\varphi}(x,0) dx + \iint\limits_{Q_T} f^m(t) \vec{g} \vec{\varphi} dxdt \tag{11}$$

for all $m = 1,2,...,$ and for any $\vec{\varphi} \in \overset{\circ}{W}{}_2^{1,1}(Q_T) \cap J(Q_T)$ such that $\vec{\varphi}_{xt} \in L_2(Q_T),\ \vec{\varphi}(x,T) = \varphi_x(x,T) = 0$.

Let us substitute $f^{(m)}(t)$ from (10) into (11), whence, by the theory of equations of motion of Kelvin-Voigt [6] it follows, that there is a unique weak solution of $\vec{\upsilon}^m \in \overset{0}{J}(Q_T)$, which $\vec{\upsilon}^m, \vec{\upsilon}_x^m \in L_2(Q_T)$ for (1)-(4). Since the integral identity (11) represents a weak solution of direct problem for the equation of the Kelvin-Voigt with right side $\vec{F}(x,t) = \vec{g}(x,t) f(t)$. Under the assumption (8), we know that $\vec{F}(x,t) \in L_2(Q_T)$ and consequently from [6] it follows that there is a unique weak generalized solution of the direct problem (11) for the equation of the motion of Kelvin-Voigt.

Thus the sequence of pairs $(\vec{\upsilon}^m, f^m)$ is well defined. If we prove that the sequence $(\vec{\upsilon}^m, f^m)$ is a sequence of the Cauchy, then by the completeness of the space $V_2(Q_T) \times L_2(0,T)$ follows, that a pair of functions $(\vec{\upsilon}, f)$ is the limit for the sequence $\{(\vec{\upsilon}^m, f^m)\}$, i.e. $(\vec{\upsilon}^m, f^m) \to (\vec{\upsilon}, f)$ as $m \to \infty$. Therefore, $(\vec{\upsilon}, f)$ is the desired weak solution of the inverse problem of the (1)-(4), (7).

Let us introduce the notations

$$U^{m+1} = \vec{\upsilon}^{m+1} - \vec{\upsilon}^m \text{ and } F^{m+1} = f^{m+1} - f^m$$

from (10), (11) we get the relations

$$F^{m+1} = (\vec{u}, g)_{2,\Omega}^{-1}\left[\int\limits_\Omega (\nu\nabla\vec{u} - \chi\nabla\vec{u}_t)\vec{U}_x^m - \int\limits_\Omega \vec{u}_t \vec{U}^m dx - \int\limits_\Omega (\upsilon_k^m \vec{U}^m + U_k^m \vec{\upsilon}^{m-1}) \vec{u}_{x_k} dx \right], \tag{12}$$

$$\iint\limits_{Q_T} \left[-\vec{U}^{m+1}\vec{\varphi}_t + \nu \vec{U}_x^{m+1} \vec{\varphi}_x - \chi \vec{U}_x^{m+1} \vec{\varphi}_{xt} - \left(U_k^{m+1} \vec{\upsilon}^{m+1} + \upsilon_k^m \vec{U}^{m+1} \right) \vec{\varphi}_{x_k} \right] dxdt = \iint\limits_{Q_T} F^{m+1} \vec{g} \vec{\varphi} dxdt \tag{13}$$

with $\vec{U}^m(x,0) = 0$ initial conditions for $m = 1,2,3.....$ and $\vec{\varphi} \in \overset{\circ}{W}{}_2^{1,1}(Q_T) \cap J(Q_T)$ such, that $\vec{\varphi}_{xt} \in L_2(Q_T),\ \vec{\varphi}(x,T) = \varphi_x(x,T) = 0$.

We will estimate each term in the right-hand side of (12) by using Holder, Cauchy inequalities and $\|\vec{v}\|_{2,\Omega} \le C(\Omega) \|\vec{v}_x\|_{2,\Omega}$, $v(x) \in \overset{\circ}{W}{}_2^1(\Omega)$ Poincare inequality [8] as follows.

$$\left|F^{m+1}(t)\right| = \left|(u,g)_{2,\Omega}^{-1}\right| \left\| \left[\nu\int_\Omega \nabla U^m \nabla u dx - \chi\int_\Omega \nabla U^m \nabla u_t dx - \int_\Omega U^m u_t dx - \int_\Omega \left(\vec{\upsilon}_k^m U^m + U^m \vec{\upsilon}^{m-1}\right) u_{x_k} dx \right] \right\| \le$$

$$\left|(u,g)_{2,\Omega}^{-1}\right| \left\{ \left\|\nabla U^m\right\| \left(\nu\|\nabla u\| + \chi\|\nabla u_t\|\right) + \left\|U^m\right\| \cdot \|u_t\| + \left\|u_{x_k}\right\| \left(\left\|\vec{\upsilon}_k^m\right\|_{4,\Omega} \left\|U^m\right\|_{4,\Omega} + \left\|U^m\right\|_{4,\Omega} \left\|\vec{\upsilon}^{m-1}\right\|_{4,\Omega} \right) \right\} \le$$

$$\left|(u,g)_{2,\Omega}^{-1}\right| \left\{ \left\|\nabla \vec{U}^m\right\| \left(\nu\|\nabla \vec{u}\|_{2,\Omega} + \chi\|\nabla \vec{u}_t\| + C(\Omega)\|\vec{u}_t\|\right) + \frac{64}{27}\|u_x\| \cdot \left\|\vec{\upsilon}^m\right\|^{\frac{1}{4}} \left\|\vec{\upsilon}_x^m\right\|^{\frac{3}{4}} \left\|U^m\right\|^{\frac{1}{4}} \left\|\nabla U^m\right\|^{\frac{3}{4}} \right\} \le$$

$$\left|(u,g)_{2,\Omega}^{-1}\right| \left\{ \left(\nu\|\nabla \vec{u}\| + \chi\|\nabla \vec{u}_t\| + C(\Omega)\|\vec{u}_t\|\right) \left\|\nabla U^m\right\| + \frac{64}{27}\left\|\vec{u}_{x_k}\right\| \cdot \left\|\vec{\upsilon}_x^m\right\| \cdot \left\|\nabla U^m\right\| \right\} \le$$

$$\left|(u,g)_{2,\Omega}^{-1}\right| \left\{ \nu\|\nabla \vec{u}\| + \chi\|\nabla \vec{u}_t\| + C(\Omega)\|\vec{u}_t\| + \frac{64}{27}\|\nabla u\| \cdot \left\|\vec{\upsilon}_x^m\right\| \right\} \left\|\nabla U^m\right\|_{2,\Omega} \le C_1 \left\|\nabla U^m\right\|_{2,\Omega}.$$

After integrating it by τ from 0 to t, we obtain the relation

$$\int_0^t \left|F^{m+1}\right|^2 d\tau \le C_1 \left\|\nabla U^m\right\|_{2,Q_t}^2, \tag{14}$$

where $\sqrt{C_1} \equiv \left|(u,g)_{2,\Omega}^{-1}\right| \left\{ \nu\|\nabla \vec{u}\| + \chi\|\nabla \vec{u}_t\| + C(\Omega)\|\vec{u}_t\| + \frac{64}{27}\|\nabla \vec{u}\| \cdot \left\|\vec{\upsilon}_x^m\right\| \right\}$ is a positive constant which independent on m, U^m and f^m.

Further, from the relation (13) assuming therein $\varphi \equiv U^{m+1}$, we get the identity

$$\frac{1}{2}\int_\Omega \left(\left|U^{m+1}\right|^2 + \chi\left|\nabla U^{m+1}\right|^2 \right) dx + \nu\iint_{Q_t} \left|\nabla U^{m+1}\right|^2 dxdt = \iint_{Q_t} U_k^{m+1} \cdot \vec{\upsilon}_{x_k}^{m+1} U^{m+1} dxdt + \iint_{Q_t} F^{m+1} \vec{g} U^{m+1} dxdt. \tag{15}$$

Let estimate the right-hand side of the last identity by Cauchy and (14) inequalities

$$\left| \iint_{Q_T} F^{m+1} \vec{g} U^{m+1} dxdt \right| \le \underset{t\in(0,T)}{vrai\max}\|g\| \cdot \left\|U^{m+1}\right\|_{2,Q_T} \left(\int_0^t \left|F^{m+1}\right|^2 dt \right)^{\frac{1}{2}} \le \frac{\varepsilon}{2}\int_0^t \left|F^{m+1}\right|^2 d\tau + \frac{1}{2\varepsilon} C_2^2 \left\|U^{m+1}\right\|_{2,Q_T}^2,$$

$$\left| \iint_{Q_T} U_k^{m+1} \vec{\upsilon}_{x_k}^{m+1} U^{m+1} dxdt \right| \le \int_0^t \left\|\vec{\upsilon}_x^{m+1}\right\| \cdot \left\|U^{m+1}\right\|_{4,\Omega}^2 dt \le \left(\frac{4}{3}\right)^{\frac{3}{2}} C(\Omega) \int_0^t \left\|\vec{\upsilon}_x^{m+1}\right\| \cdot \left\|\nabla U^{m+1}\right\|_{2,\Omega}^2 dt \le C_3 \left\|\nabla U^{m+1}\right\|_{2,Q_t}^2$$

where $C_2 \equiv \|g(x,t)\|_{L_\infty(0,T;L_2(\Omega))}$, $C_3 \equiv \frac{8}{\sqrt{27}} C(\Omega) \left\|\vec{\upsilon}_x^{m+1}\right\|_{L_\infty(0,T;L_2(\Omega))}$ are positive constants.

Consequently, from (15), we get the estimation

$$\left\|U^{m+1}\right\|^2 + \chi\left\|U_x^{m+1}\right\|_{2,\Omega}^2 + \nu\int_0^t \left\|U_x^{m+1}\right\|_{2,\Omega}^2 d\tau \le \frac{C_2^2}{2\varepsilon}\int_0^t \left\|U^{m+1}\right\|^2 d\tau + \frac{C_3}{\chi}\chi\int_0^t \left\|U_x^{m+1}\right\|^2 d\tau + \varepsilon\int_0^t \left|F^{m+1}\right|^2 d\tau \le$$

$$\frac{C_4}{\varepsilon}\int_0^t\left(\left\|U^{m+1}\right\|^2+\chi\left\|U_x^{m+1}\right\|^2\right)d\tau+\varepsilon\int_0^t\left|F^{m+1}\right|^2 d\tau. \tag{16}$$

From the last relation by the Gronwall lemma [8] we get the estimate

$$\max_{t\in[0,T]}\left(\left\|\vec{U}^{m+1}\right\|^2+\chi\left\|\vec{U}_x^{m+1}\right\|_{2,\Omega}^2\right)+\nu\left\|\vec{U}_x^{m+1}\right\|_{2,Q_t}^2\le\varepsilon\exp\left(C_4\varepsilon^{-1}t\right)\int_0^t\left|F^{m+1}\right|^2 d\tau\cdot \tag{17}$$

Considering together the (14), (17), we note that the following inequalities are true

$$\left\|F^{m+1}\right\|_{L_2(0,t)}^2\le C_1\varepsilon\nu^{-1}\exp\left[C_4\varepsilon^{-1}t\right]\left\|F^m\right\|_{L_2(0,t)}^2, \tag{18}$$

$$\max_{t\in[0,T]}\left(\left\|\vec{U}^{m+1}\right\|_{2,\Omega}^2+\chi\left\|\vec{U}_x^{m+1}\right\|_{2,\Omega}^2\right)+\nu\left\|\vec{U}_x^{m+1}\right\|_{2,Q_t}^2\le C_1\varepsilon\exp\left(C_4\varepsilon^{-1}t\right)\left\|\nabla\vec{U}^m\right\|_{2,Q_t}^2$$

or

$$\left\|\vec{U}^{m+1}\right\|_{V_2(Q_T)}^2\le C_1\varepsilon\nu^{-1}\exp\left(C_4\varepsilon^{-1}t\right)\left\|\vec{U}^m\right\|_{V_2(Q_t)}, \tag{19}$$

for any $m=0,1,2,\dots$.

By virtue of the arbitrariness of ε and t, choosing the ε_0 and t_1 such that satisfied the inequality

$$C_1\varepsilon_0\nu^{-1}\exp\left(C_4\varepsilon_0^{-1}t_1\right)\le q<1 \tag{20}$$

and from (18)-(19) follows the estimates

$$\left\|F^{m+1}\right\|_{L_2(0,t_1)}^2\le q\left\|F^m\right\|_{L_2(0,t_1)}^2,\quad \left\|\vec{U}^{m+1}\right\|_{V_2(Q_{t_1})}^2+\left\|\nabla\vec{U}^{m+1}\right\|_{2,\Omega}^2\le q\left\|\vec{U}^m\right\|_{V_2(Q_{t_1})} \tag{21}$$

for any $m=1,2,\dots,$ where $Q_{t_1}=\Omega\times(0,t_1]$.

Consequently, from the foregoing estimate (21) and convergence of the geometric progression it follows that $\{(\vec{\upsilon}^m,f^m)\}$ is a Cauchy sequence in the functional space $V_2(Q_{t_1})\times L_2(0,t_1)$. By virtue of the foregoing reasoning there is a unique pair of the functions $(\vec{\upsilon},f)\in V_2(Q_{t_1})\times L_2(0,t_1)$ with $\vec{\upsilon}_x\in L_2(\Omega)$, for all $t\in(0,t_1]$, such that

$$\vec{\upsilon}^m\to\vec{\upsilon}\text{ in }V_2(Q_{t_1}),\ \vec{\upsilon}_x^m\to\vec{\upsilon}_x\text{ in }L_2(\Omega)\text{ and }f^m\to f\text{ in }L_2(0,t_1)\text{ as }m\to\infty. \tag{22}$$

The passing to the limit as $m\to\infty$ in relations of (10)-(11) due to the strong convergence of $\vec{\upsilon}^m$ and f^m, we obtain that the limit functions $\vec{\upsilon}$ and f are a weak generalized solution of the inverse problem (1)-(5) in $Q_{t_1}=\Omega\times(0,t_1]$.

Now, let us prove the uniqueness of the solution of the inverse problem (1)-(5) in Q_{t_1}.

We assume to the contrary that there are two distinct solutions $(\vec{\upsilon}_k,f_k)$, $k=1,2$ in Q_{t_1}. Then due to relation (21), we get the estimates

$$\|f_1-f_2\|_{L_2(0,t_1)}^2\le q\|f_1-f_2\|_{L_2(0,t_1)}^2,\quad \|\vec{\upsilon}_1-\vec{\upsilon}_2\|_{V_2(Q_{t_1})}^2\le q\|\vec{\upsilon}_1-\vec{\upsilon}_2\|_{V_2(Q_{t_1})}^2, \tag{23}$$

where it known that the q<1. From the relation (23) it follows that $f_1=f_2$ and $\vec{\upsilon}_1=\vec{\upsilon}_2$.

So, we have proved that the existence and uniqueness of solutions $(\vec{\upsilon}, f)$ only in interval $(0,t_1)$, $t_1 < T$.

Next, we extend the proof of the theorem, i.e. give a proof in all $(0,T)$.

We have a constants ε_0, C_i and t_1 which independent on initial given function $\vec{\upsilon}_0(x)$. Therefore, if such t are unexhausted all interval $(0,T)$, then repeating the reasoning for $t \in [t_1, t_2]$, where t_1 is such that $\vec{\upsilon}_0(x) \equiv \vec{\upsilon}(x,t_1)$, etc., in a finite number of steps we see that, the inverse problem (1)-(5) has a unique generalized solution in all $Q_T = \Omega \times [0,T]$. Thereby, we complete proving theorem.

References

[1] A. I. Prilepko, I. A. Vasin, J. Differential equations. 25,1(1989), 106-117.

[2] A.I. Prilepko, D.G. Orlovsky, I.A. Vasin, Methods for Solving Inverse Problems in Mathematical Physics, Marcel Dekker Inc., 2000.

[3] I.A. Vasin, J. Mathematical Notes. 54(1993), 34-44.

[4] U. U. Abylkairov, Mathematical Journal, Almaty, 4,10(2003) 5-12.

[5] V.L.Kamynin, J. Mathematical Notes. 73,2(2003)371-387.

[6] A. P. Oskolkov, J. Zap. Nauchn. Sem. LOMI 38(1973) 98-136.

[7] A. P. Oskolkov, J. Problemy Math. Anal., 5(1973) 78-87.

[8] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gorden and Breach, New York, 1969.