On spectral problems for loaded two-dimension Laplace operator

Meiramkul Amangaliyeva, Muvasharkhan Jenaliyev, Kanzharbek Imanberdiyev, and Murat Ramazanov

Citation: AIP Conference Proceedings **1759**, 020049 (2016); doi: 10.1063/1.4959663
View online: http://dx.doi.org/10.1063/1.4959663
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1759?ver=pdfcov
Published by the AIP Publishing

On spectral problems for loaded two-dimension Laplace operator

Meiramkul Amangaliyeva*, Muvasharkhan Jenaliyev*, Kanzharbek Imanberdiyev*
and Murat Ramazanov†

*Institute of Mathematics and Mathematical Modeling, 050010, Almaty, Kazakhstan
†Buketov Karaganda State University, 100028, Karaganda, Kazakhstan

Abstract. The investigated spectral problems arise in the study of the stabilization problem for a loaded heat equation. The dimension of the space variable is equal to two. Stabilization of the solution of the equation is carried out by means of boundary control actions. The solution of this problem can be solved by separation of variables.

Keywords: Loaded Laplace equation, Spectrum, Eigenfunction
PACS: 02.30.Jr, 02.70.Hm

INTRODUCTION

Let $\Omega = \{x, y: -\pi/2 < x, y < \pi/2\}$ be domain with boundary $\partial\Omega$. It is considered in cylinder $Q = \Omega \times \{t > 0\}$ with the side surface $\Sigma = \partial\Omega \times \{t > 0\}$.

We consider the boundary value problem for a loaded heat equation

$$\begin{aligned} u_t - \Delta u + \alpha u(0, y, t) + \beta u(x, 0, t) &= 0, \ \{x, y, t\} \in Q, & (1)\\ u(x, y, 0) &= u_0(x, y), \ \{x, y\} \in \Omega, & (2)\\ u(x, y, t) &= p(x, y, t), \ \{x, y, t\} \in \Sigma. & (3) \end{aligned}$$

For boundary value problem (1)–(3) we investigate the stabilization problem. It is required to find such functions $p(x, y, t)$, so that the solution $u(x, y, t)$ satisfies the following inequality

$$\|u(x, y, t)\|_{L_2(\Omega)} \leq C_0 e^{-\sigma t}, \ \sigma = \text{const} > 0, \ C_0 = \text{const} > 0, \ t > 0. \qquad (4)$$

We introduce the auxiliary boundary value problem. Let $\Omega_1 = \{x, y: -\pi < x, y < \pi\}$ and $Q_1 = \Omega_1 \times \{t > 0\}$. We obtain

$$\begin{aligned} z_t - \Delta z + \alpha z(0, y, t) + \beta z(x, 0, t) &= 0, \ \{x, y, t\} \in Q_1, & (5)\\ z(x, y, 0) &= z_0(x, y), \ \{x, y\} \in \Omega_1, & (6)\\ \frac{\partial^{(j)} z(-\pi, y, t)}{\partial x^{(j)}} &= \frac{\partial^{(j)} z(\pi, y, t)}{\partial x^{(j)}}, \ \{y, t\} \in (-\pi, \pi) \times \{t > 0\}, & \\ \frac{\partial^{(j)} z(x, -\pi, t)}{\partial y^{(j)}} &= \frac{\partial^{(j)} z(x, \pi, t)}{\partial y^{(j)}}, \ \{x, t\} \in (-\pi, \pi) \times \{t > 0\}, \ j = 0, 1. & (7) \end{aligned}$$

For the auxiliary boundary value problem (5)–(7) we investigate the stabilization problem. It is required to find such initial functions $z_0(x, y, t)$, so that the solution $z(x, y, t)$ satisfies the following inequality

$$\|z(x, y, t)\|_{L_2(\Omega_1)} \leq C_0 e^{-\sigma t}, \ t > 0, \qquad (8)$$

where the constants C_0 and σ are the same as in the original problem (1)–(4).

We will to define the function $z_0(x, y)$ as continuation of function $u_0(x, y)$ from Ω to a large domain Ω_1.

Thus, if we solve the problem (5)–(8), we can find a solution to the original problem (1)–(4). The restriction of the solution $z(x, y, t)$ to domain Q gives us a function $u(x, y, t)$ as solution of boundary value problem (1)–(3), satisfying

International Conference on Analysis and Applied Mathematics (ICAAM 2016)
AIP Conf. Proc. 1759, 020049-1–020049-4; doi: 10.1063/1.4959663
Published by AIP Publishing. 978-0-7354-1417-4/$30.00

to condition (4). Further, the restriction of the solution $z(x,y,t)$ on the boundaries Σ determines the required function $p(x,y,t)$.

To solving of the problem (5)–(8), we use the separation method of variables. Let $\mathbf{Z} = \{0, \pm 1, \pm 2, ...\}$. We have

$$z(x,y,t) = \sum_{k,l \in \mathbf{Z}} Z_{kl}(t)\varphi_{kl}(x,y), \tag{9}$$

where the functions $Z_{kl}(t)$ is a solution of the Cauchy problem:

$$Z'_{kl}(t) + \lambda Z_{kl}(t) = 0,\ t > 0,\ Z_{kl}(0) = Z_{0kl},$$

and the functions $\{\varphi_{kl}(x,y),\ (k,l) \in \mathbf{Z}\}$, is the solution of the corresponding spectral problem discussed below. Note that in the Cauchy problem numbers Z_{0kl} are unknown.

STATEMENT OF PROBLEMS

In the domain $Q = \{x, y : -\pi < x < \pi,\ -\pi < y < \pi\}$ we consider the following two spectral problems:

$$\begin{cases} -\triangle\varphi(x,y) + \alpha\varphi(0,y) = \lambda\varphi(x,y),\ \{x,y\} \in Q, \\ \dfrac{\partial^j\varphi(-\pi,y)}{\partial x^j} = \dfrac{\partial^j\varphi(\pi,y)}{\partial x^j},\ \dfrac{\partial^j\varphi(x,-\pi)}{\partial y^j} = \dfrac{\partial^j\varphi(x,\pi)}{\partial y^j},\ j = 0,1; \end{cases} \tag{10}$$

$$\begin{cases} -\triangle\varphi(x,y) + \alpha\varphi(0,y) + \beta\varphi(x,0) = \lambda\varphi(x,y),\ \{x,y\} \in Q, \\ \dfrac{\partial^j\varphi(-\pi,y)}{\partial x^j} = \dfrac{\partial^j\varphi(\pi,y)}{\partial x^j},\ \dfrac{\partial^j\varphi(x,-\pi)}{\partial y^j} = \dfrac{\partial^j\varphi(x,\pi)}{\partial y^j},\ j = 0,1, \end{cases} \tag{11}$$

where Δ is Laplace operator, $\alpha,\ \beta \in \mathbf{C}$ are given complex numbers, $\lambda \in \mathbf{C}$ is spectral parameter.

We introduce some of the definitions from [1]. Let $\Omega - n$-dimensional cube with an edge lengths 2π. Let

$$\mathbf{A}(s) \quad = \quad \sum_{|\gamma| \le m} a_\gamma s^\gamma,\ s^\gamma = s_1^{\gamma_1} \cdots s_n^{\gamma_n},\ |\gamma| = \gamma_1 + \cdots \gamma_n,\ \mathbf{A}(-iD), \tag{12}$$

$$\mathbf{A}(-iD)e^{is\cdot x} \quad = \quad \mathbf{A}(s)e^{is\cdot x},\ s \cdot x = s_1x_1 + ... + s_n \cdot x_n. \tag{13}$$

$P^\infty(\Omega)$ is linear manifold of infinitely smooth periodic in all variables functions. The operator $\mathbf{A} : L_2(\Omega) \to L_2(\Omega)$, defined as the closure in $L_2(\Omega)$ the differential expression $\mathbf{A}(-iD)$, defined originally on the functions of the $P^\infty(\Omega)$, is called $\prod$-operator.

Let $\mathscr{S} = \{s :\ s_k = 0, \pm 1, \pm 2, ...,\ k = 1, ..., n\}$. The eigenfunctions and eigenvalues of $\prod$-operator give a following proposition.

Proposition 1. *The set of exponents $\left\{e^{is\cdot x},\ s \in \mathscr{S}\right\}$ form an orthogonal basis in $L_2(\Omega)$ and is the set of eigenfunctions for each of the $\prod$-operators $\mathbf{A}$. A numbers $\{\mathbf{A}(s),\ s \in \mathscr{S}\}$ are the eigenvalues.*

According to these introductions, our operators of the spectral problems (10)–(11) are loaded $\prod$-operators.

MAIN RESULTS

Let $\mathbf{Z} = \{0, \pm 1, \pm 2, ...\}$. The following lemmas are valid.

Lemma 2. *(a) Let $\forall\, l \in \mathbf{Z} :\ \alpha \neq l^2$. Then spectral problem (10) has the system of eigenvalues and eigenfunctions that defined as*

$$\left\{ \varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\alpha}{l^2 - \alpha}\right)e^{iky},\ \lambda_{kl} = l^2 + k^2,\ l \in \mathbf{Z}' \equiv \mathbf{Z} \setminus \{0\}, \right.$$

$$\left. \varphi_{k0}(x,y) = e^{iky},\ \lambda_{k0} = \alpha + k^2\ (l = 0),\ k \in \mathbf{Z} \right\}. \tag{14}$$

(b) Let $\exists l_0 \in \mathbf{Z}: \ \alpha = l_0^2$. Then spectral problem (10) has the system of eigenfunctions and associated (indicated with $\tilde{}$) functions and the eigenvalues are determined in the form

$$\left\{ \varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\alpha}{l^2-\alpha} \right) e^{iky}, \ \lambda_{kl} = l^2 + k^2, \ l \in \mathbf{Z}' \setminus \{\pm l_0\}; \right.$$

$$\left. \varphi_{kl_0}(x,y) = e^{iky}, \ \tilde{\varphi}^{\pm}_{kl_0}(x,y) = e^{\pm i l_0 x + iky}, \ \lambda_{kl_0} = \alpha + k^2, \ k \in \mathbf{Z} \right\}. \tag{15}$$

Proof. Since $\beta = 0$, then we have that [2]:

$$\varphi_{kl}(x,y) = X_{kl}(x) e^{iky}, \ k, \ l \in \mathbf{Z}, \tag{16}$$

$$\begin{cases} -X''_{kl}(x) + k^2 X_{kl}(x) + \alpha X_{kl}(0) = \lambda_{kl} X_{kl}(x), \\ X^{(j)}_{kl}(-\pi) = X^{(j)}_{kl}(\pi), \ j = 0, 1, \\ k, \ l \in \mathbf{Z}, \ \lambda_{kl} = l^2 + k^2, \end{cases} \tag{17}$$

Now, for part *(a), we have that* $\nexists l \in Z: \ \alpha = l^2$, and for part *(b)* we have that $\exists l_0 \in Z: \ \alpha = l_0^2$. □

Lemma 3. *(a) Let $\forall k, l \in \mathbf{Z}: \ \beta \neq k^2, \alpha \neq l^2$. Then spectral problem (11) has the system of eigenfunctions and eigenvalues defined as*

$$\left\{ \varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\alpha}{l^2-\alpha} \right) \left(e^{iky} + \frac{\beta}{k^2-\beta} \right), \ \lambda_{kl} = k^2 + l^2, \ k, \ l \in \mathbf{Z}'; \right.$$

$$\left. e^{ilx} + \frac{\alpha}{l^2-\alpha}, \ \lambda_{0l} = \beta + l^2, \ l \in \mathbf{Z}'^{iky} + \frac{\beta}{k^2-\beta}, \ \lambda_{k0} = k^2 + \alpha, \ k \in \mathbf{Z}' \right\}. \tag{18}$$

(b) Let $\forall k \in \mathbf{Z}: \ \beta \neq k^2$; and $\exists l_0 \in \mathbf{Z}: \ \alpha = l_0^2$. Then spectral problem (11) has the system of eigenfunctions and associated (indicated with $\tilde{}$) functions and eigenvalues determined in the form (where $\mathbf{Z}'_1 = \mathbf{Z}' \setminus \{\pm l_0\}$)

$$\left\{ \varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\alpha}{l^2-\alpha} \right) \left(e^{iky} + \frac{\beta}{k^2-\beta} \right), \ \lambda_{kl} = k^2 + l^2, \ k \in \mathbf{Z}', \ l \in \mathbf{Z}'_1; \right.$$

$$\varphi_{kl_0}(x,y) = e^{iky} + \frac{\beta}{k^2-\beta},$$

$$\tilde{\varphi}^{\pm}_{kl_0}(x,y) = e^{\pm i l_0 x} \left(e^{iky} + \frac{\beta}{k^2-\beta} \right), \ \lambda_{kl_0} = k^2 + \alpha, \ \alpha = l_0^2, \ k \in \mathbf{Z}';$$

$$\left. \varphi_{0l_0}(x,y) = 1, \ \tilde{\varphi}_{0l_0}(x,y) = e^{\pm i l_0 x}, \ \lambda_{0l_0} = \alpha + \beta \right\}. \tag{19}$$

(c) Let $\forall l \in \mathbf{Z}: \ \alpha \neq l^2$; and $\exists k_0 \in \mathbf{Z}: \ \beta = k_0^2$. Then problem spectral (11) has the system of eigenfunctions and associated (indicated with $\tilde{}$) functions and eigenvalues determined in the form (where $\mathbf{Z}'_2 = \mathbf{Z}' \setminus \{\pm k_0\}$):

$$\left\{ \varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\alpha}{l^2-\alpha} \right) \left(e^{iky} + \frac{\beta}{k^2-\beta} \right), \ \lambda_{kl} = k^2 + l^2, \ k \in \mathbf{Z}'_2, \ l \in \mathbf{Z}'; \right.$$

$$\varphi_{k_0 l}(x,y) = e^{ilx} + \frac{\alpha}{l^2-\alpha},$$

$$\tilde{\varphi}^{\pm}_{k_0 l}(x,y) = e^{\pm i k_0 y} \left(e^{ilx} + \frac{\alpha}{l^2-\alpha} \right), \ \lambda_{k_0 l} = \beta + l^2, \ \beta = k_0^2, \ l \in \mathbf{Z}';$$

$$\left. \varphi_{k_0 0}(x,y) = 1, \ \tilde{\varphi}_{k_0 0}(x,y) = e^{\pm i k_0 y}, \ \lambda_{k_0 0} = \alpha + \beta \right\}. \tag{20}$$

(d) Let $\exists k_0, l_0 \in \mathbf{Z}: \beta = k_0^2, \alpha = l_0^2$. Then spectral problem (11) has the system of eigenfunctions and associated (indicated with ˜) functions and eigenvalues determined in the form of

$$\Big\{ \varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\alpha}{l^2-\alpha}\right)\left(e^{iky} + \frac{\beta}{k^2-\beta}\right),\ \lambda_{kl} = k^2 + l^2,\ k \in \mathbf{Z}_2',\ l \in \mathbf{Z}_1';$$

$$\varphi_{k_0 l}(x,y) = e^{ilx} + \frac{\alpha}{l^2-\alpha},$$

$$\tilde{\varphi}^{\pm}_{k_0 l}(x,y) = e^{\pm i k_0 y}\left(e^{ilx} + \frac{\alpha}{l^2-\alpha}\right),\ \lambda_{k_0 l} = \beta + l^2,\ \beta = k_0^2,\ l \in \mathbf{Z}';$$

$$\varphi_{k_0 0}(x,y) = 1,\ \tilde{\varphi}_{k_0 0}(x,y) = e^{\pm i k_0 y},\ \varphi_{0 l_0}(x,y) = e^{\pm i l_0 x},$$

$$\varphi_{k_0 l_0}(x,y) = e^{\pm i(k_0 y \pm l_0 x)},\ \lambda_{k_0 l_0} = \alpha + \beta \Big\}. \tag{21}$$

Proof. We will find a solution of spectral problem (11) in form

$$\varphi_{kl}(x,y) = X_{kl}(x) Y_k(y),\ k, l \in \mathbf{Z}. \tag{22}$$

Next, using the separation method of variables, we have

$$\frac{-X''(x) + \alpha X(0) - \lambda X(x)}{X(x)} = \frac{Y''(y) - \beta Y(0)}{Y(y)} = -\mu. \tag{23}$$

This problem (23) is reduced to the solution of the following auxiliary spectral problems

$$\begin{cases} -Y_k''(y) + \beta Y_k(0) = \mu_k Y_k(y),\ k \in \mathbf{Z}, \\ Y_k^{(j)}(-\pi) = Y_k^{(j)}(\pi),\ j = 0, 1, \end{cases} \tag{24}$$

and

$$\begin{cases} -X_{kl}''(x) + \mu_k X_{kl}(x) + \alpha X_{kl}(0) = \lambda_{kl} X_{kl}(x),\ k, l \in \mathbf{Z}, \\ X_{kl}^{(j)}(-\pi) = X_{kl}^{(j)}(\pi),\ j = 0, 1. \end{cases} \tag{25}$$

Now, for part (a) we have that $\nexists k, l \in Z: \alpha = l^2, \beta = k^2$; for part *(b)* we have that $\nexists k \in Z: \beta = k^2, \exists l_0 \in Z: \alpha = l_0^2$; for part *(c)* we have that $\nexists l \in Z: \alpha = l^2, \exists k_0 \in Z: \beta = k_0^2$; and finally for part *(d)* we have that $\exists k_0, l_0 \in Z: \alpha = l_0^2$, $\beta = k_0^2$. □

A one-dimensional problem we consider in [2].

Note that the system of eigenfunctions and associated functions constructed in Lemma 2 and 3, according to Paley-Wiener theorem is complete [3].

ACKNOWLEDGMENTS

This publication is supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan under the project number 0823/GF4.

REFERENCES

1. A. A. Dezin, *General Problems in the Theory of Boundary Value Problems*, Nauka, Moscow, 1980, (in Russian, p. 208).
2. M. T. Jenaliyev, and M. I. Ramazanov, *Matematicheskii Zhurnal* **15**, 33–53 (2015), (in Russian).
3. F. Riesz, and B. Sz.-Nagy, *Lecons D'analyse Fonctionnelle*, Akademiai Kiado, Budapest, 1972, (p. 588).