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An inverse coefficient problem of heat conductivity with a nonlocal Samarskii-Ionkin type condition

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Abstract. We consider the problem of modeling the process of determining the distribution function of temperature and time varying structure of homogeneous bar of a given law changes in the average temperature. So there is an inverse problem for the heat equation in which together with finding the solution of the equation it is required to find unknown coefficient depending only on the time variable. The specific features of the considered problems is that the system of eigenfunctions of the multiple differentiation operator subject to boundary conditions of the initial problem does not have the basis property. In this paper, it is proved that this inverse problem has a unique generalized solution.

Keywords: Inverse problem, Heat equation, Not strengthened regular boundary conditions, Samarskii-Ionkin boundary conditions, Biorthogonal Fourier series, Riesz basis
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INTRODUCTION

Problems of determining coefficients or the right-hand side of a differential equation simultaneously with its solution are called inverse problems of the mathematical physics. In this paper we consider one family of problems modeling the process of determining the function of temperature distribution and time varying structure of a homogeneous bar by a given law of changing of medium temperature. In the process of mathematical modeling there arises an inverse problem for a heat equation where alongside with a solution of the problem it is required to find unknown coefficient depending only on a time variable.

The solvability of various inverse problems for parabolic equations was studied in papers of Anikonov Yu.E. and Belov Yu.Ya., Bubnov B.A., Prilepko A.I. and Kostin A.B., Monakhov V.N., Kozhanov A.I., Sabitov K.B. and many others. We note [1-11] from recent papers close to the theme of our article.

Unlike the preceding works, we study the inverse problem for a heat equation subject to boundary conditions with respect to a spatial variable under which the system of eigenfunctions of the corresponding spectral problem for an ordinary differential operator does not form a basis.

Paper [12] is most close to the subject of this one. In this paper the existence of the classical solution of an inverse problem analogous to our investigated problem was justified. However, due to the fact that boundary conditions in [12] are regular, but not strengthened regular, the improvement of the smoothness and satisfaction of additional conditions have been required from the input data of the problem. In the present paper these conditions are completely removed and it is shown that an inverse problem has a unique generalized solution.

STATEMENT OF THE PROBLEM

In the domain $\Omega = \{(x,t) : 0 < x < 1, 0 < t < T\}$ consider a problem on finding unknown coefficient $p(t)$ of the heat equation

$$u_t = u_{xx}(x,t) - p(t)u(x,t) + f(x,t) \tag{1}$$

subject to the initial condition

$$u(x,0) = \varphi(x),\ 0 \le x \le 1, \tag{2}$$

the nonlocal boundary conditions

$$u_x(0,t) = u_x(1,t) + \alpha u(1,t),\ u(0,t) = 0,\ 0 \le t \le T, \tag{3}$$

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and the overdetermination conditions

$$\int_0^1 u(x,t)dx = E(t), E(t) \neq 0, 0 \leq t \leq T, \tag{4}$$

where $E(t) \in W_2^1(0,T)$. Here the parameter α is any positive number, and $f(x), \varphi(x)$ and $E(t)$ are given functions. At $\alpha = 0$ boundary conditions (3) are well- known and called in literature as **Samarskii-Ionkin** conditions.

Direct problem (1)-(3) in case when $p(t) \equiv 0$ was investigated in [20].

The most close to the theme of the present paper is [12]. The existence of classical solution of an inverse problem analogous to our investigated problem has been justified in this paper. However, due to the fact that boundary conditions in [12] are regular, but not strengthened regular, from the input data of the problem there have been required the improvement of the smoothness and satisfaction to additional conditions:

$$(A_1)\ \varphi \in C^2[0,1]; \varphi'(0) - \alpha\varphi(0) = 0,\ \varphi(0) = \varphi(1);$$

$$\varphi_0 > 0, \varphi_{2n-1} \geq 0, n = 1,2,3...., \text{ if } \alpha < 0;$$

$$\varphi_1 < 0, \varphi_{2n-1} \leq 0, n = 2,3...., \text{ if } \alpha > 0.$$

$$(A_2)\ E(t) \in C^1[0,T];\ E(0) = \int_0^1 \varphi(x)dx;\ \ E(t) > 0,\ \forall t \in [0,1];$$

$$(A_3)\ f(x,t) \in C^2[\bar{D_T}]; f(x,t) \in C^2[0,1], \forall t \in [0,1];$$

$$f_x(1,t) - \alpha f(0,t) = 0,\ f(0,t) = f(1,t);$$

$$f_0(\tau) > 0, f_{2n-1}(\tau) \geq 0, n = 1,2,3...., \text{ if } \alpha < 0;$$

$$f_{2n-1}(\tau) \leq 0, n = 2,3...., \text{ if } \alpha > 0.$$

In the present paper, these conditions are completely removed and it is shown that the inverse problem has a unique generalized solution.

For solution of the problem it is necessary to use the apparatus of nonlocal non-selfadjoint differential operators and results on basis properties of its root vectors, developed in papers of T.Sh. Kal'menov, M.A. Sadybekov, A.M. Sarsenbi, N.S. Imanbaev, D. Suragan [13]-[19].

AUXILIARY SYSTEM

By the *solution of the problem* we will call a pair of functions $\{u(x,t), p(t)\}$ that turn equation (1) and the condition (2)-(4) into identity in a corresponding class of functions $u(x,t)$ and $p(t)$.

Using of the Fourier method for the solution of problem (1)-(3) leads to a spectral problem for the operator l given by the differential expression and the boundary conditions

$$l(y) \equiv -y''(x) = \lambda y(x),\ 0 < x < 1,\ y'(0) = y'(1) + \alpha y(1),\ y(0) = 0. \tag{5}$$

The boundary conditions in (5) are regular but not strengthened regular [10]. Eigenfunctions of the operator l has the form

$$y_k^{(1)}(x) = sin(2\pi kx),\ k = 1,2,....,\ y_k^{(2)}(x) = sin(2\pi\beta x)\ k = 0,1,2,....$$

This system is almost normalized but does not form even a unconditional basis in $L_2(0,1)$.

However, as shown in [20], the auxiliary system

$$\begin{aligned} y_0(x) &= y_0^{(2)}(x)(2\beta_0)^{-1},\ y_{2k}(x) = y_k^{(1)}(x), \\ y_{2k-1}(x) &= (y_k^{(2)}(x) - y_k^{(1)}(x))(2\delta_k)^{-1},\ k = 1,2,... \end{aligned}$$

constructed from the mentioned above, forms a Riesz basis in $L_2(0,1)$. If the function u satisfies boundary conditions (5), then its Fourier series according to the system $\{y_k(x)\}$ converges in sense of the space $W_2^2(0,1)$, in particular, uniformly. This fact allows to use the method of separation of variables for the solution of an initial-boundary value problem with boundary condition (3).

It is easy to calculate that

$$\begin{aligned} y_0''(x) &= -\lambda_0^{(2)} y_0(x),\ y_{2k}''(x) = -\lambda_k^{(1)} y_{2k}(x), \\ y_{2k-1}''(x) &= -\lambda_k^{(2)} y_{2k-1}(x) - \frac{\lambda_k^{(2)} - \lambda_k^{(1)}}{2\delta_k} y_{2k}(x). \end{aligned} \tag{6}$$

SOLUTION OF THE DIRECT PROBLEM

Consequently any solution and the input data of problem (1)-(3) can be represented in the form of the biorthogonal series

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) y_k(x),\ f(x,t) = \sum_{k=0}^{\infty} f_k(t) y_k(x),\ \varphi(x) = \sum_{k=0}^{\infty} \varphi_k y_k(x), \tag{7}$$

where $u_k(t) = (u(x,t), \nu_k(x))$, $f_k = (f(x), \nu_k(x))$ and $\varphi_k = (\varphi(x), \nu_k(x))$.

In the paper, we justify that the solution of the direct problem exists, is unique and may formally be written in the form of biorthogonal expansion (7). For the research completion of the direct problem (analogous to the Fourier method) we justify the smoothness of the obtained formal solution and the convergence of all encountered series.

Let $\|u\|_0$ be a norm of the space $L_2(\Omega)$. By $W_2^{2,1}(\Omega)$ denote the space of the function $u(x,t)$, for which almost everywhere there exist the generalized derivatives $u_{xx}(x,t)$, $u_t(x,t)$ belonging to $L_2(\Omega)$ with the norm

$$\|u(x,t)\|_{2,1}^2 = \|u(x,t)\|_0^2 + \|u_{xx}(x,t)\|_0^2 + \|u_t(x,t)\|_0^2.$$

Under *the generalized solution* of problem (1)-(3) we mean a function $u(x,t) \in W_2^{2,1}(\Omega)$, that turn the problem into identity.

For the direct problem (1)-(3) the following theorem takes place.

Theorem 1. *Let $f(x,t) \in L_2(\Omega)$ and $p(t) \in L_2(0,T)$. If function $\varphi(x)$ belongs to $W_2^2(0,1)$ and satisfies boundary conditions (5), then there exists a unique generalized solution $u(x,t) \in W_2^{2,1}(\Omega)$ of problem (1)-(3).*

SOLUTION OF THE INVERSE PROBLEM

The main result of the paper is theorem on the existence and uniqueness of a generalized solution of problem (1)-(4).

Theorem 2. *If functions φ and f belong to classes $\varphi \in W_2^2(0,1)$, $f \in L_2(\Omega)$, $E(t) \neq 0$ and $E(t) \in W_2^1(0,T)$, then a unique generalized solution $u(x,t) \in W_2^{2,1}(\Omega)$, $p(t) \in L_2(0,1)$ of problem (1)-(4) exists.*

Proof. We substitute the solution of direct problem (1)-(3) in the form of biorthogonal expansion

$$u(x,t) = u_0(t) y_0(x) + \sum_{k=1}^{\infty} \Big(u_{2k}(t) y_{2k}(x) + u_{2k-1}(t) y_{2k-1}(x) \Big) \tag{8}$$

into the overdetermination conditions (4). We get under all $0 \le t \le T$

$$E(t) = u_0(t) \int_0^1 y_0(x)dx + \sum_{k=1}^{\infty} \Big(u_{2k}(t) \int_0^1 y_{2k}(x)dx + u_{2k-1}(t) \int_0^1 y_{2k-1}(x)dx \Big).$$

Separately calculate integrals from basis functions

$$\begin{aligned} \int_0^1 y_{2k}(x)dx &= \int_0^1 \sin 2\pi x dx = 0,\ k = 0,1,2,..., \\ \int_0^1 y_0(x)dx &= (2\beta_0)^{-1} \int_0^1 y_0^{(2)}(x)dx = (2\beta_0)^{-1} \int_0^1 \sin 2\beta_0 x dx = \frac{1-\cos 2\beta_0}{(2\beta_0)^2}. \end{aligned}$$

Since $\tan\beta_k = \frac{\alpha}{2\beta_k}$, then

$$1-\cos 2\beta_k = 2\frac{\tan^2\beta_k}{1+\tan^2\beta_k} = \frac{2\alpha^2}{(2\beta_k)^2+\alpha^2}, k=1,2,.... \tag{9}$$

Therefore

$$\int_0^1 y_0(x)dx = \frac{2\alpha^2}{(2\beta_0)^2(2\beta_k)^2+\alpha^2)}.$$

Analogically calculate

$$\begin{aligned}\int_0^1 y_{2k-1}(x)dx &= (2\delta_k)^{-1}\int_0^1\Big(y_k^{(2)}(x)-y_k^{(1)}(x)\Big)dx\\ &= (2\delta_k)^{-1}\int_0^1\Big(\sin(2\beta_k x)-\sin(2\pi kx)\Big)dx\\ &= (2\delta_k)^{-1}\frac{1-\cos 2\beta_k}{(2\beta_k)^2} = \frac{2\alpha^2}{\delta(2\beta_k)^2(2\beta_k)^2+\alpha^2)}.\end{aligned}$$

For convenience we introduce notations $q(t)=\exp\{\int_0^t p(s)ds\}$.

It is easy to see that

$$p(t) = \frac{q^{'}(t)}{q(t)}. \tag{10}$$

Finally we have

$$\begin{aligned}E(t) &= \Big(\varphi_0 e^{-\lambda_0^{(2)}t-\int_0^t p(s)ds}+\int_0^t f_0(\tau)e^{-\lambda_0^{(2)}(t-\tau)-\int_\tau^t p(s)ds}d\tau\Big)\frac{2\alpha^2}{(2\beta_0)^2((2\beta_k)^2+\alpha^2)}\\ &+\frac{2\alpha^2}{\delta(2\beta_k)^2(2\beta_k)^2+\alpha^2)}\sum_{k=1}^{\infty}\Big(\varphi_{2k-1}e^{-\lambda_k^{(2)}t-\int_0^t p(s)ds}+\int_0^t f_{2k-1}(\tau)e^{-\lambda_k^{(2)}(t-\tau)-\int_\tau^t p(s)ds}d\tau\Big)\\ &= \frac{\alpha^2}{q(t)}\left[\frac{2e^{-\lambda_0^{(2)}t}}{(2\beta_0)^2[(2\beta_k)^2+\alpha^2]}\varphi_0+\sum_{k=1}^{\infty}\frac{e^{-\lambda_k^{(2)}t}}{\delta_k(2\beta_k)^2((2\beta_k)^2+\alpha^2)}\varphi_{2k-1}\right]\\ &+\alpha^2\int_0^t\frac{q(\tau)}{q(t)}\left[\frac{2f_0e^{-\lambda_0^{(2)}(t-\tau)}}{(2\beta_0)^2[(2\beta_k)^2+\alpha^2]}+\sum_{k=1}^{\infty}\frac{e^{-\lambda_k^{(2)}(t-\tau)}}{\delta_k(2\beta_k)^2((2\beta_k)^2+\alpha^2)}f_{2k-1}(\tau)\right]d\tau.\end{aligned}$$

If we introduce new notations

$$F(t) = \frac{\alpha^2}{E(t)}\left[\frac{2e^{-\lambda_0^{(2)}t}}{(2\beta_0)^2[(2\beta_k)^2+\alpha^2]}\varphi_0+\sum_{k=1}^{\infty}\frac{e^{-\lambda_k^{(2)}t}}{\delta_k(2\beta_k)^2((2\beta_k)^2+\alpha^2)}\varphi_{2k-1}\right],$$

$$K(t,\tau) = \frac{\alpha^2}{E(t)}\left[\frac{2f_0e^{-\lambda_0^{(2)}(t-\tau)}}{(2\beta_0)^2[(2\beta_k)^2+\alpha^2]}+\sum_{k=1}^{\infty}\frac{e^{-\lambda_k^{(2)}(t-\tau)}}{\delta_k(2\beta_k)^2((2\beta_k)^2+\alpha^2)}f_{2k-1}(\tau)\right],$$

then for $q(t)$ we have the integral equation

$$q(t)-\int_0^t K(t,\tau)q(\tau)d\tau = F(t). \tag{11}$$

This equation is an integral equation of Volterra type of the second kind. It is well-known that equation (10) has a unique solution $q(t)$ in $L_2(0,T)$.

Since $E(t)\neq 0$ and $E(t)\in W_2^1(0,T)$, we have $q\in W_2^1(0,T)$ from (10). Substituting $q(t)$ into (9), we find the unknown coefficient of the diffusion $p\in L_2(0,T)$. The theorem is proved. □

CONCLUSION

In this work, we considered one family of problems of modeling the process of determining the distribution function of temperature and time varying structure of homogeneous bar of a given law changes in the average temperature. So there is an inverse problem for the heat equation in which together with finding the solution of the equation required to find unknown coefficient depending only on the time variable. The specific features of the considered problems is that the system of eigenfunctions of the multiple differentiation operator subject to boundary conditions of the initial problem does not have the basis property. We proved the unique existence of a generalized solution to the mentioned problem.

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