# Numerical implementation of the method of fictitious domains for elliptic equations

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**Abstract.** In the paper we study the elliptical type equation with strongly changing coefficients. We are interested in studying such equation because the given type equations are yielded when we use the fictitious domain method. In this paper we suggest a special method for numerical solution of the elliptic equation with strongly changing coefficients. We have proved the theorem for the assessment of developed iteration process convergence rate. We have developed computational algorithm and numerical calculations have been done to illustrate the suggested method effectiveness.

**Keywords:** Elliptical Equation, Dirichlet Problem, The Equation with rapidly Changing Coefficients, Computational Algorithm, Iterative Process, Fictitious Domain Method, Boundary Conditions.
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## INTRODUCTION

It is effective to use the fictitious domain method in irregular shape domains for numerical solution for elliptical type equations.

Reference [1] suggests efficient (due to operations number) difference scheme of the second order accuracy, alternately-triangular scheme for numerical solution of the elliptical equation. Modified alternately-triangular iteration method with Chebyshev parameters of Dirichlet differential problem solution for the elliptical equation of the second-order accuracy was made in reference [2]. Lebedev V.I. in his monograph [3] studied the use of composition method for the solution of problems on characteristic constants, nonstationary problems, Dirichlet problems for biharmonic equation and domain problems. Reference [4] studies stationary differential problem for Poisson's equation with piecewise constant coefficients in subdomains. Poisson's equation on the boundary approximates in a specific way, that is, the differential equation coefficients are selected as quotient which denominator contains coefficients sum in subdomains. Two-phase iteration process based on domain partitioning method has been built.

Solving the Poisson equation for pressure is the main computing unit in the problems of hydrodynamics of an incompressible fluid. In [5] a parallel implementation of the method of fictitious domains for the Poisson equation in a three-dimensional region with a stepped-back is proposed. This method is based on the parallel implementation of the fast algorithm for solving the Poisson equation in a parallelepiped. In [6] , a variant of the method of collocation and least residuals for the numerical solution of the Poisson equation in polar coordinates on a non-uniform grid is proposed. In [7, 8] the method of fictitious domains is used for the numerical solution of elliptic equations with complex geometry. Method of fictitious domain for the equations of mathematical physics was studied by Bugrov A.N., Konovalov A.N., Smagulov S.S., Orunhanov M.K., Glowinski R., Girauit V. and others [5–10]. In these studies various modifications of the method of fictitious domain for the Navier-Stokes equations were investigated.

In the given paper we suggest a specific method for numerical solution of the elliptical equation with strongly changing coefficients. The basis of the suggested method is in the special replacement of variables which brings the problem with second order discontinuous coefficients to the problem with first order discontinuous coefficients. We have built the iteration process with two parameters which takes into account the equation coefficient ration in subdomains. We have proved the theorem for the developed iteration process convergence rate assessment. We have developed computational algorithm and made numerical calculations to illustrate the suggested method efficiency.

## PROBLEM SETTING

Let's $\Omega$ is the bounded domain from $R^2$ with piecewise-smooth boundary $\partial\Omega$. For determination let's $\Omega = Q_1 \bigcup Q_2,\ Q_1 \bigcap Q_2 = \Gamma$, $Q_2$ is strictly internal subdomain. We will study the elliptical equation in $\Omega$

$$-div(k\nabla u) = f(\vec{x}),\ \vec{x} \in \Omega, \tag{1}$$

with boundary condition

$$u(\vec{x}) = 0,\ \vec{x} \in \partial\Omega, \tag{2}$$

where

$$k(\vec{x}) = \begin{cases} k_1 = const,\ \vec{x} \in Q_1, \\ k_2 = const, (\vec{x}),\ x \in Q_2. \end{cases}$$

Function $f(\vec{x})$ is suggested as belonging to Hilbert space of real functions $L_2(\Omega)$ and are determined in subdomains with the following ways

$$f(\vec{x}) = \begin{cases} f^{(1)}(x),\ x \in Q_2, \\ 0,\ x \in Q_1. \end{cases}$$

We will make the replacement of variables $u = 2\nu/k_1$ in (1), simple transformations, and get

$$\Delta\nu + div(\omega\nabla\nu) = -f(\vec{x}), \tag{3}$$

where $\omega = \frac{2k(x)}{k_1} - 1$. Let's designate $\theta = \frac{2k_2}{k_1} - 1$.

We will introduce symbol $\vec{p} = \left(\omega\frac{\partial\nu}{\partial x_1},\ \omega\frac{\partial\nu}{\partial x_2}\right)$ and equation (3) will be written as equation system

$$\begin{cases} \Delta\nu + \nabla\vec{p} = -f(\vec{x}), \\ p_1/\omega - \frac{\partial\nu}{\partial x_1} = 0, \\ p_2/\omega - \frac{\partial\nu}{\partial x_2} = 0. \end{cases} \tag{4}$$

## COMPUTATIONAL ALGORITHM

For the numerical solution of equation system (4) with boundary conditions $\nu|_{\partial\Omega} = 0$ let's study the iteration method

$$B\nu_t^{n+1} + \Delta_h\nu^{n+1} + \nabla_h\vec{p}^{n+1} = -f(\vec{x}),\quad \beta(\vec{p}^{n+1} - \vec{p}^n) + \frac{\vec{p}^{n+1}}{\omega} - \nabla_h\nu^{n+1} = 0, \tag{5}$$

where $B$- is iteration method operator, $\beta$-is iteration parameter, index $h$ means difference analogue of differentiation operator. Operator $B$ in iteration method (5) is selected by the following way

$$B = (1-\tau)\Delta_h - \tau div(\rho\nabla_h), \tag{6}$$

where $\rho = (\beta + 1/\omega)^{-1}$.

Let's suppose that $\nu^0 \in \overset{o}{W}{}_2^1(\Omega)$ and $p^0 = \nabla q$,where $q \in \overset{o}{W}{}_2^1$.This condition is satisfied if $(\nu^0, p^0) = 0$.

Further, we will consider that $B > 0$ on $\overset{o}{W}{}_2^1$. It is enough to do inequality

$$1 - \tau - \frac{\tau}{\beta} > 0. \tag{7}$$

In case of $B$ on $\overset{o}{W}{}_2^1$, it satisfies operator inequality

$$-\chi_1\Delta \le B \le -\chi_2\Delta. \tag{8}$$

Constants $\chi_1$ and $\chi_2$ may be selected independent upon $\theta \geq 1$. Substituting operator $B$ determined as (6) and (5) we get

$$\Delta_h v^{n+1} = F(x), \tag{9}$$

$$\vec{p}^{n+1} = \beta\rho\vec{p}^n + \rho\nabla_h v^{n+1}, \tag{10}$$

where

$$F(\vec{x}) = (1-\tau)\Delta_h v^n - \tau div(\beta\rho\nabla_h v^n) - \tau div_h(\beta\rho\vec{p}^n). \tag{11}$$

We present numerical algorithm of method (9), (10). One step of iteration method (9), (10) consists in finding value $v^{n+1}$ due to known $v^n$, $\vec{p}^n$. For this, it is necessary to solve Dirichlet problem for Poisson's equation (9) in $\Omega$. After that value $\vec{p}^{n+1}$ upon the known $\vec{p}^n$ and $v^{n+1}$ is counted using formula (10).

## CONVERGENCE STUDYING

We assess convergence rate method (9), (10). Let's designate $\{y,r\} = \{y^n,r^n\} = \{v-v^n, p-p^n\}$, $\{\hat{y},\hat{r}\} = \{y^{n+1},r^{n+1}\}$. Then equations (5) can be rewritten as

$$(By_t, v) + (\nabla_h\hat{y}, \nabla_h v) + (\nabla_h\hat{r}, v) = 0, \ \forall v \in \overset{o}{W}{}_2^1, \tag{12}$$

$$\beta\tau r_t + \hat{r}/\omega - \nabla_h\hat{y} = 0, \tag{13}$$

$\{y^0, r^0\} \in \overset{o}{W}{}_2^1 \times L_2$, here $y_t = (\hat{y}-y)/\tau$.

Let's call function $\psi$ from as $L_2$ piecewise gradient if it can be presented as

$$\psi = \nabla g_i \ in \ Q_i; \ where \ g_i \in W_2^1(Q_i), \tag{14}$$

$$g_i|_{\partial\Omega\bigcap\partial\Omega_i} = 0, \ i = 1,2,...,N$$

and call function $\psi$ gradient if it is as the following

$$\psi = \nabla g \ in\Omega, \ where \ g \in \overset{o}{W}{}_2^1(\Omega).$$

As $p^0 = \nabla g$, $g \in \overset{o}{W}{}_2^1$ and $\omega$-is piecewise constant, $r^0$ is piecewise gradient.

Let's multiply both parts of equation (20) scalarly in $L_2$ on $2\tau\hat{r}$ and put $v = 2\tau\hat{y}$ in correlation (19). Adding up the got equalities we have

$$\|\hat{y}\|_B^2 - \|y\|_B^2 + \tau^2\|y_t\|_B^2 + 2\tau\|\nabla_h\hat{y}\|^2 + \beta\tau\|\hat{r}\|^2 - \beta\tau\|r\|^2 + \beta\tau^3\|r_t\|^2 + \frac{2\tau}{\omega}\|\hat{r}\|^2 = 0. \tag{15}$$

Let's study $r^n$. As

$$\hat{\tau} = \frac{\beta}{\beta+1/\omega}r + \frac{1}{\beta+1/\omega}\nabla\hat{y}$$

and $r$ is piecewise gradient function, $\hat{\tau}$ is also piecewise gradient. Thus, all $r^n$ are piecewise gradient.

Let's $G$ is the space of piecewise gradient functions, $G_1$ is the space of gradient functions. It is obvious that $G_1 \subseteq G$. We will show that there is rigid embedding $G_1 \subset G$ and find the orthogonality in $L_2$ of add-ins $G_1$ to $G$. If $\psi$ is orthogonal in $L_2$ to all elements $G_1$, we have $(\psi, \nabla q)_\Omega = 0$ for any element $\nabla q \in G_1$. If function $\nabla g$ is smooth enough and has carrier in $Q_i$,

$$(\psi, \nabla g)_\Omega = (\psi, \nabla g)_{Q_i} = -(div\psi, g)_{Q_i} = -(\Delta g_i, g)_{Q_i} = 0.$$

Due to $g$-is arbitrary, the last correlation means that

$$\Delta g_i = 0 \ in \ Q_i. \tag{16}$$

It is evident that the correlation is done in each $Q_i,\ i=1,2$. Thus, element $\psi \in G$, orthogonal to all elements $G_1$, will be presented as (21), where $q_i$ is harmonic in $Q_i$ function. We will find the conditions to be satisfied by function $\psi$, orthogonal to $G_1$ on $\Gamma$.

Let's $\nabla q \in G_1$, then

$$0=(\psi,\nabla q)_\Omega=(\nabla q_1,\nabla q)_{Q_1}+(\nabla q_2,\nabla q)_{Q_2}=\int\limits_\Gamma g\frac{\partial q_1}{\partial n_1}ds+\int\limits_\Gamma g\frac{\partial q_2}{\partial n_2}ds=\int\limits_\Gamma g\left(\frac{\partial q_1}{\partial n_1}-\frac{\partial q_2}{\partial n_1}\right)ds$$

(here $n_i$ are inner normal vectors on $\partial Q_i$); that is, values of normal compounds $\psi_1=\nabla q_1$ and $\psi_2=\nabla q_2$ on $\Gamma$ coincide. Thus, normal compound of vector-function $\psi$ is continuous (in integral sense) at the transfer through $\Gamma$. It follows that orthogonal in $L_2$ add-ins $G_2$ of space $G_1$ to $G$ consists of all vector-functions of type (21), normal compound of which is continuous during the transfer through adjacent boundaries, and the forming functions $g_i$ are harmonic in $Q_i$.

We continue to study iteration method convergence (9), (10). It was stated that $\hat{r}\in G$. Let's present $\hat{r}$ as $\hat{r}=\hat{q}+\hat{h}$, where $\hat{q}\in G_1$ and $\hat{h}\in G_2$. Correlation (19) then is as the following

$$(By_t,\nu)+(\nabla\hat{y},\nabla\nu)+(\hat{q},\nabla\nu)+(\hat{h},\nabla\nu)=0, \tag{17}$$

for $\forall\nu\in \overset{o}{W}{}_2^1$.

The last scalar product in (24) equals nought because $\nabla\nu\in G_1$. Having divided both parts (24) in $\|\nabla\nu\|$ and having assessed the term containing $\hat{q}$ we get

$$\frac{|(\hat{q},\nabla\nu)|}{\|\nabla\nu\|}\le\frac{|(By_t,\nu)|}{\|\nabla\nu\|}+\frac{|(\nabla\hat{y},\nabla\nu)|}{\|\nabla\nu\|}\le\sqrt{\chi_2}\|y_t\|_B+\|\nabla_h\hat{y}\|.$$

As soon as the right part of this inequality is independent upon $\nu\in\overset{o}{W}{}_2^1$, and $\hat{q}\in G_1$, that is, it is presented as $\hat{q}=\nabla g(g\in\overset{o}{W}{}_2^1)$, taking sup on $\nu$ in the left part of the inequality, we get the assessment

$$\|\hat{q}\|\le\sqrt{\chi_2}\|y_t\|_B+\|\nabla_h\hat{y}\|,$$

where $\|\nabla_h\hat{y}\|=\|\hat{y}\|_1$. Let's square both parts of this inequality and assess the right part

$$\|\hat{q}\|^2\le 2(\chi_2\|y_t\|_B^2+\|\nabla_h\hat{y}\|^2).$$

Let's multiply the last inequality on $\beta\tau^2\mu$ ( $\lambda>0$ is arbitrary) and add to (22). In the result we have

$$\begin{gathered}\|\hat{y}\|_B^2+\tau^2(1-2\beta\lambda\chi_2)\|y_t\|_B^2+2\tau(1-\beta\tau\lambda)\|\nabla_h\hat{y}\|^2+\\ \beta\tau^2\lambda\|\hat{q}\|^2+2\tau\left(\hat{r},\tfrac{\hat{r}}{\omega}\right)+\beta\tau\|\hat{r}\|^2\le\|y\|_B^2+\beta\tau\|r\|^2.\end{gathered} \tag{18}$$

Let's assess scalar product $(\hat{r},\hat{r}/\omega)$. At any $\delta,\ 0<\delta<1$ the following inequality is true

$$\begin{gathered}\left(\tfrac{\hat{r}}{\omega},\hat{r}\right)\ge\left(\tfrac{\hat{q}}{\omega},\hat{q}\right)+\left(\tfrac{\hat{h}}{\omega},\hat{h}\right)-2\left|\left(\tfrac{\hat{q}}{\omega},\hat{h}\right)\right|\ge\\ (1-\delta)\left(\tfrac{\hat{h}}{\omega},\hat{h}\right)+\left(1-\tfrac{1}{\delta}\right)\left(\tfrac{\hat{q}}{\omega},\hat{q}\right)\ge\\ (1-\delta)\left[\|\hat{h}\|_{Q_1}^2+\tfrac{1}{\theta}\|\hat{h}\|_{Q_2}^2\right]+\left(1-\tfrac{1}{\delta}\right)\|\hat{q}\|^2.\end{gathered} \tag{19}$$

Due to $\hat{h}\in G$, according to lemma 1 there is the assessment

$$\|\hat{h}\|_{Q_2}\le c_3\|\hat{h}\|_{Q_1}^2.$$

Thus

$$\|\hat{h}\|_\Omega^2=\|\hat{h}\|_{Q_1}^2+\|\hat{h}\|_{Q_2}^2\le(1+c_3)\|\hat{h}\|_{Q_1}^2.$$

That is why from (26) we get the following assessment

$$\left(\frac{\hat{r}}{\omega},\hat{r}\right)\ge c_4(1-\delta)\|\hat{h}\|^2+\left(1-\frac{1}{\delta}\right)\|\hat{q}\|^2,\ c_4=(1+c_3)^{-1}.$$

Using the last inequality we will take (25) and get

$$\begin{gathered}\|\hat{y}\|_B^2+\tau^2(1-2\beta\lambda\chi_2)\|y_t\|_B^2+2\tau(1-\beta\tau\lambda)\|\hat{y}\|_1^2+\\ \beta\tau\|\hat{r}\|^2+\beta\tau^2\lambda\|\hat{q}\|^2+2\tau(1-\delta)c_4\|\hat{h}\|^2+\\ 2\tau\left(1-\frac{1}{\delta}\right)\|\hat{q}\|^2\leq\|y\|_B^2+\beta\tau\|r\|^2.\end{gathered}\tag{20}$$

Let's fix $\beta>0$ and select $\tau>0$ so that for all $\theta>1$ the condition $\beta>0$ were satisfied. Let's select $\lambda$, satisfying the conditions

$$1-2\beta\lambda\chi_2>0,\ 1-\beta\tau\lambda>0$$

and suppose $\delta=\frac{4}{4+\beta\tau\lambda}<1$.

Then

$$\beta\tau^2\lambda+2\tau(1-1/\delta)=\beta\tau^2\lambda-2\tau\frac{\beta\tau\lambda}{4}=\frac{\beta\tau^2\lambda}{2},\ 1-\delta=\frac{\beta\tau\lambda}{4+\beta\tau\lambda}.$$

Inequality (27) at such $\delta$ is the following

$$\left(1+\frac{c_5\tau}{\chi_2}\right)\|\hat{y}\|_B^2+\beta\tau(1+c_6\tau)\|\hat{r}\|^2\leq\|y|_B^2+\beta\tau\|r\|^2,\tag{21}$$

where $c_5=\tau(1-2\beta\lambda\chi_2),\ c_6=min\left\{\frac{\lambda}{2},\frac{2c_4\lambda}{4+\beta\tau\lambda}\right\}$ it is easy to see that constants $\beta,\tau,\chi_2,\lambda$ can be selected the same for all $\theta,\ 1\leq\theta\leq\infty$. Thus, the following theorem is proved.

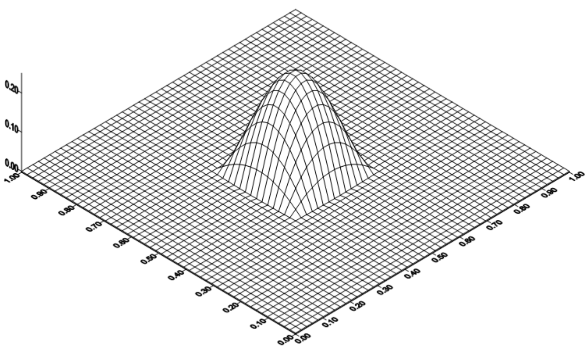
**Theorem 1.** For any $\beta>0$ there is $\overline{\tau}=\overline{\tau}(\beta)$ independent upon $\omega\geq 1$ such, that $-\chi_1\Delta\leq B\leq-\chi_2\Delta$ at $\tau\leq\overline{\tau}$ constants $\chi_1,\chi_2$ are independent upon $\omega$.

In this case iteration process (9), (10) converges with geometric sequence rate, and convergence rate is independent upon $\omega$.

## NUMERICAL CALCULATIONS

Test problem (1)-(2) is solved with the above described method. Subdomain $Q_2$ was selected as a square $Q_2=\{x_{1,k_1}\leq x_1\leq x_{1,k_2};x_{2,m_1}\leq x_2\leq x_{2,m_2}\}$, where $x_{1,k_1}=0.25,\ x_{1,k_2}=0.75,\ x_{2,m_1}=0.25,\ x_{2,m_2}=0.75$. Domain $\Omega$ cover subdomain $Q_2$, $\Omega=\{0\leq x_1\leq 1;0\leq x_2\leq 1\}$. Subdomain $Q_1$ is determined as $Q_1=\Omega\setminus Q_2$ the right part is given in $Q_2$ with the following way $f(x_1,x_2)=2(x_2^2-(x_{2,m_1}+x_{2,m_2})x_2+x_{2,m_1}x_{2,m_2})+2(x_1^2-(x_{1,k_1}+x_{1,k_2})x_1+x_{1,k_1}x_{1,k_2})$, where $x_{1,k_1}=0.25,\ x_{1,k_2}=0.75,\ x_{2,m_1}=0.25,\ x_{2,m_2}=0.75$.

Function $f(x_1,x_2)=0$ equals zero in subdomain $Q_1$. Iteration parameter $\tau$ was selected $\tau=10^{-3}\div 10^{-5}$, and parameter $\beta$ was determined so that to satisfy condition (7). And it is necessary to watch the parameter symbol $\omega$ in the sub-domains because $-1\leq\omega\leq 1$.



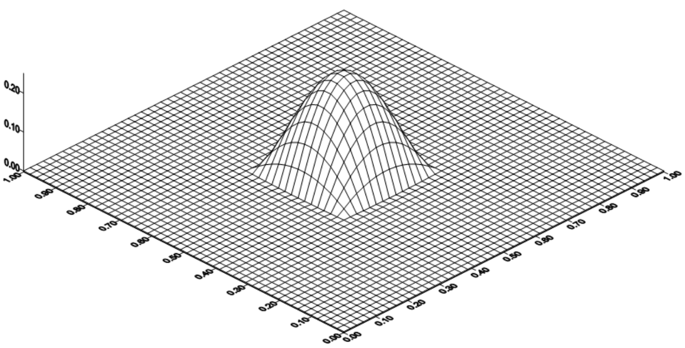


Figure 1. The diagram of exact solution having mesh points 101x101. Figure 2. The diagram of approximate solution having mesh points 101x101.

The set problem of elliptical type with strongly changing coefficients was solved by fictitious domain method continued with leading coefficients. Figures 1-2 show the corresponding results of exact and approximate solution having mesh points 101x101.

We used uniform mesh 101x101, 501x501, 1001x1001for our calculations. To do the computing experiment on the small mesh numerical experiment was done on the super computer URSA on the basis of 128 four-core processors Intelő Xeonő series E5335 2.00GHz at Al Farabi KazNU. The developed method is based on building computational algorithm for the elliptical equation with strongly changing coefficients. The developed algorithm uniformly converges at a certain iteration quantity, and the results are accurate within $10^{-10}$. Numerical computation results are presented by graphics editor Surfer.

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