Investigation of the stability and convergence of difference schemes for three-dimensional equations of the atmospheric boundary layer

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Abstract. In this article we construct finite-difference scheme for three-dimensional equations of the atmospheric boundary layer. The solvability of the mathematical model is proved and quality properties of solutions are studied. A priori estimates are derived for the solution of differential equations. The mathematical questions of difference schemes for equations of the atmospheric boundary layer are studied. Nonlinear terms are approximated in such a way that this integral term of the identity vanishes when it is scalar multiplied. This property of the difference scheme is formulated as a lemma. The main a priori estimates for the solution of the difference problem are derived. Approximation properties are investigated and the theorem of convergence of the difference solution to the solution of the difference problem to the solution of the differential problem is proved.

Keywords: Differential equations, difference sceme,equations of the atmospheric boundary layer, Navier-Stokes equations, Young's inequality, Cauchy-Schwarz inequality
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INTRODUCTION

Mathematical models of computational fluid dynamics serves as the basis for the study of various natural phenomena, technological processes and environmental problems. In this regard, the development and study of efficient and stable numerical algorithms for solving the system of equations of the atmospheric boundary layer and their practical implementation is relevant. There are various methods for the numerical solution of differential equations, new techniques are has been developing, the work on their improvement has been continuously performed, reassessing the methods is carried out. Basic methods for solving grid equations are systematized and described in detail in [1]. When solving the Navier-Stokes equations, explicit schemes are inefficient due to hard restrictions on the ratio of temporal and spatial steps of the computational grid, especially on finding stationary solutions to establish. Therefore, the most frequently used implicit differencing scheme, unconditionally stable or have weaker constraints on the stability. An overview of the most commonly used numerical algorithms presented e.g. in papers [2–8].

In [9] a new symmetric method of approximation of the non-stationary Navier-Stokes system of equations of the Cauchy-Kovalevskaya type is proposed. The properties of the modified problem are studied. The convergence of the solution of modified problem to the solution of the original problem is proved on the infinite time interval when ε. In [10] the convergence of finite-difference scheme, approximating the primitive equations with the second order in the spatial variables, to the solution of the differential problem under the natural assumption of smoothness of the solution of the original problem. The reference [11] studies difference schemes by time, the accuracy order of which can be arbitrarily high. Difference schemes by time for solving the Navier-Stokes equations are presented. The impact of the scheme order on the calculations accuracy is examined. In [12–14] numerical algorithms for solving the Navier-Stokes equations using the finite element method are proposed. The analysis of stability and convergence of the proposed methods is conducted.

PROBLEM SETTING

Consider the three-dimensional equations of the atmospheric boundary layer in a domain $\Omega = \{0 < x_i < l_i,\ i = 1,\ 2,\ 3\}$ with a border S:

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x_1}+\upsilon\frac{\partial u}{\partial x_2}+\omega\frac{\partial u}{\partial x_3}+\frac{\partial p}{\partial x_1}=\frac{1}{De}\upsilon+\\ +\frac{1}{Re_T}\left(\frac{\partial}{\partial x_1}\left(a_{x_1}\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(a_{x_2}\frac{\partial u}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(a_{x_3}\frac{\partial u}{\partial x_3}\right)\right)+f_1(\vec{x},t)\end{gathered} \tag{1}$$

$$\begin{gathered}\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x_1}+\upsilon\frac{\partial \upsilon}{\partial x_2}+\omega\frac{\partial \upsilon}{\partial x_3}+\frac{\partial p}{\partial x_2}=-\frac{1}{De}u+\\ +\frac{1}{Re_T}\left(\frac{\partial}{\partial x_1}\left(a_{x_1}\frac{\partial \upsilon}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(a_{x_2}\frac{\partial \upsilon}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(a_{x_3}\frac{\partial \upsilon}{\partial x_3}\right)\right)+f_2(\vec{x},t)\end{gathered} \tag{2}$$

$$\begin{gathered}\frac{\partial \omega}{\partial \bar{t}}+u\frac{\partial \omega}{\partial x_1}+\upsilon\frac{\partial \omega}{\partial x_2}+\omega\frac{\partial \omega}{\partial x_3}+\frac{\partial p}{\partial x_3}=\overline{\lambda}+\\ +\frac{1}{Re_T}\left(\frac{\partial}{\partial x_1}\left(a_{x_1}\frac{\partial \omega}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(a_{x_2}\frac{\partial \omega}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(a_{x_2}\frac{\partial \omega}{\partial x_3}\right)\right)+f_3(\vec{x},t)\end{gathered} \tag{3}$$

$$div\vec{V}=\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}+\frac{\partial \omega}{\partial z}=0 \tag{4}$$

where t – is time, x_1, x_2, x_3 – are Cartesian coordinates, $\vec{V}$ - the wind velocity vector with components u, υ, ω, p – is pressure, De – is a dimensionless characteristic deviations of the wind from geostrophic, Re_T – is a the number of dimensionless turbulent exchange, $\overline{\lambda}$ – is the dimensionless parameter of convection, a_{x_1}, a_{x_2} – are horizontal coefficients of atmospheric turbulence, the amount of movement, a_{x_3} – is a atmospheric vertical coefficient of turbulent exchange for momentum. The system of equations (1)-(4) is complemented by the following initial-boundary conditions:

$$\vec{V}(x,0)=\vec{V}^0(x),\quad x\in\Omega;\ \ \vec{V}(x,t)=0,\ x\in S. \tag{5}$$

In Ω the function $\vec{V}^0(x)$ is set as follows: $div\vec{V}^0(x)=0$ For the numerical solution of the equations of the atmospheric boundary layer (1) -(4), a grid with distributed velocities are used. In the domain Ω we build the grids Ω_H, $\Omega_H = \Omega_h\bigcup\Omega_x\bigcup\Omega_y\bigcup\Omega_z$, where

$$\begin{gathered}\Omega_h=\{(x_{1i},x_{2j},x_{3k}),\ x_{1i}=ih_1,\ x_{2j}=jh_2,\ x_{3k}=kh_3,\\ i=0,1,\dots,N_1;\ j=0,1,\dots,N_2;\ k=0,1,\dots,N_3,\ h_1=l_1/N_1,\ h_2=l_2/N_2,\ h_3=l_3/N_3\}\\ \Omega_x=\{(x_{1i+1/2},x_{2j},x_{3k}),x_{1i+1/2}=(i+1/2)h_1,x_{2j}=jh_2,\ x_{3k}=kh_3,\\ i=0,1,\dots,N_1-1;\ j=0,1,\dots,N_2;\ k=0,1,\dots,N_3,\ h_1=l_1/N_1,\ h_2=l_2/N_2,\ h_3=l_3/N_3\}\\ \Omega_y=\{(x_{1i},x_{2j+1/2},x_{3k}),\ x_{1i}=ih_1,\ x_{2j+1/2}=(j+1/2)h_2,\ x_{3k}=kh_3\\ i=0,1,\dots,N_1;\ j=0,1,\dots,N_2-1;\ k=0,1,\dots,N_3,\ h_1=l_1/N_1,\ h_2=l_2/N_2,\ h_3=l_3/N_3\}\\ \Omega_z=\{(x_{1i},x_{2j},x_{3k+1/2}),\ x_{1i}=ih_1,\ x_{2j}=jh_2,x_{3k+1/2}=(k+1/2)h_3,\\ i=0,1,\dots,N_1;\ j=0,1,\dots,N_2;\ k=0,1,\dots,N_3-1,\ h_1=l_1/N_1,\ h_2=l_2/N_2,\ h_3=l_3/N_3\}.\end{gathered} \tag{6}$$

Thus, the following difference scheme is constructed:

$$\begin{gathered}\frac{u^{n+1}_{i+1/2,j,k}-u^{n}_{i+1/2,j,k}}{\tau}+L^{(1)}_{1h}u^n_{i+1/2,j,k}+P^{n+1}_{x_1,i,j,k}=\frac{1}{De}\upsilon^n_{i,j+1/2,k}+\\ +\frac{1}{Re_T}\left[(a_{i,j,k}u^n_{x_1,i+1/2,j,k})_{\bar{x}_1}+(a_{i+1/2,j+1/2,k}u^n_{x_2,i+1/2,j,k})_{\bar{x}_2}+(a_{i+1/2,j,k+1/2}u^n_{x_3,i+1/2,j,k})_{\bar{x}_3}\right]+f^0_{i+1/2,j,k},\\ i=\overline{1,N_1-2},\ j=\overline{1,N_2-1},\ k=\overline{1,N_3-1}\end{gathered} \tag{7}$$

$$\begin{gathered}\frac{\upsilon^{n+1}_{i,j+1/2,k}-\upsilon^{n}_{i,j+1/2,k}}{\tau}+L^{(2)}_{1h}\upsilon^n_{i,j+1/2,k}+P^{n+1}_{x_2,i,j,k}=-\frac{1}{De}u^n_{i+1/2,j,k}+\\ +\frac{1}{Re_T}\left[(a_{i+1/2,j+1/2,k}\upsilon^n_{x_1,i+1/2,j+1/2,k})_{\bar{x}_1}+(a_{i,j+1,k}\upsilon^n_{x_2,i,j,k})_{\bar{x}_2}+(a_{i,j+1/2,k+1/2}\upsilon^n_{x_3,i,j+1/2,k+1/2})_{\bar{x}_3}\right]+f^0_{i,j+1/2,k},\end{gathered} \tag{8}$$

$$i=\overline{1,N_1-2},\ \ j=\overline{1,N_2-1},\ \ k=\overline{1,N_3-1}$$

$$\frac{\omega^{n+1}_{i,j,k+1/2}-\omega^{n}_{i,j,k+1/2}}{\tau}+L^{(3)}_{1h}\omega^{n}_{i,j,k+1/2}+P^{n+1}_{x_3,i,j,k}=\overline{\lambda}+$$
$$+\frac{1}{Re_T}\left[(a_{i+1/2,j,k+1/2}\omega^{n}_{x_1,i+1/2,j,k+1/2})_{\bar{x}_1}+(a_{i,j+1/2,k}\omega^{n}_{x_2,i,j+1/2,k})_{\bar{x}_2}+(a_{i,j,k+1}\omega^{n}_{x_3,i,j,k})_{\bar{x}_3}\right]+f^{0}_{i,j,k+1/2}, \tag{9}$$
$$i=\overline{1,N_1-2},\ \ j=\overline{1,N_2-1},\ \ k=\overline{1,N_3-1}$$

The continuity equation in a difference form is written as follows:

$$div_h\vec{V}^{n+1}=u^{n+1}_{\bar{x}_1,i+1/2jk}+\upsilon^{n+1}_{\bar{x}_2,ij+1/2k}+\omega^{n+1}_{\bar{x}_3,ijk+1/2}=0 \tag{10}$$

The following initial-boundary conditions are satisfied:

$$u^0_{i+1/2j,k}=V^0(x_{1i}+0,5h_1,x_{2j},x_{3k}),\ \ \upsilon^0_{i,j+1/2,k}=V^0(x_{1i},x_{2,j}+0,5h_2,x_{3k}),\ \ \omega^0_{i,j,k+1/2}=V^0(x_{1i},x_{2j},x_{3k}+0,5h_3)$$
$$\upsilon^{n+1}_{0,j+1/2,k}=\upsilon^{n+1}_{N_1,j+1/2,k}=u^{n+1}_{1/2j,k}=u^{n+1}_{N_1-1/2j,k}=\omega^{n+1}_{0,j,k+1/2}=\omega^{n+1}_{N_1,j,k+1/2}=0,\ j=\overline{0,N_2-1},\ \ k=\overline{0,N_3-1}$$
$$\upsilon^{n+1}_{i,1/2,k}=\upsilon^{n+1}_{N_1,j-1/2,k}=u^{n+1}_{i+1/2,0,k}=u^{n+1}_{i+1/2N_2,k}=\omega^{n+1}_{i,0,k+1/2}=\omega^{n+1}_{i,N_2,k+1/2}=0,\ \ i=\overline{0,N_1-1},\ k=\overline{0,N_3-1} \tag{11}$$
$$\upsilon^{n+1}_{i,j+1/20}=\upsilon^{n+1}_{i,j+1/2,N_3}=u^{n+1}_{i+1/2,j,0}=u^{n+1}_{i+1/2,j,N_3}=\omega^{n+1}_{i,j,1/2}=\omega^{n+1}_{i,j,N_3-1/2}=0,\ i=\overline{0,N_1-1},\ \ j=\overline{0,N_2-1}$$

Lemma. For any grid functions $u_{i+1/2j,k}\in\Omega_x,\ \upsilon_{i,j+1/2,k}\in\Omega_y,\ \omega_{i,j,k+1/2}\in\Omega_z$, satisfying conditions (10), (11), the following identities hold

$$(L^{(1)}_{1h}u_{i+1/2,j,k},u_{i+1/2j,k})=(L^{(2)}_{1h}\upsilon_{i,j+1/2,k},\upsilon_{i,j+1/2,k})=(L^{(3)}_{1h}\omega_{i,j,k+1/2},\omega_{i,j,k+1/2})=0 \tag{12}$$

where the summation is performed on the internal nodes of the mesh $\Omega_x\bigcup\Omega_y\bigcup\Omega_z$. We define the norm of the velocity vector as follows:

$$\|\vec{V}^n\|^2=\sum_{\Omega_x}(u^n_{i+1/2j,k})^2h_1h_2h_3+\sum_{\Omega_y}(\upsilon^n_{i,j+1/2,k})^2h_1h_2h_3+\sum_{\Omega_z}(\omega^n_{i,j,k+1/2})^2h_1h_2h_3 \tag{13}$$

Multiplying the differential equation (1)-(3) by $2\tau u^{n+1}_{i+1/2j,k}h_1h_2h_3$, $2\tau\upsilon^{n+1}_{i,j+1/2,k}h_1h_2h_3$ and $2\tau\omega^{n+1}_{i,j,k+1/2}h_1h_2h_3$, respectively, then summing over the points Ω_x Ω_y, Ω_z, we obtain the following basic energy inequality:

$$\|V^{n+1}\|^2-\|V^n\|^2+\|V^{n+1}-V^n\|^2+2\tau(L_{1h}V^n,V^{n+1})+$$
$$+2\tau(\sum_{\Omega_x}p^{n+1}_{x_1}u^{n+1}_{i+1/2j,k}+\sum_{\Omega_y}p^{n+1}_{x_2}\upsilon^{n+1}_{i,j+1/2,k}+\sum_{\Omega_z}p^{n+1}_{x_3}\omega^{n+1}_{i,j,k+1/2})h_1h_2h_3+2\tau d_h\le\frac{2\tau}{De}|S_h|+2\tau|(\vec{f}^h,V^{n+1})| \tag{14}$$

Let us evaluate the quantities appearing in equation (14). Considering the conditions (11), one can make sure that

$$d_h=\frac{\tau}{Re_T}\Bigg[\sum_{\Omega_x}(a_{i,j,k}u^n_{x_1,i+1/2j,k}u^{n+1}_{x_1,i+1/2j,k}+a_{i+1/2j+1/2,k}u^n_{x_2,i+1/2j,k}u^{n+1}_{x_2,i+1/2j,k}+$$
$$+a_{i+1/2,j,k+1/2}u^n_{x_3,i+1/2j,k}u^{n+1}_{x_3,i+1/2j,k})h_1h_2h_3+\sum_{\Omega_y}(a_{i+1/2,j+1/2,k}\upsilon^n_{x_1,i+1/2j,k}\upsilon^{n+1}_{x_1,i+1/2j,k}+a_{i,j+1,k}\upsilon^n_{x_2,i,j,k}\upsilon^{n+1}_{x_2,i,j,k}+ \tag{15}$$
$$+a_{i,j+1/2,k+1/2}\upsilon^n_{x_3,i,j+1/2,k+1/2}\upsilon^{n+1}_{x_3,i,j+1/2,k+1/2})h_1h_2h_3+\sum_{\Omega_z}(a_{i+1/2,j,k+1/2}\omega^n_{x_1,i+1/2,j,k+1/2}\omega^{n+1}_{x_1,i+1/2,j,k+1/2}+$$
$$+a_{i,j+1/2,k}\omega^n_{x_2,i,j+1/2,k}\omega^{n+1}_{x_2,i,j+1/2,k}+a_{i,j,k+1}\omega^n_{x_3,i,j,k}\omega^{n+1}_{x_3,i,j,k})h_1h_2h_3\Bigg]$$

Using Young's inequality and the boundedness of the coefficient $a(x_{1i},x_{2j},x_{3k})$ from below, we have

$$|d_h|\ge C_1(\|\nabla_h\vec{V}^n\|^2+\|\nabla_h\vec{V}^{n+1}\|^2-\|\nabla_h(\vec{V}^{n+1}-\vec{V}^n)\|^2), \tag{16}$$

where

$$\|\nabla\vec{V}\|^2 = \|\vec{V}_{x_1}\|^2 + \|\vec{V}_{x_2}\|^2 + \|\vec{V}_{x_3}\|^2, \tag{17}$$

$C_1 = 0,5a\tau/Re_T$. The term S_h can be written as follows:

$$S_h = \sum_{\Omega_x} \upsilon^n_{i+1/2j,k} u^{n+1}_{i+1/2j,k} h_1h_2h_3 - \sum_{\Omega_y} u^n_{i,j+1/2,k} \upsilon^{n+1}_{i,j+1/2,k} h_1h_2h_3 =$$

$$= \sum_{\Omega_x} \upsilon^n_{i+1/2j,k}(u^{n+1}_{i+1/2j,k} - u^n_{i+1/2j,k})h_1h_2h_3 - \sum_{\Omega_y} u^n_{i,j+1/2,k}(\upsilon^{n+1}_{i,j+1/2,k} - \upsilon^n_{i,j+1/2,k})h_1h_2h_3 \tag{18}$$

further, using young's inequality, we have

$$|S_h| \le \left(\sum_{\Omega_y} (\upsilon^n_{i,j+1/2,k})^2 h_1h_2h_3 + \sum_{\Omega_x} (u^n_{i+1/2j,k})^2 h_1h_2h_3 \right) +$$

$$+ \left(\sum_{\Omega_x} (u^{n+1}_{i+1/2j,k} - u^n_{i+1/2j,k})^2 h_1h_2h_3 + \sum_{\Omega_y} (\upsilon^{n+1}_{i,j+1/2,k} - \upsilon^n_{i,j+1/2,k})^2 h_1h_2h_3 \right). \tag{19}$$

We add non-negative summands

$$\sum_{\Omega_z} (\omega^n_{i,j,k+1/2})^2 h_1h_2h_3 + \sum_{\Omega_z} (\omega^{n+1}_{i,j,k+1/2} - \omega^n_{i,j,k+1/2})^2 h_1h_2h_3 \tag{20}$$

to the right-hand side of the inequality to obtain

$$|S_h| \le \|\vec{V}^n\|^2 + \|\vec{V}^{n+1} - \vec{V}^n\|^2. \tag{21}$$

By virtue the Lemma proved above:

$$2\tau(L_{1h}\vec{V}^n, \vec{V}^{n+1}) = 2\tau^2(L_{1h}\vec{V}^n, \vec{V}^n_t) \tag{22}$$

Using the Cauchy-Schwarz inequality we obtain:

$$|2\tau^2(L_{1h}V^n, V^n_t)| \le C_2\tau^2 \left\{ \sum_{\Omega_h} [(u^n_{i+1/2j,k})^2 + (\upsilon^n_{i,j+1/2,k})^2 + +(\omega^n_{i,j,k+1/2})]^2 h_1h_2h_3 \right\}^{1/2} \cdot \|\vec{V}^n_t\| \tag{23}$$

$$= C_2\tau^2 \| |\vec{V}^n|^2 \| \cdot \|\vec{V}^n_t\|$$

The term $\||\vec{V}^n|^2\|$ is evaluated as follows [15]:

$$\||\vec{V}^n|^2\| \le (\frac{4}{3})^{\frac{3}{4}} \|\vec{V}^n\|^{1/2} \|\nabla_h\vec{V}^n\|^{3/2}. \tag{24}$$

Then

$$|2\tau^2(L_{1h}\vec{V}^n, \vec{V}^n_t)| \le C_2(4/3)^{3/4}\tau^2 \|\vec{V}^n\|^{1/2} \|\nabla_h\vec{V}^n\|^{3/2} \|\vec{V}^n_t\| \le \|\vec{V}^n_t\|^2 + C_3\|\vec{V}^n\| \, \|\nabla_h\vec{V}^n\|^3, \tag{25}$$

where $C_3 = \frac{2^{\frac{5}{2}} C_2\tau^2}{3^{3/4}}$.

$$\begin{aligned} &\|\vec{V}^{n+1}\|^2 - \|\vec{V}^n\|^2 + \tfrac{1}{2}\|\vec{V}^{n+1} - \vec{V}^n\|^2 + C_1(\|\nabla_h\vec{V}^n\|^2 + \|\nabla_h\vec{V}^{n+1}\|^2 - \|\nabla_h(\vec{V}^{n+1} - \vec{V}^n)\|^2) - C_3\|\vec{V}^n_t\|^2 - \\ &-C_3\|\vec{V}^n\| \, \|\nabla_h\vec{V}^n\|^3 \le \tfrac{2\tau}{De}(\|\vec{V}^n\|^2 + \|\vec{V}^{n+1} - \vec{V}^n\|^2) + 2\tau|(\vec{f}^n, \vec{V}^{n+1})| \end{aligned} \tag{26}$$

Hence we have

$$\|\vec{V}^{n+1}\|^2 + C_1 \sum_{k=0}^{n} \|\nabla_h\vec{V}^{n+1}\|^2 \le \|V^0\|^2 + 2\tau \left(\sum_{k=0}^{n} \|\vec{f}^k\| \right) \left(\|\vec{V}^0\| + 2\tau \sum_{k=0}^{n} \|\vec{f}^k\| \right) \le 2\|\vec{V}^0\|^2 + 5\left(\tau \sum_{k=0}^{n} \|\vec{f}^k\| \right)^2 \tag{27}$$

To study the convergence of the solution of finite-difference problem to the solution of the differential problem, we consider the finite-difference equations for equations of the atmospheric boundary layer

$$u^n_{t,h,i+1/2j,k} + L^{(1)}_{1h} u^n_{x_1,h,i+1/2,j,k} + P^{n+1}_{x_1,h,i,j,k} = \tfrac{1}{De} \upsilon^n_{h,i,j+1/2,k} + \\ + \tfrac{1}{Re_T}\left[(a_{i,j,k} u^n_{x_1,h,i+1/2j,k})_{\bar{x}_1} + (a_{i+1/2j+1/2,k} u^n_{x_2,h,i+1/2j,k})_{\bar{x}_2} + (a_{i+1/2,j,k+1/2} u^n_{x_3,h,i+1/2,k})_{\bar{x}_3}\right] + f^0_{i+1/2j,k}, \\ i = \overline{1, N_1-2},\ j = \overline{1, N_2-1},\ k = \overline{1, N_3-1} \tag{28}$$

$$\upsilon_{t,h,i,j+1/2,k} + L^{(2)}_{1h} \upsilon^n_{i,j+1/2,k} + P^{n+1}_{x_2,h,i,j,k} = -\tfrac{1}{De} u^n_{h,i+1/2,j,k} + \\ + \tfrac{1}{Re_T}\left[(a_{i+1/2,j+1/2,k} \upsilon^n_{x_1,h,i+1/2,j+1/2,k})_{\bar{x}_1} + (a_{i,j+1,k} \upsilon^n_{x_2,h,i,j,k})_{\bar{x}_2} + (a_{i,j+1/2,k+1/2} \upsilon^n_{x_3,h,i,j+1/2,k+1/2})_{\bar{x}_3}\right] + f^0_{i,j+1/2,k}, \\ i = \overline{1, N_1-2},\ j = \overline{1, N_2-1},\ k = \overline{1, N_3-1} \tag{29}$$

$$\omega^n_{t,h,i,j,k+1/2} + L^{(3)}_{1h} \omega^n_{h,i,j,k+1/2} + P^{n+1}_{x_3,h,i,j,k} = \overline{\lambda} + \\ + \tfrac{1}{Re_T}\left[(a_{i+1/2,j,k+1/2} \omega^n_{x_1,h,i+1/2,j,k+1/2})_{\bar{x}_1} + (a_{i,j+1/2,k} \omega^n_{x_2,h,i,j+1/2,k})_{\bar{x}_2} + (a_{i,j,k+1} \omega^n_{x_3,h,i,j,k})_{\bar{x}_3}\right] + f^0_{i,j,k+1/2}, \\ i = \overline{1, N_1-2},\ j = \overline{1, N_2-1},\ k = \overline{1, N_3-1} \tag{30}$$

$$u^{n+1}_{\bar{x}_1,h,i+1/2j,k} + \upsilon^{n+1}_{\bar{x}_2,h,i,j+1/2,k} + \omega^{n+1}_{\bar{x}_3,h,i,j,k+1/2} = 0. \tag{31}$$

with the following initial-boundary conditions:

$$u^0_{i+1/2j,k} = V^0(x_{1i} + 0,5h_1, x_{2j}, x_{3k}),\ \ \upsilon^0_{i,j+1/2,k} = V^0(x_{1i}, x_{2,j} + 0,5h_2, x_{3k}), \omega^0_{i,j,k+1/2} = V^0(x_{1i}, x_{2j}, x_{3k} + 0,5h_3)$$

$$\begin{aligned} &\upsilon^{n+1}_{0,j+1/2,k} = \upsilon^{n+1}_{N_1,j+1/2,k} = u^{n+1}_{1/2j,k} = u^{n+1}_{N_1-1/2,j,k} = \omega^{n+1}_{0,j,k+1/2} = \omega^{n+1}_{N_1,j,k+1/2} = 0,\ \ j = \overline{0, N_2-1},\ \ k = \overline{0, N_3-1} \\ &\upsilon^{n+1}_{i,1/2,k} = \upsilon^{n+1}_{N_1,j-1/2,k} = u^{n+1}_{i+1/2,0,k} = u^{n+1}_{i+1/2,N_2,k} = \omega^{n+1}_{i,0,k+1/2} = \omega^{n+1}_{i,N_2,k+1/2} = 0,\ \ i = \overline{0, N_1-1},\ \ k = \overline{0, N_3-1} \\ &\upsilon^{n+1}_{i,j+1/2,0} = \upsilon^{n+1}_{i,j+1/2,N_3} = u^{n+1}_{i+1/2,j,0} = u^{n+1}_{i+1/2,j,N_3} = \omega^{n+1}_{i,j,1/2} = \omega^{n+1}_{i,j,N_3-1/2} = 0,\ \ i = \overline{0, N_1-1},\ \ j = \overline{0, N_2-1} \end{aligned} \tag{32}$$

We define the error of the solutions of differential problem (1)-(5) and the difference scheme (28)-(32) as follows:

$$\begin{aligned} &\phi^{(1)n}_{i+1/2j,k} = u^n_{h,i+1/2j,k} - u^n_{i+1/2j,k} \\ &\phi^{(2)n}_{i,j+1/2,k} = \upsilon^n_{h,i,j+1/2,k} - \upsilon^n_{i,j+1/2,k} \\ &\phi^{(3)n}_{i,j,k+1/2} = \omega^n_{h,i,j,k+1/2} - \omega^n_{i,j,k+1/2} \\ &\pi^{n+1}_{i,j,k} = p^{n+1}_{h,i,j,k} - p^{n+1}_{i,j,k} \end{aligned} \tag{33}$$

Defining from (33) $u^n_{h,i+1/2j,k}$, $\upsilon^n_{h,i,j+1/2,k}$, $\omega^n_{h,i,j,k+1/2}$, $p^{n+1}_{h,i,j,k}$ through $\phi^{(1)n}_{i+1/2j,k}$, $\phi^{(2)n}_{i,j+1/2,k}$, $\phi^{(3)n}_{i,j,k+1/2}$, $\pi^n_{i,j,k}$ and substituting into (28)-(31), we obtain

$$\phi^{(1)n}_t + L^{(1)}_{1h} \phi^{(1)n} + \pi^n_{i,j,k} = \frac{1}{De} \phi^{(2)n}_{i+1/2j,k} + \tag{34}$$

$$+\frac{1}{Re_T}\left[(a_{i,j,k} \phi^{(1)}_{x_1,i+1/2j,k})_{\bar{x}_1} + (a_{i+1/2j+1/2,k} \phi^{(1)n}_{x_2,i+1/2j,k})_{\bar{x}_2} + (a_{i+1/2,j,k+1/2} \phi^{(1)n}_{x_3,i+1/2j,k})_{\bar{x}_3}\right] + A^{(1)}_{i+1/2j,k} + \psi^{(1)}_{i+1/2j,k},$$

$$\phi^{(2)n}_t + L^{(2)}_{1h} \phi^{(2)n} + \pi^n_{i,j,k} = -\frac{1}{De} \phi^{(1)n}_{i,j+1/2,k} + \tag{35}$$

$$+\frac{1}{Re_T}\left[(a_{i+1/2,j+1/2,k} \phi^{(2)}_{x_1,i+1/2,j+1/2,k})_{\bar{x}_1} + (a_{i,j+1,k} \phi^{(2)n}_{x_2,i,j,k})_{\bar{x}_2} + (a_{i,j+1/2,k+1/2} \phi^{(2)n}_{x_3,i,j+1/2,k+1/2})_{\bar{x}_3}\right] + A^{(2)}_{i,j+1/2,k} + \psi^{(2)}_{i,j+1/2,k},$$

$$\phi^{(3)n}_t + L^{(3)}_{1h} \phi^{(2)n} + \pi^n_{i,j,k} = \overline{\lambda} + \tag{36}$$

$$+\frac{1}{Re_T}\left[(a_{i+1/2,j,k+1/2} \phi^{(3)}_{x_1,i+1/2,j,k+1/2})_{\bar{x}_1} + (a_{i,j+1/2,k} \phi^{(3)n}_{x_2,i,j+1/2,k})_{\bar{x}_2} + (a_{i,j,k+1} \phi^{(3)n}_{x_3,i,j,k})_{\bar{x}_3}\right] + A^{(3)}_{i,j,k+1/2} + \psi^{(3)}_{i,j,k+1/2},$$

$$\phi^{(1)(n+1)}_{\bar{x}_1,i+1/2j,k} + \phi^{(2)(n+1)}_{\bar{x}_2,i,j+1/2,k} + \phi^{(3)(n+1)}_{\bar{x}_3,i,j,k+1/2} = 0. \tag{37}$$

where the error of approximation of difference scheme (28)-(32)on the exact solution of the differential problem (1)-(5) is defined as

$$\psi^{(1)}_{i+1/2j,k} = \frac{1}{De}\upsilon^n_{i,j+1/2,k} + \frac{1}{Re_T}\left[(a_{i,j,k}u^n_{x_1,i+1/2j,k})_{\bar{x}_1} + (a_{i+1/2j+1/2,k}u^n_{x_2,i+1/2j,k})_{\bar{x}_2} + (a_{i+1/2,j,k+1/2}u^n_{x_3,i+1/2j,k})_{\bar{x}_3}\right]$$
$$-u^n_{t,i+1/2j,k} - L^{(1)}_{1h}u^n - P^n_{x_1,i,j,k} \tag{38}$$

$$\psi^{(2)}_{i,j+1/2,k} = -\frac{1}{De}u^n_{i+1/2j,k} + \frac{1}{Re_T}\left[(a_{i+1/2,j+1/2,k}\upsilon^n_{x_1,i+1/2,j+1/2,k})_{\bar{x}_1} + (a_{i,j+1,k}\upsilon^n_{x_2,i,j,k})_{\bar{x}_2} + (a_{i,j+1/2,k+1/2}\upsilon^n_{x_3,i,j+1/2,k+1/2})_{\bar{x}_3}\right]$$
$$-\upsilon^n_{t,i,j+1/2,k} - L^{(2)}_{1h}\upsilon^n - P^n_{x_2,i,j,k} \tag{39}$$

$$\psi^{(3)}_{i,j,k+1/2} = \overline{\lambda} + \frac{1}{Re_T}\left[(a_{i+1/2,j,k+1/2}\omega^n_{x_1,i+1/2,j,k+1/2})_{\bar{x}_1} + (a_{i,j+1/2,k}\omega^n_{x_2,i,j+1/2,k})_{\bar{x}_2} + (a_{i,j,k+1}\omega^n_{x_3,i,j,k})_{\bar{x}_3}\right]$$
$$-\omega^n_{t,i,j,k+1/2} - L^{(3)}_{1h}\omega^n - P^n_{x_3,i,j,k} \tag{40}$$

and has the second order of approximation by h and the first order by τ [16]. The initial-boundary conditions for the problem of error (34)-(37) are defined as follows

$$\begin{array}{l}
\phi^{(1)0}_{i+1/2j,k} = 0,\ \phi^{(2)0}_{i,j+1/2,k} = 0,\ \phi^{(3)0}_{i,j,k+1/2} = 0 \\
\phi^{(2)n+1}_{0,j+1/2,k} = \phi^{(2)n+1}_{N_1,j+1/2,k} = \phi^{(1)n+1}_{1/2j,k} = \phi^{(1)n+1}_{N_1-1/2,j,k} = \phi^{(3)n+1}_{0,j,k+1/2} = \phi^{(3)n+1}_{N_1,j,k+1/2} = 0,\ \ j = \overline{0,N_2-1},\ \ k = \overline{0,N_3-1} \\
\phi^{(2)n+1}_{i,1/2,k} = \phi^{(2)n+1}_{N_1,j-1/2,k} = \phi^{(1)n+1}_{i+1/2,0,k} = \phi^{(1)n+1}_{i+1/2,N_2,k} = \phi^{(3)n+1}_{i,0,k+1/2} = \phi^{(3)n+1}_{i,N_2,k+1/2} = 0,\ \ i = \overline{0,N_1-1},\ \ k = \overline{0,N_3-1} \\
\phi^{(2)n+1}_{i,j+1/2,0} = \phi^{(2)n+1}_{i,j+1/2,N_3} = \phi^{(1)n+1}_{i+1/2,j,0} = \phi^{(1)n+1}_{i+1/2,j,N_3} = \phi^{(3)n+1}_{i,j,1/2} = \phi^{(3)n+1}_{i,j,N_3-1/2} = 0,\ \ i = \overline{0,N_1-1},\ \ j = \overline{0,N_2-1}
\end{array} \tag{41}$$

Multiplying the differential equation (34)-(37) by $2\tau\phi^{(1)(n+1)}_{i+1/2j,k}h_1h_2h_3$, $2\tau\phi^{(2)(n+1)}_{i,j+1/2,k}h_1h_2h_3$, $2\tau\phi^{(3)(n+1)}_{i,j,k+1/2}h_1h_2h_3$, respectively, then summing by grid domains Ω_x, Ω_y, Ω_z, we obtain

$$\begin{array}{l}
\|\vec{\phi}^{n+1}\|^2 - \left(1+\frac{5\tau}{2}C_4 - \frac{2\tau}{De}\right)\|\vec{\phi}^n\|^2 + (1-2\tau C_4)\|\nabla_h\vec{\phi}^{n+1}\|^2 + \left(1-\frac{24\tau C_1}{h^2} - \frac{2}{\tau} - \frac{2\tau}{De}\right)\|\vec{\phi}^{n+1}-\vec{\phi}^n\|^2 + \\
+2\tau(C_1 - C_3\|\vec{\phi}^n\|\|\nabla_h\vec{\phi}^n\|)\|\nabla_h\vec{\phi}^n\|^2 \le 2\tau\|\vec{\psi}^n\|\|\vec{\phi}^{n+1}\|.
\end{array} \tag{42}$$

Let us denote $a = 1$; $b = 1+\frac{5\tau}{2}C_4 - \frac{2\tau}{De}$ and rewrite (42) as follows:

$$\begin{array}{l}
a\|\vec{\phi}^{n+1}\|^2 - b\|\vec{\phi}^n\|^2 + \left(1-\frac{24\tau C_1}{h^2} - \frac{2}{\tau} - \frac{2\tau}{De}\right)\|\vec{\phi}^{n+1}-\vec{\phi}^n\|^2 + (1-2\tau C_4)\|\nabla_h\vec{\phi}^{n+1}\|^2 + \\
+2\tau(C_1 - C_3\|\vec{\phi}^n\|\|\nabla_h\vec{\phi}^n\|)\|\nabla_h\vec{\phi}^n\|^2 \le 2\tau\|\vec{\psi}^n\|\|\vec{\phi}^{n+1}\|
\end{array} \tag{43}$$

Let $a \ge b$. Then is follows that

$$\frac{2}{De} - \frac{5}{2}C_4 \ge 0. \tag{44}$$

Let

$$C_1 - C_3\|\vec{\phi}^n\|\|\nabla_h\vec{\phi}^n\| \ge 0;\ \ 1 - \frac{24\tau C_1}{h^2} - \frac{2}{\tau} - \frac{2\tau}{De} > 0;\ \ 1-2\tau C_4 > 0. \tag{45}$$

Then, considering that the third and fifth terms in the left-hand side of (43) are nonnegative, we obtain

$$a(\|\vec{\phi}^{n+1}\|^2 - \|\vec{\phi}^n\|^2) + C_5\|\nabla_h\vec{\phi}^{n+1}\|^2 \le 2\tau\|\vec{\psi}^n\|\ \|\vec{\phi}^{n+1}\| \tag{46}$$

where $C_5 = 1-2\tau C_4$. Considering that $a = 1$, we have

$$\|\vec{\phi}^{n+1}\|^2 - \|\vec{\phi}^n\|^2 + C_5\|\nabla_h\vec{\phi}^{n+1}\|^2 \le 2\tau\|\vec{\psi}^n\|\ \|\vec{\phi}^{n+1}\|.$$

Considering similarly as for problem (1)-(5), we obtain the following estimation for the problem (34)-(37), (41):

$$\|\vec{\phi}^{n+1}\|^2 + C_5\sum_{k=0}^{n}\|\nabla_h\vec{\phi}^{n+1}\|^2 \le 5\tau^2\left(\sum_{k=0}^{n}\|\vec{\psi}^k\|\right)^2, \tag{47}$$

Further, given according to (38), that $\|\vec{\psi}^n\| = O(h^2)$, we finally have

$$\|\vec{\phi}^{n+1}\|^2 + C_5 \sum_{k=0}^{n} \|\nabla_h \vec{\phi}^{n+1}\|^2 \leq C_6(\tau^2 + h^4). \tag{48}$$

which proves the convergence of the solution of the difference problem (28)-(32) to the solution of the differential problem (1)-(5). **Theorem.** Let the conditions (45). Then the solution of the difference scheme (28)-(32) is stable and converges to the solution of the differential problem (1)-(5) with the speed determined by the inequality(48).

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