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Niyaz Tokmagambetov and Gulzat Nalzhupbayeva

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Operator perturbed Cauchy problem for the Gellerstedt equation

Niyaz Tokmagambetov and Gulzat Nalzhupbayeva

Al-Farabi Kazakh National University, 050040, Almaty, Kazakhstan

Abstract. In this paper, we consider boundary value problems with perturbations operator for a degenerate first order hyperbolic equation in the characteristic triangle with Cauchy conditions on the curve of degeneracy. The unique solvability of the boundary value problems with operator perturbations is proved.

Keywords: Gellerstedt operator, Degenerate hyperbolic equation, Perturbation, Cauchy problem, Nonlocal boundary condition, Hypergeometric function
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INTRODUCTION

Nonlocal boundary value problems of various kinds for partial differential equations are of great current interest in several fields of application. In a typical nonlocal problem, the partial differential equation (resp. boundary conditions) for an unknown function u at any point in a domain Ω involves not only the local behavior of u in a neighborhood of that point but also the nonlocal behavior of u elsewhere in Ω. For example, at any point in Ω the partial differential equation and/or boundary conditions may contain integrals of u over parts of Ω, values of u elsewhere in Ω or, generally speaking, some nonlocal operator on u. Indeed, nonlocal conditions can be more useful than standard conditions to describe some physical phenomena.

Boundary value problems with shift is a simple class of nonlocal problems. This class of nonlocal problems have been studied by V.A. Steklov, F.I. Frankl, A.V. Bitsadze, A.M. Nakhushev, A.N. Zarubin, O.A. Repin, E.A. Utkina and their students (see [1]).

Integral boundary conditions are generalization of the problems with shift. The study of boundary value problems for parabolic and hyperbolic equations with integral conditions, initiated by J. Cannon [2] and L.I. Kamynin [3], has been developed by N.I. Ionkin [4], L.A. Muravei and A.V. Filinovsky [5], S. Mesloub and S.A. Messaoudi [6], A. Bouziani [7], A.I. Kozhanov [8, 9], L.S. Pulkina [9, 10] and the others.

STATEMENT OF PROBLEMS

Let $\Omega \subset \mathbb{R}^2$ be a domain bounded by characteristics

$$AC:\, \xi = x - \frac{2}{m+2} y^{\frac{m+2}{2}} = 0, BC:\, \eta = x + \frac{2}{m+2} y^{\frac{m+2}{2}} = 1$$

of the following degenerate second order hyperbolic equation

$$L(u) := y^m u_{xx} - u_{yy} = f(x,y),\; m = const > 0, \tag{1}$$

and by the segment $AB: 0 \le x \le 1$ of the line $y = 0$. Hereinafter, by I we denote unit interval $(0,1)$, and by $\overline{\Omega}$ we denote closure of Ω. Let $\beta = \frac{m}{2m+4}$.

Problem OPCP$_1$. *To find in Ω the solution $u(x,y)$ of equation* (1) *from $C(\overline{\Omega}) \cap C^1(\Omega)$, satisfying the conditions*

$$u|_{AB} \equiv u(x,0) = \tau(x),\; 0 < x < 1, \tag{2}$$

$$u_y|_{AB} \equiv u_y(x,0) = \mathscr{S}_1 u(x) + \nu(x),\; 0 < x < 1, \tag{3}$$

where $\mathscr{S}_1 : C(\overline{\Omega}) \to C^{1-\beta}[0,1]$.

Advancements in Mathematical Sciences
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Problem OPCP$_2$. *To find in Ω the solution $u(x,y)$ of equation (1) from $C(\overline{\Omega})\cap C^1(\Omega)$, satisfying the conditions*

$$u|_{AB} \equiv u(x,0) = \mathscr{S}_2 u(x) + \tau(x),\ 0<x<1, \tag{4}$$

$$u_y|_{AB} \equiv u_y(x,0) = \nu(x),\ 0<x<1, \tag{5}$$

where $\mathscr{S}_2 : C(\overline{\Omega}) \to C^\beta[0,1]$.

Problem OPCP$_3$. *To find in Ω the solution $u(x,y)$ of the equation*

$$L(u) := y^m u_{xx} - u_{yy} + \mathscr{S}_3 u(x) = f(x,y),\ m = const > 0 \tag{6}$$

from $C(\overline{\Omega})\cap C^1(\Omega)$, satisfying conditions (2)–(5), where $\mathscr{S}_3 : C(\overline{\Omega}) \to C(\overline{\Omega})$.

MAIN RESULTS

Let us consider the operators

$$\mathscr{T}_1 h \quad : \quad = \Gamma\left(\frac{m+4}{m+2}\right)/\Gamma^2\left(\frac{m+4}{2m+4}\right) y \int_0^1 h\left[x + \frac{2(1-2t)}{m+2} y^{\frac{m+2}{2}}\right][t(1-t)]^{-\frac{m}{2m+4}} dt,$$

$$\mathscr{T}_2 \nu \quad : \quad = \Gamma\left(\frac{m+4}{m+2}\right)/\Gamma^2\left(\frac{m+4}{2m+4}\right) \int_0^1 \nu\left[x + \frac{2(1-2t)}{m+2} y^{\frac{m+2}{2}}\right][t(1-t)]^{-\frac{m+4}{2m+4}} dt,$$

$$\mathscr{T}_3 g \quad : \quad = -\int\limits_{x-\frac{2}{m+2}y^{\frac{m+2}{2}}}^{x+\frac{2}{m+2}y^{\frac{m+2}{2}}} \int\limits_{x_1}^{x+\frac{2}{m+2}y^{\frac{m+2}{2}}} H(t,x_1,x-\frac{2}{m+2}y^{\frac{m+2}{2}},x+\frac{2}{m+2}y^{\frac{m+2}{2}})g(t,x_1)dt dx_1,$$

where $H(t,x_1,x,y)$ is a Riemann function of (1), which has an explicit form [11, p. 264]. Denote $\mathscr{V}_i = -\mathscr{S}_i \circ \mathscr{T}_i$, $i = 1,2,3$.

Theorem 1. *Let $\tau(x) \in C^\beta[0,1]$, $\nu(x) \in C^{1-\beta}[0,1]$ and $f(x,y) \in C(\overline{\Omega})$. If $V_1 : C[0,1] \to C^{1-\beta}[0,1]$ is a Volterra integral operator with weak singularity in the kernel, then problem OPCP$_1$ is uniquely solvable.*

Proof. We denote $\phi(x) = \mathscr{S}_1 u(x),\ x \in \bar{I}$. Then, condition (3) can be written as

$$u_y|_{AB} = \phi(x) + \nu(x), x \in \bar{I}. \tag{7}$$

From ([11, p. 264], [12, p. 12]) it follows that if $\phi(x) \in C^{1-\beta}[0,1]$, then the solution of equation (1) with Cauchy initial data (2), (7) has the form

$$u = \mathscr{T}_1 \nu + \mathscr{T}_1 \phi + \mathscr{T}_2 \tau + \mathscr{T}_3 f. \tag{8}$$

By applying the operator $\mathscr{S}_1$ to the both sides of equation (8), we get integral equation

$$\phi + \mathscr{V}_1 \phi = F, \tag{9}$$

where

$$F = \mathscr{S}_1(\mathscr{T}_1 \nu + \mathscr{T}_2 \tau + \mathscr{T}_3 f),$$

and $F(x) \in C^{1-\beta}[0,1]$. By virtue of this, equation (9) is a Volterra integral equation of the second kind with weak singularity in the kernel, which is uniquely solvable in the space $C[0,1]$ and $\phi(x) \in C[0,1]$. From $\mathscr{V}_1 : C[0,1] \to C^{1-\beta}[0,1]$, it follows that $\phi(x) \in C^{1-\beta}[0,1]$. □

The following theorems are proved similarly.

Theorem 2. *Let $\tau(x) \in C^\beta[0,1]$, $\nu(x) \in C^{1-\beta}[0,1]$ and $f(x,y) \in C(\overline{\Omega})$. If $V_2 : C[0,1] \to C^\beta[0,1]$ is a Volterra integral operator with weak singularity in the kernel, then problem OPCP$_2$ is uniquely solvable.*

Theorem 3. *Let $\tau(x) \in C^\beta[0,1]$, $\nu(x) \in C^{1-\beta}[0,1]$ and $f(x,y) \in C(\overline{\Omega})$. If $V_3 : C(\overline{\Omega}) \to C(\overline{\Omega})$ is a Volterra integral operator with weak singularity in the kernel, then the problem OPCP$_3$ is uniquely solvable.*

EXAMPLE OF $\mathscr{S}_1$

Let the operator $\mathscr{S}_1$ be of the following form

$$\mathscr{S}_1 r = \int_0^x dx_1 \int_0^{\kappa(x,x_1)} K(x_1,y_1,x) r(x_1,y_1) dy_1,$$

where

$$\kappa(x,x_1) = \left(\frac{m+2}{2} x_1\right)^{\frac{2}{m+2}} \text{ for } 0 < x_1 < \frac{x}{2},$$

and

$$\kappa(x,x_1) = \left(\frac{m+2}{2}(x-x_1)\right)^{\frac{2}{m+2}} \text{ for } \frac{x}{2} < x_1 < x.$$

Here the kernel $K(x_1,y_1,x)$ has the following structure

$$K(x_1,y_1,x) := \left(x_1 - \frac{2}{m+2} y_1^{\frac{m+2}{2}}\right)^{-a} K_1(x_1,y_1,x),$$

where

$$K_1(x_1,y_1,x) \in C(\overline{\Omega} \times \bar{I}), \ \ \frac{\partial K_1}{\partial x}(x_1,y_1,x) \in C(\overline{\Omega} \times \bar{I}),$$

and a is some real number. If $a < \frac{2}{m+2}$, then $\mathscr{V}_1$ is a Volterra integral operator with weak singularity in the kernel as the operator acting from $C[0,1]$ to $C^{1-\beta}[0,1]$. For more details see [13].

Note that the problems with integral perturbations of boundary conditions have been investigated in [14–16] by the author in collaboration with T.Sh. Kalmenov, B.E. Kanguzhin and D. Suragan. Also see [17–26].

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