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On the ill-posed problem for the Poisson equation

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Abstract. In this paper we consider the boundary value problem in two-dimensional rectangular domain for the Poisson equation. The studied ill-posed boundary value problem is reduced to the optimal control problem. In terms of solutions of the adjoint boundary value problem, the necessary and sufficient conditions of optimality are established. It is found criterion for strong solvability of the ill-posed boundary value problem.

Keywords: Poisson equation, Ill-posed problem, Inverse problem, Optimal control
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INTRODUCTION

Recently among the experts on equations of mathematical physics interest in problems that are ill-posed by J. Hadamard has significantly increased [1]. Due to the ill-posed problems classic works by J. Hadamard [1], A. N. Tikhonov [2], M. M. Lavrent'ev [3] and many others can be noted, which have drawn the attention of researchers for ill-posed problems and have made a significant contribution to the development of this important area of mathematics. In this paper we study the ill-posed problem [1–8] for the Poisson equation in two-dimensional rectangular domain. The correctness criterion of homogeneous mixed Cauchy problem for the Poisson equation in a rectangular domain was established in the papers of T. Sh. Kalmenov, U. A. Iskakova [6, 7]. In paper [8] the ill-posed problem for the heat equation is considered. The general regularization method for constructing an approximate solution of ill-posed problems of mathematical physics was proposed by A. N. Tikhonov [2]. In the book R. Lattes, J. -L. Lions [4] for regularization of ill-posed boundary value problems the quasiinversion method is proposed. Features and questions of the regularization of Cauchy problems for abstract differential equations with the operator coefficients are studied by I. V. Mel'nikova and U. A. Anufrieva [8].

STATEMENT OF THE PROBLEM

We consider the boundary value problem

$$y_{tt}(x,t) + y_{xx}(x,t) = f(x,t), \tag{1}$$

$$y(0,t) = 0,\ y(\pi,t) = 0, \tag{2}$$

$$y(x,-1) = \varphi_1(x),\ y_t(x,-1) = \varphi_2(x), \tag{3}$$

in the domain $\Omega = \{x, t \,|\, 0 < x < \pi, -1 < t < 1\}$ with the additional condition

$$y_t(x,1) \in \mathscr{U}_g, \text{ where } \mathscr{U}_g \text{ is a closed convex set of } L_2(0,\pi). \tag{4}$$

It is assumed that the data in problem (1)–(3) satisfies the following conditions:

$$f \in L_2(\Omega),\ \varphi_1 \in H_0^1(0,\pi),\ \varphi_2 \in L_2(0,\pi). \tag{5}$$

In the book R. Lattes, J. -L.Lions [4], it is indicated that problem (1)–(3) is ill-posed in the space $L_2(\Omega)$. In this paper for solving the ill-posed problem we apply methods of optimal control.

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THE OPTIMIZATION PROBLEM AND ITS REGULARIZATION

For the investigation of problem (1)–(4), we formulate according to it the following optimization problem:

$$y_{tt}(x,t)+y_{xx}(x,t)=f(x,t), \tag{6}$$

$$y(0,t)=y(\pi,t)=0, \tag{7}$$

$$y_t(x,-1)=\varphi_2(x),\ y_t(x,1)=\psi(x), \tag{8}$$

with functional of optimality:

$$\mathscr{J}(\psi)=\int_0^\pi |y_x(x,-1)-\varphi_1'(x)|^2dx\to\min_{\psi\in\mathscr{U}_g}. \tag{9}$$

We note, in optimization problem (6)–(9) the function $\psi(x)$ plays the role of control function. In addition, further in the work it will be shown that boundary value problem (6)–(8) is well-posed, namely it is uniquely solvable for any given functions $\psi\in\mathscr{U}_g\subset L_2(0,\pi)$, $f\in L_2(\Omega)$. As it is known from the theory of optimal control optimization problem (6)–(9) is also ill-posed. To study our problem, we will use stabilizer of Tikhonov [2]. Effective tool is the method of regularization. In our case

$$\alpha\int_0^\pi |\psi(x)|^2\,dx,\ \ (\alpha>0)$$

will serve as a stabilizer. We consider the problem of minimizing the following functional

$$\mathscr{J}_\alpha(y,\psi)=\int_0^\pi |y_x(x,-1)-\varphi_1'(x)|^2dx+\alpha\int_0^\pi |\psi(x)|^2dx\to\min_{\psi\in\mathscr{U}_g}. \tag{10}$$

Thus, we have regularized optimization problem (6)–(8), (10). Due to the presence of the stabilizer the problem has become strictly convex, namely we get well-posed optimization problem. Therefore, for each value $\alpha>0$ this problem has the unique optimal solution that delivers the minimum value to minimized functional (10). However, it does not exclude the fact that the minimum value problem of functional (10) can be strictly greater than zero. For optimal control problem (6)–(8), (10) we will establish optimality conditions. We introduce the concept of optimal control.

Definition 1. *An element $\overline{\psi}\in L_2(0,\pi)$ which satisfies the condition*

$$\mathscr{J}_\alpha(\overline{\psi})=\inf_{\psi\in\mathscr{U}_g}\mathscr{J}_\alpha(\psi)$$

is called the optimal control.

We denote the solution of problem (6)–(8) by $y(x,t;\psi)$ corresponding to the given control $\psi(x)\in\mathscr{U}_g$. So $y(x,t;0)$ corresponds to the solution of problem (6)–(8) when $\psi(x)\equiv 0$. Then, we get

$$\pi(\psi_1,\psi_2)=\int_0^\pi [y_x(x,-1;\psi_1)-y_x(x,-1;0)][y_x(x,-1;\psi_2)-y_x(x,-1;0)]dx+\alpha\cdot\int_0^\pi \psi_1(x)\cdot\psi_2(x)dx,$$

$$L(\psi_1)=\int_0^\pi [\varphi_1'(x)-y_x(x,-1;0)][y_x(x,-1;\psi_1)-y_x(x,-1;0)]dx.$$

Here, $\pi(\psi_1,\psi_2)$ is the bilinear functional on $\mathscr{U}_g$, $L(\psi_1)$ is the continuous linear functional on admissible set of controls $\mathscr{U}_g$, as it will be shown below, that the solution $y(x,t;\psi)$ of problem (6)–(8) is not only continuous but it is continuously differentiable on control ψ. Using the notation, functional (10) can be rewritten as

$$\mathscr{J}_\alpha(\psi)=\pi(\psi,\psi)-2L(\psi)+\int_0^\pi |y_x(x,-1;0)-\varphi_1'(x)|^2dx.$$

THE EXISTENCE OF SOLUTION OF THE REGULARIZED PROBLEM AND THE VARIATIONAL INEQUALITY

The following theorem holds [9]:

Theorem 1. *As $\pi(\psi,\psi)$ is the continuous symmetric quadratic functional on a $\mathscr{U}_g$ and satisfies the condition*

$$\pi(\psi,\psi) \geq c\|\psi\|^2, \ (c = \text{const} > 0), \tag{11}$$

then for problem (6)–(8), (10) exists only for $\overline{\psi} \in \mathscr{U}_g$:

$$\mathscr{J}_\alpha(\overline{\psi}) = \inf_{\psi\in\mathscr{U}_g} \mathscr{J}_\alpha(\psi).$$

Proposition 2 (Variational inequality)**.** *The function $\overline{\psi} \in \mathscr{U}_g$ is a function of the optimal control if and only if the following inequality holds:*

$$\langle \mathscr{J}_{\alpha\psi}(\overline{\psi}), \psi - \overline{\psi}\rangle \geq 0 \textit{ for all } \psi \in \mathscr{U}_g,$$

namely we have

$$\int_0^\pi \left[y_x(x,-1;\overline{\psi}) - \varphi_1'(x)\right] \frac{\partial}{\partial x}\left\{y_\psi(x,-1;\overline{\psi})\left[\psi(x)-\overline{\psi}(x)\right]\right\} dx$$
$$+\alpha \int_0^\pi \overline{\psi}(x)\left[\psi(x)-\overline{\psi}(x)\right] dx \geq 0 \textit{ for all } \psi \in \mathscr{U}_g. \tag{12}$$

We now carry out the necessary further transformations of variational inequality (12). For this purpose, we rewrite the boundary value problem (6)–(8) in the operator form $\mathscr{A}y = \mathscr{F} = \{f, \varphi_2, \psi\}$. As for any admissible controls boundary value problem (6)–(8) is uniquely solvable, then its solution $y(x,t;\psi)$ can be written in the following form $y(x,t;\psi) = \mathscr{A}^{-1}\mathscr{F} = \mathscr{A}_0^{-1}f + \mathscr{A}_1^{-1}\varphi_2 + \mathscr{A}_2^{-1}\psi$. Next, we take the derivative of this solution in the direction of $\psi - \overline{\psi}$. We have

$$y_\psi(x,t;\overline{\psi})\cdot[\psi-\overline{\psi}] = \mathscr{A}^{-1}(\psi-\overline{\psi}) = \mathscr{A}_0^{-1}f + \mathscr{A}_1^{-1}\varphi_2 + \mathscr{A}_2^{-1}\psi - [\mathscr{A}_0^{-1}f + \mathscr{A}_1^{-1}\varphi_2 + \mathscr{A}_2^{-1}\overline{\psi}] = y(x,t;\psi) - y(x,t;\overline{\psi}).$$

Thus inequality (12) has the form:

$$\int_0^\pi \left[y_x(x,-1;\overline{\psi}) - \varphi_1'(x)\right]\cdot\left[y_x(x,-1;\psi) - y_x(x,-1;\overline{\psi})\right] dx + \alpha\cdot\int_0^\pi \overline{\psi}(x)\cdot\left[\psi(x)-\overline{\psi}(x)\right] dx \geq 0 \ \text{for all } \psi \in \mathscr{U}_g. \tag{13}$$

THE ADJOINT BOUNDARY VALUE PROBLEM

For further study of regularized optimization problem (6)–(8), (10), we introduce the adjoint boundary value problem

$$\begin{cases} v_{tt}(x,t) + v_{xx}(x,t) = 0, \ x \in (0,\pi), \ t \in (-1,1), \\ v(0,t) = v(\pi,t) = 0, \ t \in (-1,1), \\ \int_\eta^x v_t(\xi,-1)d\xi = -y_x(x,-1;\overline{\psi}) + \\ +\varphi_1'(x) + y_\eta(\eta,-1;\overline{\psi}) - \varphi_1'(\eta) \ \text{for all } 0 < \eta < x < \pi, \\ v_t(x,1) = 0. \end{cases} \tag{14}$$

THE OPTIMALITY CONDITIONS

As

$$\int_{-1}^{1}\int_{0}^{\pi}[\tilde{y}_{tt}(x,t)+\tilde{y}_{xx}(x,t)]\,v(x,t;\overline{\psi})dxdt = \int_{0}^{\pi}[\psi(x)-\overline{\psi}(x)]v(x,1;\overline{\psi})dx + \int_{0}^{\pi}[y(x,-1;\psi)-y(x,-1;\overline{\psi})]v_t(x,-1;\overline{\psi})dx = 0. \quad (15)$$

Then from relation (15), we finally obtain the desired variational inequality

$$\int_{0}^{\pi}[-v(x,1;\overline{\psi})+\alpha\cdot\overline{\psi}(x)]\cdot[\psi(x)-\overline{\psi}(x)]\,dx \geq 0 \text{ for all } \psi\in\mathscr{U}_g. \quad (16)$$

Thus, on the basis of Proposition 2 we have established the optimality conditions, which can be formulated as the following proposition:

Proposition 3. *The element $\overline{\psi}(x)$ is the optimal solution to the problem (6)–(8), (10), if and only if it satisfies boundary value problems (6)–(8), (14) and variational inequality (16).*

APPLICATION OF THE METHOD OF SEPARATION OF VARIABLES

For resolving the conditions of an optimality (6)–(8), (14) and (16) we use a method of separation of variables. We will search solutions of boundary value problems (6)–(8) and (14) in the form

$$y(x,t)=\sum_{k=1}^{\infty}y_k(t)X_k(x),\;\; v(x,t)=\sum_{k=1}^{\infty}v_k(t)X_k(x),$$

where

$$X_k(x)=\frac{\sin kx}{\sqrt{\pi/2}},\quad \lambda_k=k^2,\; k=1,2,\ldots \quad (17)$$

are systems orthonormalized eigenfunctions and eigenvalues for a spectral problem:

$$X''(x)=\lambda\cdot X(x),\quad X(0)=X(\pi)=0.$$

From (6)–(8), (14) and (16) we accordingly obtain

$$\begin{cases} y_k''(t)-k^2y_k(t)=f_k(t),\, t\in(-1,1), \\ y_k'(-1)=\varphi_{2k};\; y_k'(1)=\overline{\psi}_k;\; k=1,2,\ldots, \end{cases} \quad (18)$$

$$\begin{cases} v_k''(t)-k^2v_k(t)=0,\, t\in(-1,1), \\ v_k'(-1)=k^2[y_k(-1)-\varphi_{1k}];\; v_k'(1)=0;\; k=1,2,\ldots, \end{cases} \quad (19)$$

$$[-v_k(1)+\alpha\cdot\overline{\psi}_k]\cdot[\psi_k-\overline{\psi}_k]\geq 0,\;\text{for}\,\forall\;\psi_k,\; k=1,2,\ldots, \quad (20)$$

where $f_k(t),\ \varphi_{1k},\ \varphi_{2k},\ \overline{\psi}_k,\ \psi_k,\ k=1,2,\ldots$ are Fourier-coefficients of functions $f(x,t),\ \varphi_1(x),\ \varphi_2(x)$ and $\overline{\psi}(x),\ \psi(x)$ on system (17).

Assume us write solutions of boundary value problems (18) and (19):

$$y_k(t)=\overline{\psi}_k\cdot\frac{\cosh k(t+1)}{\sinh 2k}-\varphi_{2k}\cdot\frac{\cosh k(1-t)}{k\sinh 2k}+\int_{-1}^{1}G_k(t,\tau)\cdot f_k(\tau)d\tau, \quad (21)$$

$$v_k(t) = -[y_k(-1) - \varphi_{1k}] \cdot \frac{k \cosh k(1-t)}{\sinh 2k}, \tag{22}$$

where

$$G_k(t,\tau) = \begin{cases} -\frac{\cosh k(1-t)\cdot \cosh k(1+\tau)}{\sinh 2k}, & -1 < \tau < t < 1; \\ -\frac{\cosh k(1-\tau)\cdot \cosh k(1+t)}{\sinh 2k}, & -1 < t < \tau < 1. \end{cases}$$

From (20) and (21)–(22) we find

$$-v_k(1) = [y_k(-1) - \varphi_{1k}] \cdot \frac{k}{\sinh 2k},$$

$$y_k(-1; \overline{\psi}_k) = -\varphi_{2k}\frac{\coth 2k}{k} + \overline{\psi}_k \frac{1}{\sinh 2k} + \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau,$$

$$\left[A_{k\alpha}\overline{\psi}_k - \varphi_{1k} - \varphi_{2k}\frac{\coth 2k}{k} + \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau \right] \cdot [\psi_k - \overline{\psi}_k] \geq 0 \text{ for all } \psi_k, \tag{23}$$

where $A_{k\alpha} = \frac{k+\alpha \sinh^2 2k}{k \sinh 2k}$, $k = 1, 2, \cdots$.

Now we put, that $\mathscr{U}_g \equiv L_2(0,\pi)$. Since the functions $\psi(x)$ do not have any restrictions except for belonging to the space $L_2(0,\pi)$, from (23) we can find the optimal values of Fourier coefficients $\overline{\psi}_k$, $k = 1, 2, \ldots$:

$$\overline{\psi}_k = A_{k\alpha}^{-1} \left[\varphi_{1k} + \varphi_{2k}\frac{\coth 2k}{k} - \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau \right]. \tag{24}$$

Further, as $\alpha \to 0$ (21) and (24) imply that

$$\begin{aligned} y_{k0}(t) &= \lim_{\alpha\to 0} y_k(t) = \varphi_{1k}\cosh k(1+t) + \varphi_{2k}\frac{\sinh k(1+t)}{k} \\ &- \cosh k(1+t) \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau + \int_{-1}^{1} G_k(t,\tau) \cdot f_k(\tau) d\tau, \end{aligned} \tag{25}$$

$$\overline{\psi}_{k0} = \lim_{\alpha\to 0} \overline{\psi}_k = \varphi_{1k}\sinh 2k + \varphi_{2k}\frac{\cosh 2k}{k} - \sinh 2k \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau. \tag{26}$$

Additionally, the solutions $y_k(t)$ found under formula (21) according to optimal Fourier coefficients $\overline{\psi}_k$, $k = 1, 2, \ldots$ (24) must satisfy limiting relations: $\lim_{\alpha\to 0} y_k(-1) = \varphi_{1k}$, which really hold. And it is coordinated with a condition $y(x,-1) = \varphi_1(x)$ from (3).

Thus, for finding of the exact solution of problem (6)–(8) according to (26) we construct the following series:

$$\overline{\psi}(x) = \sum_{k=1}^{\infty} \sqrt{2/\pi} \sinh 2k \left[\varphi_{1k} + \varphi_{2k}\frac{\coth 2k}{k} - \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau \right] \sin kx,$$

and for initial Cauchy-Dirichlet problem (1)–(3) we obtain the solution on the basis of formulas (25).

CONCLUSION

From equalities (25) and (26) the following directly holds:

Firstly, with growth of index k and at $\alpha \to 0$ the Fourier-coefficients of the function $\overline{\psi}(x)$ and, respectively, the solution $y_k(t)$ can increase without limit if this growth is not be "suppressed" with corresponding more rapid decrease of the absolute values of the coefficients φ_{1k}, φ_{2k} and values of norms $\|f_k(t)\|_{L_2(-1,1)}$.

Secondly, boundary value problem (1)–(3) under conditions (5) has unique L_2-strong solution [10] if and only if

$$\{\exp\{2k\}\cdot\varphi_{1k}\}_{k=1}^{\infty}, \quad \{k^{-1}\exp\{2k\}\cdot\varphi_{2k}\}_{k=1}^{\infty}, \quad \{\exp\{2k\}\cdot\|f_k(\tau)\|_{L_2(-1,1)}\}_{k=1}^{\infty} \subset l_2. \tag{27}$$

Thus, it is clear not only the meaning of regularization in problem (6)–(8) and (10), but also the nature of incorrectness in Cauchy-Dirichlet problem (1)–(3) [6, 7]. And regularization allows us to find an approximate solution.

Thirdly, we consider the example of Hadamard (see p.37, [11]). To receive analogue of the Hadamard example in problem (1)–(3) it is necessary to accept:

$$f(x,t) = 0, \ \varphi_1(x) = 0, \ \varphi_2(x) = \sqrt{2/\pi}\cdot k\cdot\exp\{-\sqrt{k}\}\sin kx, \ k \in \mathbf{N}.$$

Really, the solution of Cauchy-Dirichlet problem for Laplace equation has the form:

$$y(x,t) = \sqrt{2/\pi}\cdot\exp\{-\sqrt{k}\}\sin kx\cdot\sinh k(t+1), \ k \in \mathbf{N}. \tag{28}$$

This solution of a problem in example of Hadamard considered by us is unique. Moreover, as $k \to \infty$ the function $\varphi_2(x)$ approaches uniformly zero and that not only, but also all its derivatives approache zero and it belongs to space $L_2(0,\pi)$. However the solution (28) at any $t > -1$ has the form of a sinusoid with an arbitrarily large amplitude and does not belong to space $L_2((0,\pi)\times(-1,1))$.

In order to the function $\varphi_2(x)$ satisfied to condition (27), it is necessary and sufficient, that the Fourier-coefficients φ_{2k} had the asymptote behavior for large k of order $\exp\{-(2+\varepsilon)k\}$ where $\varepsilon > 0$. In example of Hadamard considered by us we have asymptote which is only equal to $\exp\{-\sqrt{k}\}$, and it is obviously not enough for correctness of Cauchy-Dirichlet problem for Poisson equation.

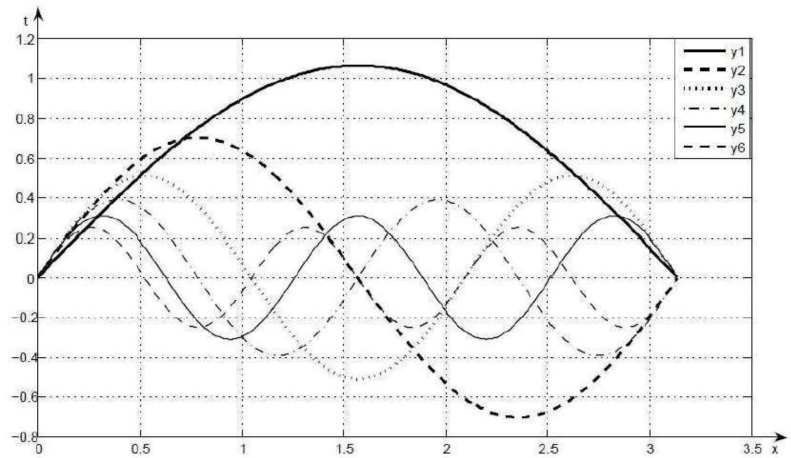

FIGURE 1. Graph of solution $y_k(x,t)$ at $k = \overline{1,6}$ of (28).

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