Partial discretization method for stability analysis of dynamic systems

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Abstract – This paper investigates stability of motion of elastic systems with nonlinear characteristics. Model of steady motion of elastic systems in the absence of oscillatory processes is considered. The model is based on Lyapunov stability criterion. Analysis of the perturbation equation is carried out by use of the partial discretization method. Partial discretization of the Hill equation in a class of generalized functions (Dirac's delta function) is employed to considerably simplify the Hill parametrical equation and obtain its quasi-analytical solutions. Efficiency of the offered method is shown on the example of stability of resonant oscillations of physically nonlinear systems.

Keywords – Nonlinear system, partial discretization method, resonance, stability.

I. INTRODUCTION

This paper considers the issues of stability of motion of nonlinear mechanical systems and methods of their analysis which is of practical interest.

Stability of motion is one of the main problems of modern machinery dynamics. Under the influence of large inertial forces and technological loadings in links of machines as a result of their elastic deformation, difficult oscillatory processes connected with modulation of frequencies and emergence of resonant phenomena occur. These undesirable processes significantly affect strength characteristics of separate elements, and also functionality of machines. Misalignment of links and their deviation from the set trajectories can be observed. Therefore, as well as in the previous works [1]-[2], steady motion of mechanical systems is considered as their movement in the absence of oscillatory processes.

Research of stability of motion of mechanisms and machines depends on a choice of their dynamic model. Widely used model of motion of machinery elements as absolutely rigid, considerably narrows a framework of its application. As a rule, it is research of quasi-static and to resonant modes of motion. However, this model is only the first approximation for the majority of problems.

Nonlinear dynamic models of machines taking into account deformability of links are of the greatest interest. In [3]-[6] models of machine motion were developed assuming all links to be elastic. Their geometrical and physical nonlinearity was considered. Connection between elastic displacements of links was considered through reactions in hinges of adjacent links. Nonlinearity of models can cause resonances on sub- and ultra-frequencies. Therefore, ensuring steady motion of system depends on identification and elimination of frequencies causing resonant vibrations from operating modes.

Most research on stability of periodic oscillations was performed by use of asymptotic methods and methods of small parameter. They are quasi-linear and quasi-Lyapunov systems [7]-[9], etc. By means of Lyapunov's function at rather rigid restrictions on degree of nonlinearity, conditions of asymptotic global stability were obtained.

Among works on research of parametrical instability of nonlinear mechanical systems works of S. Hayashi [10], A.Tondl [11], W. Szemplinska-Stupnicka [12], etc. are well-known. In [13] questions of stability of periodic oscillations of a nonlinear system without restrictions on the size of its nonlinearity and nonautonomous terms were studied.

The objective of this paper is performing stability analysis of nonlinear dynamic deformable systems for elimination of dangerous oscillations from operating modes.

II. DYNAMIC MODELS

One method for solving problems of dynamics of elastic systems is reducing the dimension of equations of motion by applying well-known methods of separation of variables and research of dynamic processes in nonlinear mechanical systems with one degree of freedom in the form:

$$\ddot{f} + \Phi(\dot{f}, f) + \omega_0^2 f = F(\Omega t). \tag{1}$$

Degree of nonlinearity of the term $\Phi(\dot{f}, f)$ relative to generalized function of displacements $f(t)$ corresponds to assumptions of the model, and characterizes nonlinearity of elastic characteristics (geometric and physical nonlinearity) and dissipative forces.

Considering stability of the periodic solution $f_0(t)$ of (1), we set a small deviation δf from its equilibrium state:

$$f(t) = f_0(t) + \delta f \ . \tag{2}$$

Stability of the periodic solution $f_0(t)$ depends on the nature of the behavior of its small deviation δf in time, i.e. solution of the equation of the perturbed state of the system:

$$\delta \ddot{f} + \left(\frac{\partial \Phi}{\partial \dot{f}}\right)_0 \delta \dot{f} + \left(\frac{\partial \Phi}{\partial f}\right)_0 \delta f = 0\,, \tag{3}$$

where the symbol $(\ )_0$ means that the solution $f_0(t)$ is taken as argument of functions.

If the solution δf of (3) is limited at $t \to \infty$, then motion of the system is considered to be stable. If $\delta f \to \infty$ at $t \to \infty$, by definition, the motion is unstable that is identical to the criterion of Lyapunov stability.

Legitimacy of transition from (1) to (3) is given in work [10] which refers to Trefftts's research concerning properties of periodic solutions of equations in the form (1). Limitation of a solution of (1) and its asymptotic stability results in its frequency with the smallest period equaled or multiplied to the period of the external perturbing force. In [10] the Floquet theory was involved to study stability of the periodic solution of (1).

Introduce a new variable η:

$$\delta f = \eta \exp\left(-0,5 \left(\frac{\partial \Phi}{\partial \dot{f}} \right)_0 \right). \tag{4}$$

Then (3) reduces to the Hill parametrical equation relative to the variable η.

For the case of basic resonance $f(t) = r_0 + r_1 \cos(\Omega t - \varphi_1)$ the Hill equation is represented as:

$$\frac{d^2\eta}{dt^2} + \eta\left[\theta_0 + \theta_{1s}\sin\Omega t + \theta_{1c}\cos\Omega t + \theta_{2s}\sin 2\Omega t + \theta_{2c}\cos 2\Omega t\right] = 0, \tag{5}$$

where θ_0, θ_{1s}, θ_{1c}, θ_{2s}, θ_{2c} are functions of frequencies, amplitudes, and phases of oscillations of harmonic solutions of (1), Ω, r_1, φ_1 respectively.

Among methods of the dynamic analysis of vibrations of mechanical systems the methods based on creating the characteristic determinants specifying borders of instability zones of the resonant modes are widely known. For this, either the Floquet theory is used, as in work [10], or borders of instability zones are defined directly on amplitude-frequency characteristics, i.e. on resonant curves by means of the Routh-Hurvitz criterion.

Here, in contrast to the mentioned methods, the problem of stability of motion of system (1) based on applying the partial discretization method [14] to the solution of Hill's equation (5) is investigated. This method allows to obtain the analytical solution of the Hill equation characterizing the behavior of small perturbation δf in time *t*.

III. PARTIAL DISCRETIZATION OF THE HILL EQUATION

According to the method of partial discretization [14], the second term of (5) is represented discretely in a class of the generalized functions:

$$\begin{aligned}\frac{d^2\eta}{dt^2} &+ \frac{1}{2}\sum_{k=1}^{n}(t_k + t_{k+1})[(\theta_0 + \theta_{1s}\sin\Omega t_k + \theta_{1c}\cos\Omega t_k \\ &+ \theta_{2s}\sin 2\Omega t_k + \theta_{2c}\cos 2\Omega t_k)\cdot\eta(t_k)\delta(t - t_k) \\ &- (\theta_0 + \theta_{1s}\sin\Omega t_{k+1} + \theta_{1c}\cos\Omega t_{k+1} + \theta_{2s}\sin 2\Omega t_{k+1} \\ &+ \theta_{2c}\cos 2\Omega t_{k+1})\cdot\eta(t_{k+1})\delta(t - t_{k+1})] = 0,\end{aligned} \tag{6}$$

where

$\eta(t_k)$ is discrete representation of function $\eta(t)$ for the value of the argument $t = t_k$;

$k = \overline{1,n}$ the number of splitting of the argument t;

$\delta(t - t_k)$ Dirac's delta function.

Taking the following initial conditions: $\eta(0) = \eta_0$, $\dot{\eta}(0) = \dot{\eta}_0$ at $t = 0$, the solution of (6) is expressed as:

$$\begin{aligned}\eta(t) = &-\frac{1}{2}\sum_{k=1}^{n}(t_k + t_{k+1})[(\theta_0 + \theta_{1s}\sin\Omega t_k + \theta_{1c}\cos\Omega t_k \\ &+ \theta_{2s}\sin 2\Omega t_k + \theta_{2c}\cos 2\Omega t_k)\cdot\eta(t_k)H(t - t_k) \\ &- (\theta_0 + \theta_{1s}\sin\Omega t_{k+1} + \theta_{1c}\cos\Omega t_{k+1} + \theta_{2s}\sin 2\Omega t_{k+1} \\ &+ \theta_{2c}\cos 2\Omega t_{k+1})\cdot\eta(t_{k+1})H(t - t_{k+1})] + \dot{\eta}_0 t + \eta_0,\end{aligned} \tag{7}$$

where $H(t - t_k)$ denotes Heaviside step function.

Specifying t discretely, we obtain a recurrent formula for calculation of unknown $\eta(t)$ on *k*-th step of splitting of the argument *t* :

$$\begin{aligned}\eta(t_k) = &\Big[-(t_1 + t_2)(\theta_0 + \theta_{1s}\sin\Omega t_1 + \theta_{1c}\cos\Omega t_1 + \theta_{2s}\sin 2\Omega t_1 \\ &+ \theta_{2c}\cos 2\Omega t_1)\eta(t_1)\left(\frac{t_k + t_{k+1}}{2} - t_1\right)\Big] \Big/ \Big[1 + \frac{1}{2}(t_{k+1} - t_k) \\ &\cdot(t_{k+1} - t_{k-1})(\theta_0 + \theta_{1s}\sin\Omega t_k + \theta_{1c}\cos\Omega t_k + \theta_{2s}\sin 2\Omega t_k \\ &+\theta_{2c}\cos 2\Omega t_k)\Big] - \Big[\sum_{j=2}^{k-1}(t_{j+1} - t_{j-1})(\theta_0 + \theta_{1s}\sin\Omega t_j \\ &+ \theta_{1c}\cos\Omega t_j + \theta_{2s}\sin 2\Omega t_j + \theta_{2c}\cos 2\Omega t_j)\eta(t_j) \\ &\cdot\left(\frac{t_k + t_{k+1}}{2} - t_j\right)\Big] \Big/ \Big[1 + \frac{1}{2}(t_{k+1} - t_k)(t_{k+1} - t_{k-1}) \\ &\cdot(\theta_0 + \theta_{1s}\sin\Omega t_k + \theta_{1c}\cos\Omega t_k + \theta_{2c}\sin 2\Omega t_k \\ &+ \theta_{2c}\cos 2\Omega t_k)\Big] + \Big[\dot{\eta}_0\frac{t_k + t_{k+1}}{2} + \eta_0\Big] \Big/ \Big[1 + \frac{1}{2}(t_{k+1} - t_k) \\ &\cdot(t_{k+1} - t_{k-1})(\theta_0 + \theta_{1s}\sin\Omega t_k + \theta_{1c}\cos\Omega t_k \\ &+ \theta_{2s}\sin 2\Omega t_k + \theta_{2c}\cos 2\Omega t_k)\Big].\end{aligned} \tag{8}$$

In contrast to [14]-[15], where the method of partial discretization is applied to study of parametrical system oscillations, in this work it is used directly to a solution of the perturbation equation in terms of $\delta(t)$. It is possible to investigate stability of the state by analyzing the nature of behavior of $\delta(t)$, according to Lyapunov stability criterion. If the magnitude decreases with time t (decaying process)

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then $\delta f \to 0$, i.e. the state is stable. If the oscillatory process is growing then we have an unstable state.

Efficiency of the offered method will be shown below on the example of stability analysis of resonant oscillations of physically nonlinear systems.

IV. ANALYTICAL SOLUTION OF THE HILL EQUATION IN THE CASE OF PHYSICALLY NONLINEAR SYSTEMS

As an example, consider the motion of physically nonlinear systems. Equations of motion for these systems are taken in the form:

$$\frac{d^2 f}{dt^2} + k_1 \frac{df}{dt} + k_2 \left(\frac{df}{dt}\right)^2 + \alpha_1 f + \alpha_2 f^2 = F_0 + F_1 \cos \Omega t. \tag{9}$$

In (9) dissipative forces which are supposed to be nonlinear and viscous due to damping properties of physically nonlinear media (rubber and similar materials used as oscillation dampers) are taken into account.

Physical nonlinearity of the system is characterized by an arbitrary angle of rotation of cross elements that corresponds to quadratic nonlinearity of the restoring force.

Stability of a basic resonance is investigated. Solution of (9) is given by:

$$f(t) = r_0 + r_1 \cos(\Omega t - \varphi_1). \tag{10}$$

The Hill equation in this case is represented as [6]:

$$\begin{aligned} &\frac{d^2\eta}{dt^2} + \eta\left[\theta_0 + \theta_{1s} \sin \Omega t + \theta_{1c} \cos \Omega t + \theta_{2s} \sin 2\Omega t\right. \\ &\qquad \left. + \theta_{2c} \cos 2\Omega t\right] = 0, \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\theta_0 = \alpha_1 + 2\,\alpha_2 r_0 - 0.25\,k_1^2 - 0.5\,k_2^2 r_1^2 \Omega^2, \\ &\theta_{1s} = (2\,\alpha_2 r_1 + k_2 r_1 \Omega^2) \sin \varphi_1 + k_1 k_2 r_1 \Omega \cos \varphi_1, \\ &\theta_{1c} = (2\,\alpha_2 r_1 + k_2 r_1 \Omega^2) \cos \varphi_1 - k_1 k_2 r_1 \Omega \sin \varphi_1, \\ &\theta_{2s} = 0.5\,k_2^2 r_1^2 \Omega^2 \sin 2\varphi_1, \\ &\theta_{2c} = 0.5\,k_2^2 r_1^2 \Omega^2 \cos 2\varphi_1. \end{aligned} \tag{12}$$

According to the above-specified technique, under the given initial conditions, and by the method of partial discretization the analytical solution of (11)-(12) has been obtained:

$$\begin{aligned} \eta(t_k) = &\Big[-(t_1 + t_2)(\theta_0 + \theta_{1s} \sin \Omega t_1 + \theta_{1c} \cos \Omega t_1 + \theta_{2s} \sin 2\Omega t_1 \\ &+ \theta_{2c} \cos 2\Omega t_1)\eta(t_1)\left(\frac{t_k + t_{k+1}}{2} - t_1\right)\Big] \Big/ \Big[1 + \frac{1}{2}(t_{k+1} - t_k) \\ &\cdot (t_{k+1} - t_{k-1})(\theta_0 + \theta_{1s} \sin \Omega t_k + \theta_{1c} \cos \Omega t_k + \theta_{2s} \sin 2\Omega t_k \\ &+ \theta_{2c} \cos 2\Omega t_k)\Big] - \Big[\sum_{j=2}^{k-1} (t_{j+1} - t_{j-1})(\theta_0 + \theta_{1s} \sin \Omega t_j \\ &+ \theta_{1c} \cos \Omega t_j + \theta_{2s} \sin 2\Omega t_j + \theta_{2c} \cos 2\Omega t_j)\eta(t_j) \\ &\cdot \left(\frac{t_k + t_{k+1}}{2} - t_j\right)\Big] \Big/ \Big[1 + \frac{1}{2}(t_{k+1} - t_k)(t_{k+1} - t_{k-1}) \\ &\cdot (\theta_0 + \theta_{1s} \sin \Omega t_k + \theta_{1c} \cos \Omega t_k + \theta_{2c} \sin 2\Omega t_k \\ &+ \theta_{2c} \cos 2\Omega t_k)\Big] + \Big[\dot{\eta}_0 \frac{t_k + t_{k+1}}{2} + \eta_0\Big] \Big/ \Big[1 + \frac{1}{2}(t_{k+1} - t_k) \\ &\cdot (t_{k+1} - t_{k-1})(\theta_0 + \theta_{1s} \sin \Omega t_k + \theta_{1c} \cos \Omega t_k \\ &+ \theta_{2s} \sin 2\Omega t_k + \theta_{2c} \cos 2\Omega t_k)\Big]. \end{aligned} \tag{13}$$

V. NUMERICAL RESULTS

Solution (13) is a recurrent formula for discrete representation of the solution $\eta(t)$ with time on k-th step of splitting of the argument t. By analyzing the nature of the behavior of $\eta(t)$, we can judge the stability of the studied state.

In this paper numerical analysis of the behavior of $\eta(t)$ giving a representation of the behavior of a small variation δf with time is realized.

Calculations were done for the parameters of the system $k_1 = 0.2;\ k_2 = 0.1;\ \alpha_1 = 5;\ \alpha_2 = 0.5;\ F_0 = 5;\ F_1 = 50.$ Step of discretization was accepted as $\Delta t = 0.05$.

Stability of the solution (13) was studied by putting on amplitude-frequency characteristics of a basic resonance (Fig.1, curve 2) three frequency areas in to-resonant, resonant and post-resonant modes of oscillations.

It is established that both to-resonant and post-resonant modes of oscillations are decaying (Fig. 2, Fig.3) that does not contradict the physical sense of the phenomena investigated. In a zone of resonant frequencies growth of oscillation amplitude is obtained that means the process is instable (Fig.4).

Here, as well as in [16] where research on stability analysis of motion of geometrically nonlinear systems was conducted by method of partial discretization, research results correspond well to graphs of amplitude-frequency characteristics of a system basic resonance (Fig.1).

Thus, application of the partial discretization method to studying stability of oscillations allows to obtain the analytical solution and determine zones of stable and unstable system oscillations. Selection of corresponding geometrical and physical parameters of the elastic system by means of their variation will help to avoid undesirable resonant phenomena in operating system modes.

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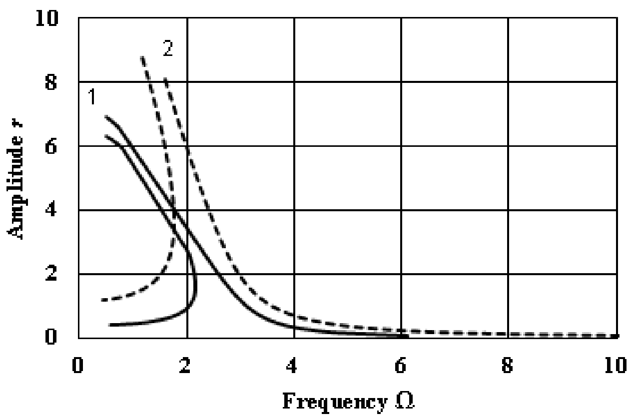

Fig. 1 amplitude-frequency characteristics of a basic resonance at $\alpha_2 = 1;\ F_1 = 10$ (curve 1), $\alpha_2 = 0.5;\ F_1 = 50$ (curve 2)

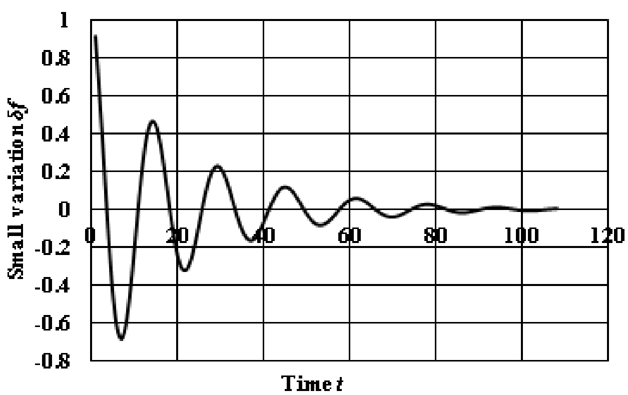

Fig. 2 behavior of the physically nonlinear system in the to-resonant zone of oscillations at $\Omega = 0.5, r = 1.5$

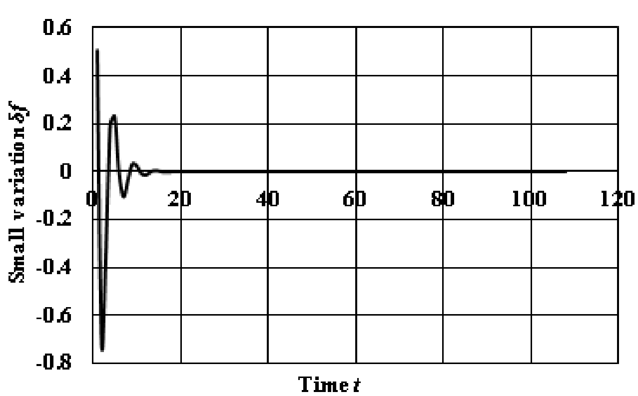

Fig. 3 behavior of the physically nonlinear system in the post-resonant zone of oscillations at $\Omega = 7.26, r = 0.15$

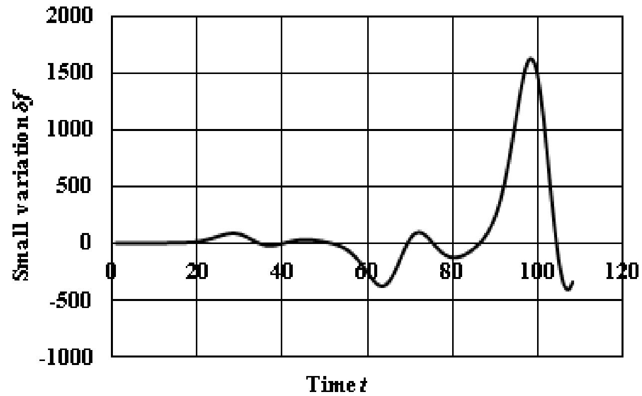

Fig. 4 behavior of the physically nonlinear system in the resonant zone of oscillations at $\Omega = 1.8, r = 8$

CONCLUSION

In this work, according to the offered criterion of dynamic stability of elastic systems, stability of motion of nonlinear mechanical systems and methods of their analysis have been considered.

Steady motion of nonlinear systems is considered as their movement in the absence of oscillatory processes. These requirements are identical to determining of Lyapunov stability. Therefore, the technique of stability analysis of nonlinear systems is based on the analysis of solutions for perturbation equations. As a method for solution to the problem, the partial discretization method is offered. The essence of this method consists in discretization of variable coefficients of the Hill equation in a class of the generalized functions. The solution of the Hill equation is considerably simplified by identifying its variable coefficients as constants on each step of discretization. The obtained analytical solution of the perturbation equation is a recurrent formula for calculation of oscillation amplitudes. It allows to predict parametrical instability of resonant modes of motion of nonlinear systems. Efficiency of the used method is shown on the example of physically nonlinear systems. Research results correspond well to the known results obtained by other methods.

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