A method of damping waveguide eigenmodes in solving axisymmetric diffraction problems

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***Abstract*—The problem of TE wave diffraction by a semi-infinite pin is solved using this method, which combines the Wiener-Hopf technique with the known solution to the diffraction problem for a semi-infinite thin cylinder.**

***Index Terms*—Factorization, Wiener-Hopf method, diffraction, circular waveguide, pin**

I. Introduction

ANALYTICAL solutions to axisymmetric problems of diffraction on volumetric structures inside a circular waveguide still attract attention of specialists. For example, wave diffraction on a semi-infinite pin is considered in [1], where the solution to the linear initial problem is reduced to the solution of a system of nonlinear algebraic equations, which is extremely cumbersome for numerical calculations. A similar problem was also considered in [2], [3] using the interpolation method of factorization, related to the Wiener-Hopf (W-H) method, where a numerical-analytical method of generalized stitching was proposed for solving systems of linear algebraic equations, the awkwardness and complexity of which significantly increases with an increase in the accuracy of the obtained solution.

This paper considers the boundary value problem of wave diffraction on a semi-infinite ideally conducting pin, coaxially located inside an infinite waveguide.

The main idea in solving the boundary value problem is placement of an analytical source at the end of the pin to suppress all eigenmodes inside a semi-infinite cylinder, previously obtained by the W-H method in solving the problem of diffraction on a semi-infinite ideally conducting thin cylinder.

Therefore, the W-H method enables us to use the exact solution of the problem of diffraction on a semi-infinite thin cylinder, where the expressions for the fields of scattered waves for each region have the form:

Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP19676900)

a) $0 \le r \le a,\ z < 0$

$$E_0^a = \frac{2\pi i}{a} K_0 \sum_{n=1}^{\infty} \frac{e^{-iw_n^a z} v_n^a \mathrm{J}_1(v_n^a r)(a_1', a')_{v_n^a}}{w_n^a (w_n^a + h) L_-(a_1, -w_n^a) \mathrm{J}_0(v_n^a a)}; \quad (1)$$

b) $0 \le r \le a_1,\ z > 0$

$$E_0^b = -\frac{2\pi i}{a_1} K_0 \sum_{m=1}^{\infty} e^{iw_m^b z} \frac{v_m^b L_+(a_1, w_m^b) \mathrm{J}_1(v_m^b r)}{w_m^b (w_m^b - h) \mathrm{J}_0(v_m^b a_1)}; \quad (2)$$

c) $a_1 \le r \le a,\ z > 0$

$$E_0^c = -2\pi i K_0 \sum_{n=1}^{\infty} e^{iw_m^c z} \frac{v_m^c L_+(a_1, w_m^c)(r', a')_{v_m^c}}{w_m^c (w_m^c - h)(a_1', a')'_{v_m^c}}, \quad (3)$$

where $v = \sqrt{k^2 - w^2}$, $(r', a') = \mathrm{J}_1(vr)\mathrm{N}_1(va) - \mathrm{J}_1(va)\mathrm{N}_1(vr)$, $(a_1', a')_v' = a_1(a_1, a') + a(a_1', a)$ is a derivative of the function with respect to v, L_- is a factorized function, analytical in the lower half-plane of the complex variable w, L_+ in the upper half-plane ($L = L_- L_+$), $L = (a_1', a')/\mathrm{J}_1(va)$, v_n^ℓ and w_n^ℓ are the roots of the Bessel functions [4] in regions $\ell = a, b, c$, $(n = 1, 2, \dots)$, $w_n^\ell = \sqrt{k^2 - v_n^{\ell 2}}$, $v_n^{a,b}$ are the roots of the Bessel functions $\mathrm{J}_1(va)$ and $\mathrm{J}_1(va_1)$, respectively, v_n^c are the roots of the combination of the Bessel function (a_1', a'), $\mathrm{N}_1(va)$ is the first order Neumann function.

Further, we present the formulation of the problem and a solution to the boundary value problem of diffraction on a semi-infinite pin, using the method of compensation of eigenmodes.

II. Statement of the boundary value problem

Let a TE wave with a unit amplitude and longitudinal wave number h be incident from the left on the end face of a perfectly conducting semi-infinite circular pin of the radius a_1, coaxially located inside an infinite circular waveguide of the radius a

$$E_\phi^i(r, z) = \mathrm{J}_1(Vr) e^{ihz}, \quad (4)$$

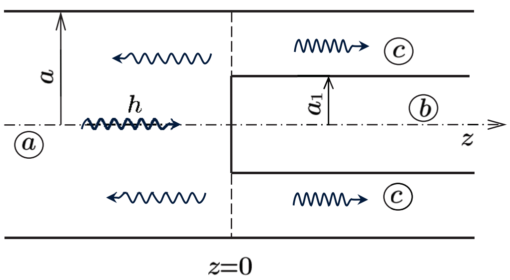

Fig. 1. Semi-infinite pin in an infinite circular waveguide.

where $V = \sqrt{k^2 - h^2}$ is one of the roots of the first-order Bessel function $\mathrm{J}_1(Va)$, $z = 0$ is the location of the end of the pin, which coincides with the origin of the coordinate system.

The boundary value problem satisfies the electrodynamic boundary conditions, i.e. there is no tangential component of the electric field on the conducting surface

$$\begin{gathered} E_\phi(r,z) + E^i_\phi(r,z) = 0, \\ 0 \le r \le a_1,\ 0 \le z;\ r = a,\ -\infty < z < \infty, \end{gathered} \tag{5}$$

including the end of the pin ($z = 0$). Here $E_\phi(r,z)$ is the electric field of the waves scattered by the semi-infinite pin.

For convenience, we divide the space inside the waveguide into three regions: $(a) : \{0 \le r \le a,\ z \le 0\}$, $(b) : \{0 \le r \le a_1,\ 0 \le z\}$ $(c) : \{a_1 \le r \le a,\ 0 \le z\}$, where natural waveguide modes with transverse and longitudinal wave numbers v^ℓ_n and w^ℓ_n, $\ell = a, b, c,$ $(n = 1, 2, \dots)$ can propagate.

III. Solution to the problem

As an initial approximation, we take the exact solution to the well-known boundary value problem of wave diffraction on a semi-infinite thin conducting circular cylinder. As the longitudinal boundary conditions of the problem are satisfied automatically, due to the eigenfunctions, it turns out to be sufficient to satisfy the boundary condition on the end surface $z = 0$.

To achieve this goal, we place an analytical source at the point $z = 0$, which should damp all the eigenmodes of the region (b) and re-radiate them in the form of eigenmodes into other regions. As the complete damping of the wave fields occurs both inside the region (b) and on the end surface, we can assume that all the boundary conditions of the boundary value problem are fulfilled.

It should also be kept in mind that the radiation field of the analytical source must be continuous at the point $z = 0$, i.e. to the left and right of $z = 0$, although the field in each region is formed only by means of eigenmodes.

Thus, each mode with the wave number w^b_n of the region (b) must be transformed into eigenmodes of region (a). For this purpose, we use the expansion of the Bessel function in a Fourier-Bessel series [5] in eigenfunctions of the region (a)

$$\mathrm{J}_1(v^b_m r) = \tfrac{2}{a}\mathrm{J}_1(v^b_m a) \sum_{n=1}^{\infty} \frac{v^a_n}{(v^{b\,2}_m - v^{a\,2}_n)} \frac{\mathrm{J}_1(v^a_n r)}{\mathrm{J}_0(v^a_n a)}\ (0 \le r \le a_1). \tag{6}$$

Substituting the function $\mathrm{J}_1(v^b_m r)$ with the opposite sign into the expression (2) for the modes of the region (b), we find the field of the analytical source, which ensures the fulfillment of the boundary condition at the end of the pin, and on the other hand, transforms the mode of the region (b) into the region (a) in the form of a series in eigenfunctions and values

$$\begin{gathered} E^a_s(r,z) = -\frac{4\pi i}{a_1 a} K_0 \sum_{n=1}^{\infty} v^a_n \frac{\mathrm{J}_1(v^a_n r)}{\mathrm{J}_0(v^a_n a)} e^{-iw^a_n z} \\ \sum_{m=1}^{\infty} e^{iw^b_m z} \frac{v^b_m}{w^b_m} \frac{L_+(a_1, w^b_m)}{(v^{b\,2}_m - v^{a\,2}_n)} \frac{\mathrm{J}_1(v^b_m a)}{(w^b_m - h)\mathrm{J}_0(v^b_m a_1)}. \end{gathered} \tag{7}$$

The resultant field of all spatial harmonics of the region (a) is defined as the sum of the fields

$$E^a(r,z) = E^a_0(r,z) + E^a_s(r,z)$$

or finally

$a)$ $0 \le r \le a,\ z < 0$

$$\begin{gathered} E^a(r,z) = \frac{2\pi i}{a} K_0 \sum_{n=1}^{\infty} v^a_n \frac{\mathrm{J}_1(v^a_n r)}{\mathrm{J}_0(v^a_n a)} e^{-iw^a_n z} \\ \Bigg(\frac{(a'_1, a')_{v^a_n}}{w^a_n (w^a_n + h) L_-(a_1, -w^a_n)} - \frac{2}{a_1} \sum_{m=1}^{\infty} e^{iw^b_m z} \frac{v^b_m}{w^b_m} \\ \frac{L_+(a_1, w^b_m)}{(v^{b\,2}_m - v^{a\,2}_n)} \frac{\mathrm{J}_1(v^b_m a)}{(w^b_m - h)\mathrm{J}_0(v^b_m a_1)} \Bigg). \end{gathered} \tag{8}$$

Using the expansion in a series of the combination of Bessel functions (r', a') in terms of eigenfunctions in the region (c)

$$\begin{gathered} (r', a')_{v^a_n} = (a'_1, a')_{v^a_n} \sum_{i=1}^{\infty} \frac{2 v^c_i\, (r', a')_{v^c_i}}{(v^{a\,2}_n - v^{c\,2}_i)(a'_1, a')'_{v^c_i}} \\ (a_1 < r \le a), \end{gathered} \tag{9}$$

and also observing the continuity of the electric field $E^a_s = E^c_s$ at $z = 0$, we find the field of the analytical source in the coaxial region

$c)$ $a_1 \le r \le a,\ 0 < z$

$$\begin{gathered} E^c_s(r,z) = \frac{4\pi^2 i}{a_1} K_0 \sum_{i=1}^{\infty} e^{iw^c_i z} \frac{v^c_i\, (r', a')_{v^c_i}}{(a'_1, a')'_{v^c_i}} \\ \sum_{n=1}^{\infty} \frac{v^{a\,2}_n}{(v^{a\,2}_n - v^{c\,2}_i)} (a'_1, a')_{v^a_n} e^{-iw^a_n z} \\ \sum_{m=1}^{\infty} e^{iw^b_m z} \frac{v^b_m}{w^b_m} \frac{L_+(a_1, w^b_m)}{(v^{b\,2}_m - v^{a\,2}_n)} \frac{\mathrm{J}_1(v^b_m a)}{(w^b_m - h)\mathrm{J}_0(v^b_m a_1)}. \end{gathered} \tag{10}$$

It should be noted that in deriving the above formula, a useful identity for Bessel functions was used [6]

$$\mathrm{J}_1(v^a_n r) \equiv -\frac{\pi}{2} a v^a_n \mathrm{J}_0(v^a_n a)(r', a'). \tag{11}$$

Finally, the resulting diffraction field in the coaxial region (c) is writes as

$$E^c(r,z) = E^c_0(r,z) + E^c_s(r,z)$$

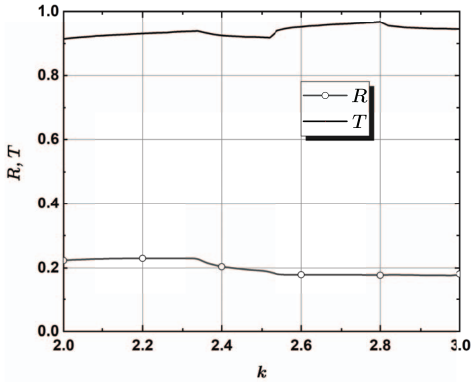

Fig. 2. Reflection and transmission coefficients. The solid black line is the reflection coefficient R, the marked line is the transmission coefficient T.

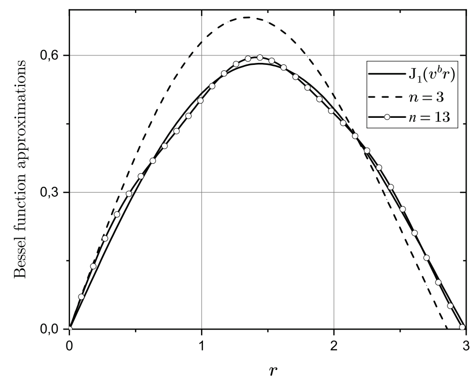

Fig. 3. The graph of the Bessel function and its expansion in a Fourier-Bessel series. The solid black line is the exact graph of the first-order Bessel function, the dotted line is its approximation for $n = 13$ in series, the marked line is for $n = 3$

or

$$E^c(r,z) = 2\pi i K_0 \sum_{n=1}^{\infty} e^{iw_m^c z} \frac{v_i^c\,(r',a')_{v_i^c}}{(a_1',a')'_{v_i^c}}$$

$$\left(\frac{L_+(a_1,w_i^c)}{w_i^c(h-w_i^c)} + \frac{2\pi}{a_1}\sum_{n=1}^{\infty} e^{-iw_n^a z}\frac{v_n^{a\,2}}{(v_n^{a\,2}-v_i^{c\,2})}(a_1',a')_{v_n^a} \right. \quad (12)$$

$$\left. \sum_{m=1}^{\infty} e^{iw_m^b z}\frac{v_m^b}{w_m^b}\frac{L_+(a_1,w_m^b)}{(v_m^{b\,2}-v_n^{a\,2})}\frac{\mathrm{J}_1(v_m^b a)}{(w_m^b-h)\mathrm{J}_0(v_m^b a_1)}\right).$$

It is obvious that the obtained expression corresponds to the continuity of the field on the common boundary between (a) and (c) $(z = 0,\ a_1 \le r \le a)$, as well as the boundary condition at the end.

Fig. 2 shows the reflection coefficients R and T of the incident wave passing through a semi-infinite pin.

Figs. 3 and 4 show the graphs of the Bessel functions $\mathrm{J}_1(vr)$ and (r',a') for fixed values of the transverse wave number $v = v_1^b$ and $v = v_1^a$, respectively, as well as their asymptotic expansions in Fourier-Bessel series with 3 and 13 initial terms. as well as their asymptotic expansions in Fourier-Bessel series with 3 and 13 initial terms.

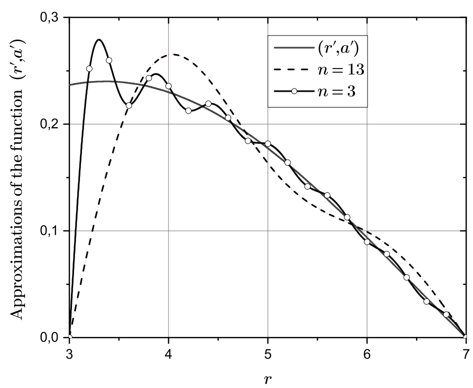

Fig. 4. The graph of the combination of Bessel functions $(r',a')_{v_n^a}$ and its Fourier-Bessel series expansion. The solid black line represents the graph of (r',a'), the marked line represents its approximation with $n = 13$ terms of the series, and the dashed line represents the approximation with $n = 13$ terms.

IV. Conclusions

A method for compensating the eigenmodes of a circular waveguide is demonstrated on the example of solving the problem of TE-wave diffraction on a perfectly conducting semi-infinite pin, coaxially located inside an infinite waveguide.

A complete damping of the eigenmodes inside a semi-infinite circular waveguide is provided by placement of an analytical source at the point of discontinuity of the boundary conditions, which transforms the waveguide mode into other regions by expanding the Bessel function into a Fourier-Bessel series.

The corresponding infinite Fourier-Bessel series are obtained by integrating the corresponding Cauchy integrals over an infinite circle. The graphs show the asymptotics of expansion of Bessel functions into a Fourier-Bessel series depending on the number of its initial terms $(n = 3, 13)$.

This technique can be used in combination with the stitching method when solving some boundary value problems.

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