Sommerfeld integrals asymptotics in dipole radiation problems

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***Abstract*—The paper proposes a method for calculating Sommerfeld integrals using the example of the problem of radiation from a Hertz dipole. An integral representation of the dipole field in the form of Sommerfeld integrals is given and its analytical expression is found in the form of an infinite power series. Qualitative and quantitative comparisons of the final results with exact expressions for the Hertz radiator obtained directly from integral representations in cylindrical and spherical coordinate systems are presented. The results of the paper may be used in the theory of diffraction and in solving the Sommerfeld problem**

***Index Terms*—Maxwell equations, convolution, Green's function, scattering of electromagnetic waves, Hertz dipole, Sommerfeld integrals**

I. Introduction

Sommerfeld integrals, introduced by A. Sommerfeld in 1909 [1], are used in solving problems related to wireless radio communication over long distances since they provide an accurate mathematical description of electromagnetic phenomena [2]–[5]. Recently, they have become widely used in mathematical models related to many electromagnetic technologies, ranging from modeling electrical discharges to plasmonic integrated devices [6]–[8]. However, numerical methods have difficulties in accurate evaluation of Sommerfeld integrals. This is caused by highly oscillatory and slowly converging behavior of the integrand and its singularities, including branch points and a pole near the real-axis path of integration. It is generally assumed that such Sommerfeld integrals cannot be calculated in a closed form.

Therefore, the calculation of Sommerfeld integrals has both practical and theoretical significance since they can be used as Green's functions in the frame of integral equation formulations. In recent years, reports on combining analytical approximations and numerical computations have been published [9]–[11].

In the first section, the integral representations of the magnetic and electric fields of the Hertz point radiator are considered using Hertz vector potentials.

In the second section, the exact analytical expressions for the Hertz dipole fields in cylindrical and spherical coordinate systems required in the third chapter are described.

Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP19676900).

In the third section, on the example of the Hertz dipole, the method for evaluating Hertz integrals is considered. The main results of the work are presented in the conclusions.

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The third section describes the exact analytical expressions for the Hertz dipole fields in cylindrical and spherical coordinate systems required in the third chapter.

In the fourth section, on the example of the Hertz dipole, the method for evaluating Hertz integrals is described.

II. Integral representations of the fields of a Hertzian point dipole

It is known that the potential of the Hertz vector

$$\boldsymbol{\Pi} = \frac{1}{i\omega\varepsilon_0\varepsilon} J * \psi, \tag{1}$$

is a solution of the Helmholtz equation

$$(k_0^2 + \triangle)\boldsymbol{\Pi} = \frac{1}{i\omega\varepsilon_0\varepsilon} J, \tag{2}$$

where the symbol $*$ is the convolution over all spatial coordinates, J is the current density, $\psi = e^{ik_0 r}/r$ is Green's function, ω is the circular frequency, ε is the dielectric permittivity.

The electromagnetic field is generally defined in terms of the Hertz vector as

$$H = -i\omega\varepsilon_0\varepsilon\nabla\times\boldsymbol{\Pi}, \tag{3}$$

$$E = \nabla\times\nabla\times\boldsymbol{\Pi} + \frac{1}{i\omega\varepsilon_0\varepsilon} J. \tag{4}$$

Note that the Hertz point dipole corresponds to the current density ($J = j$)

$$j = -i\omega p\delta(x)\delta(y)\delta(z - z_0), \tag{5}$$

where z_0 is its location along the z axis.

Integral representation of the Hertz vector of a point electric dipole

$$\boldsymbol{\Pi}^0 = \frac{ip}{4\pi\varepsilon_0\varepsilon}\int_0^\infty \frac{1}{\varkappa} e^{i\varkappa|z-z_0|} \mathrm{J}_0(k_\rho\rho) k_\rho dk_\rho, \tag{6}$$

$$\varkappa = \sqrt{k_0^2 - k_\rho^2}$$

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or the so-called inverse Hankel transform follows from the Fourier inversion of the expression (1) in the cylindrical coordinate system

$$\boldsymbol{\Pi}^0 = \frac{1}{i\omega\varepsilon_0\varepsilon}\mathbf{F}^{-1}\big[\tilde{j}\,\tilde{\psi}\,\big] = -\frac{p}{(2\pi)^3\varepsilon_0\varepsilon}\int\limits_0^\infty\int\limits_{-\infty}^\infty\int\limits_0^{2\pi} e^{i(k_\rho\rho\cos k_\alpha+k_z(z-z_0))}\frac{k_\rho}{k_0^2-k^2}dk_\rho dk_z dk_\alpha.$$

The above integrals are calculated by means of representation of the Bessel function as

$$\mathrm{J}_0(z) = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{iz\cos k_\alpha}dk_\alpha \tag{7}$$

and application of the theory of residues on the complex plane k_z.

The integral representations of the fields of the Hertz radiator are expressed in a similar way using the Hertz vector in the form of Hankel transformations

$$H_d = -e_\alpha\frac{p\omega}{4\pi}\frac{\partial}{\partial\rho}\int\limits_0^\infty e^{i\varkappa|z-z_0|}\tfrac{1}{\varkappa}\mathrm{J}_0(k_\rho\rho)k_\rho dk_\rho = e_\alpha\frac{p\omega}{4\pi}\int\limits_0^\infty e^{i\varkappa|z-z_0|}\,\tfrac{k_\rho}{\varkappa}\,\mathrm{J}_1(k_\rho\rho)k_\rho dk_\rho, \tag{8}$$

$$E_d = \frac{ip}{4\pi\varepsilon_0\varepsilon_1}\int\limits_{-\infty}^\infty e^{i\varkappa|z-z_0|}\Big(e_\rho\mathrm{sgn}(z\text{-}z_0)i\frac{\partial}{\partial\rho}\mathrm{J}_0(k_\rho\rho)+ e_z\tfrac{k_\rho^2}{\varkappa}\mathrm{J}_0(k_\rho\rho)\Big)k_\rho dk_\rho = \frac{-ip}{4\pi\varepsilon_0\varepsilon_1}\int\limits_{-\infty}^\infty e^{i\varkappa|z-z_0|}\Big(e_\rho\mathrm{sgn}(z\text{-}z_0)i\mathrm{J}_1(k_\rho\rho)- e_z\tfrac{k_\rho}{\varkappa}\mathrm{J}_0(k_\rho\rho)\Big)k_\rho^2 dk_\rho. \tag{9}$$

It is useful to represent the above integral representations also in terms of the Hankel function (see Appendix A)

$$H_d = -e_\alpha\frac{p\omega}{8\pi}\frac{\partial}{\partial\rho}\int\limits_{-\infty}^\infty e^{i\varkappa|z-z_0|}\mathrm{H}_0^{(1)}(k_\rho\rho)\frac{k_\rho}{\varkappa}dk_\rho = e_\alpha\frac{p\omega}{8\pi}\int\limits_{-\infty}^\infty e^{i\varkappa|z-z_0|}\mathrm{H}_1^{(1)}(k_\rho\rho)\frac{k_\rho^2}{\varkappa}dk_\rho, \tag{10}$$

$$E_d = \frac{-ip}{8\pi\varepsilon_0\varepsilon}\int\limits_{-\infty}^\infty e^{i\varkappa|z-z_0|}\Big(e_\rho\mathrm{sgn}(z\text{-}z_0)i\mathrm{H}_1^{(1)}(k_\rho\rho) - e_z\frac{k_\rho}{\varkappa}\mathrm{H}_0^{(1)}(k_\rho\rho)\Big)k_\rho^2 dk_\rho. \tag{11}$$

Similar integrals occur in the Sommerfeld integrals, as well as in boundary value problems.

III. Exact field formulas

Using the integral representation of a spherical wave

$$\frac{e^{i\kappa}}{r} = -\frac{1}{2i}\int\limits_{-\infty}^\infty e^{i\varkappa|z-z_0|}\mathrm{H}_0^{(1)}(k_\rho\rho)\frac{k_\rho}{\varkappa}\,dk_\rho, \tag{12}$$
$$r = \sqrt{\rho^2+|z-z_0|^2},$$

It is not difficult to calculate the integral (10) for the magnetic field of the Hertz radiator

$$H_d = e_\alpha\frac{ip\omega}{4\pi}\frac{\partial}{\partial\rho}\frac{e^{ik_0 r}}{r} = -e_\alpha\frac{\omega p k_0}{4\pi}\frac{e^{i\kappa}}{r}\Big(1+\frac{i}{\kappa}\Big)\frac{\rho}{r}, \tag{13}$$
$$\kappa = k_0 r,\ \rho = r\sin\theta, \tag{14}$$

It should be noted that the same result can be obtained by calculating the convolution (1) in (3)

$$H_d = -\nabla\times(j*\psi) = e_\alpha\frac{ip\omega}{4\pi}\frac{\partial}{\partial\rho}\frac{e^{ikr}}{r}. \tag{15}$$

It is convenient to obtain an exact expression for the electric field only by calculating the rotor of expression (13) in the cylindrical coordinate system using equations (4) and (3)

$$E_d = \frac{i}{\varepsilon_0\varepsilon\omega}\nabla\times H_d = -\frac{pk_0^2}{4\pi\varepsilon_0\varepsilon}\frac{e^{i\kappa}}{r}\Big\{e_\rho\mathrm{sgn}(z-z_0)\frac{\rho}{r}\frac{(z-z_0)}{r}\Big(1+3\big({}^i/_\kappa-{}^1/_{\kappa^2}\big)\Big)- e_z\Big(\frac{\rho^2}{r^2}-\Big(2\frac{(z-z_0)^2}{r^2}-\frac{\rho^2}{r^2}\Big)\big({}^i/_\kappa-{}^1/_{\kappa^2}\big)\Big)\Big\}. \tag{16}$$

In the spherical coordinate system, it will take the form

$$E_d = -\frac{pk_0^2}{4\pi\varepsilon_0\varepsilon}\frac{e^{i\kappa}}{r}\Big\{e_\theta\sin\theta\Big(1+{}^i/_\kappa-{}^1/_{\kappa^2}\Big)- e_r 2\cos\theta\Big({}^i/_\kappa-{}^1/_{\kappa^2}\Big)\Big\}, \tag{17}$$

due to transformations of the basis vectors

$$\begin{cases} e_\rho = e_r\sin\theta+e_\theta\cos\theta, \\ e_z = e_r\cos\theta-e_\theta\sin\theta. \end{cases} \tag{18}$$

IV. Method for calculating Sommerfeld integrals

As an example, to demonstrate the methodology for calculating the Sommerfeld integrals, consider the electric field of a dipole (11). Let $z > z_0$.

Passing to the angular integration variable θ, where

$$k_\rho = k_0\sin\theta,$$

we transform expression (11) to the Sommerfeld integral along the contour S_z (see Fig. 1)

It should be noted that to provide the convergence of the asymptotic integral, the integration contour S_z in (19) must be deformed to a line passing from top to bottom parallel to the imaginary axis, which slightly deviates to the left up to the point θ_{dir} and to the right after that. Obviously, here θ_{dir} is a saddle point.

$$E_d = -\frac{ipk_0}{8\pi\varepsilon_0\varepsilon_1}\int\limits_{S_z} e^{ik_0 r\cos(\theta-\theta_{dir})}F(\theta) \Big(e_\rho\cos\theta - e_z\sin\theta\Big)d\theta, \quad (19)$$

where

$$\cos\theta_{dir} = {(z-z_0)}/{r},\ \sin\theta_{dir} = {\rho}/{r}, \quad (20)$$

$$F(\theta) = \sqrt{\frac{2}{i\pi k_0\rho}}\sin^{\frac{3}{2}}\theta. \quad (21)$$

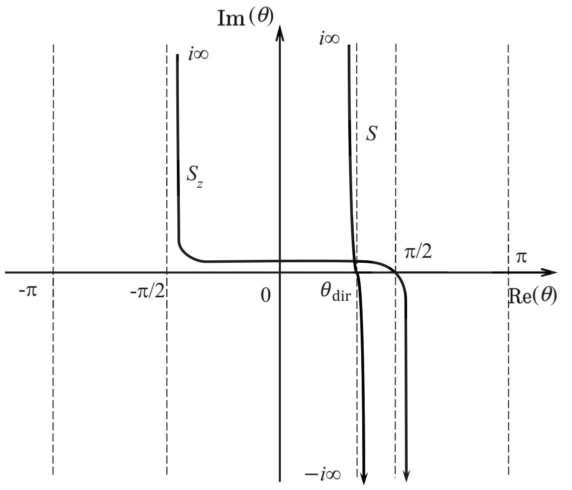

Fig. 1. The contours of integration.

To calculate the integral (19), we first move the integration contour parallel along the real axis to the origin of the coordinate system by replacing $\theta \to \theta+\theta_{dir}$, after the function

$$F(\theta+\theta_{dir})\cos(\theta+\theta_{dir})e^{i\kappa(\cos\theta+\theta^2/2)} = \sum_{m=0}^{\infty}\frac{a_m^\rho}{m!}\theta^m,$$

$$F(\theta+\theta_{dir})\sin(\theta+\theta_{dir})e^{i\kappa(\cos\theta+\theta^2/2)} = \sum_{m=0}^{\infty}\frac{a_m^z}{m!}\theta^m,$$

we make expansion into Maclaurin series, where

$$a_m^\rho = \frac{d^m}{d\theta^m}\Big(F(\theta+\theta_{dir})\cos(\theta+\theta_{dir})e^{i\kappa(\cos\theta+\frac{1}{2}\theta^2)}\Big)_0, \quad (22)$$

$$a_m^z = \frac{d^m}{d\theta^m}\Big(F(\theta+\theta_{dir})\sin(\theta+\theta_{dir})e^{i\kappa(\cos\theta+\frac{1}{2}\theta^2)}\Big)_0. \quad (23)$$

Thus, the Sommerfeld integral (19) is reduced to the calculation of the integral

$$E_d = -\frac{ipk_0^3}{8\pi\varepsilon_0\varepsilon}\sum_{m=0}^{\infty}\frac{1}{(2m)!}\Big(e_\rho a_{2m}^\rho - e_z a_{2m}^z\Big)\int\limits_S \theta^{2m}e^{-i\kappa\theta^2/2}d\theta, \quad (24)$$

which vanishes for odd values of m (see. (36) Appendix B).

Finally, we represent the dipole field as an infinite series

$$E_d(r,\theta_{dir}) = -i\sqrt{2}\frac{pk_0^3}{8\pi\varepsilon_0\varepsilon}\sum_{m=0}^{\infty}\frac{\Gamma\big(m+\frac{1}{2}\big)}{m!\,(i\kappa)^{m+\frac{1}{2}}}\big(e_\rho a_{2m}^\rho - e_z a_{2m}^z\big), \quad (25)$$

where Γ is a gamma function.

In (25) we select the leading term of the asymptotic series

$$E_{d0}(r,\theta_{dir}) = -\frac{pk_0^2}{4\pi\varepsilon_0\varepsilon}\frac{e^{i\kappa}}{r}\sin\theta_{dir}\big(e_\rho\cos\theta_{dir} - e_z\sin\theta_{dir}\big). \quad (26)$$

Note that the leading term of the series coincides with the expression for the electric dipole in the wave zone

$$E_{d0}(r,\theta_{dir}) \sim -e_\theta\frac{pk_0^2}{4\pi\varepsilon_0\varepsilon}\frac{e^{i\kappa}}{r}\sin\theta_{dir}, \quad (27)$$

where $e_\theta = e_\rho\cos\theta_{dir} - e_z\sin\theta_{dir}$ is the unit vector in the spherical coordinate system.

Let us calculate the coefficients of the series, for example, of the initial four terms

$$\{a_{2m}^\rho\} = \sqrt{\frac{2}{i\pi\kappa}}e^{i\kappa}\sin 2\theta_{dir}\{\tfrac{1}{2}; -2; 8+\tfrac{1}{2}i\kappa\},$$

$$\{a_{2m}^z\} = \sqrt{\frac{2}{i\pi\kappa}}e^{i\kappa}\{\sin^2\theta_{dir}; 2\cos 2\theta_{dir}; -8+\sin^2\theta_{dir}(16+i\kappa)\}\ (m=0,1,2,3). \quad (28)$$

As a result, we obtain an approximation of the sum of the first four terms of the series in the cylindrical coordinate system $(\kappa \gg 1)$

$$E_d(r,\theta_{dir}) \simeq -\frac{pk_0^2}{4\pi\varepsilon_0\varepsilon}\frac{e^{i\kappa}}{r}\Big\{e_\rho\Big(1+i\frac{13}{8\kappa}+\frac{209}{16\kappa^2}\Big)\sin\theta_{dir}\cos\theta_{dir} - e_z\Big(1+i\frac{13}{8\kappa}+\frac{209}{16\kappa^2}\Big)\sin^2\theta_{dir} + e_z\frac{1}{\kappa}\big(i+\frac{51}{8\kappa}-i\frac{10}{\kappa^2}\big)\Big\}. \quad (29)$$

In the spherical coordinate system, the above expression is written as

$$E_d(r,\theta_{dir}) \simeq -\frac{pk_0^2}{4\pi\varepsilon_0\varepsilon}\frac{e^{i\kappa}}{r}\Big\{e_\theta\sin\theta_{dir}\Big(1+i\frac{5}{8\kappa}+\frac{107}{16\kappa^2}\Big) + e_r\cos\theta_{dir}\Big(\frac{i}{\kappa}+\frac{51}{8\kappa^2}\Big), \quad (30)$$

due to the representation of the basis vectors as

$$\begin{cases} e_\rho = e_r\sin\theta_{dir} + e_\theta\cos\theta_{dir} \\ e_z = e_r\cos\theta_{dir} - e_\theta\sin\theta_{dir}. \end{cases} \quad (31)$$

In Figs. (2) and (3) the coefficients of the series a_m^ρ (22) and a_m^z (23) are calculated using approximate formulas (28) and (28).

Figure 2 shows a comparative estimate of the asymptotic expression in the form of a power series for the electric field of a vertical dipole (30) with the exact formula (17). Figure 3 shows the dependence, in percentage, of the modulus of the relative error of the source field $E_d(r,\theta_{dir})$ in the power series approximation $(m=3)$ in (30) on the dimensionless distance $(\kappa = k_0 r)$.

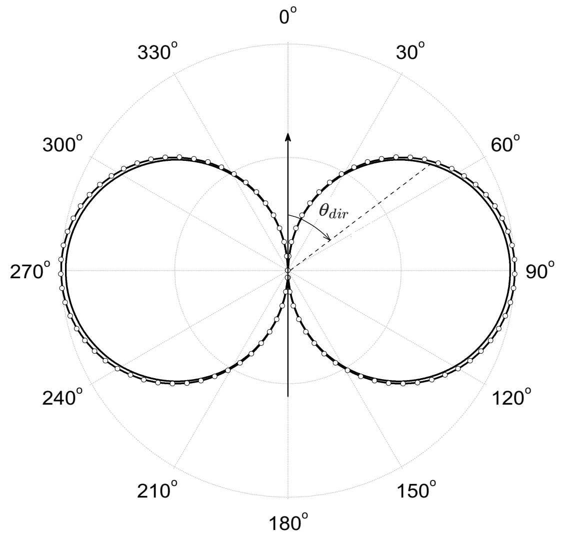

Fig. 2. Directional patterns of a vertical dipole $|E_d(r, \theta_{dir})|$. The solid line with a marker is the true diagram (17), the solid line corresponds to the approximate formula (30), $\kappa = 5, \varepsilon = 1$.

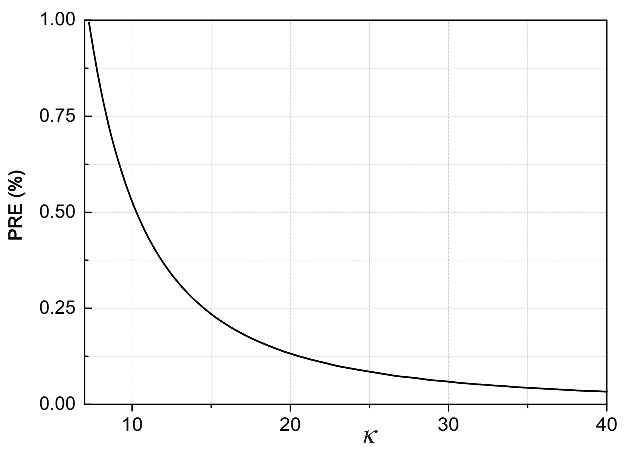

Fig. 3. Dependence of the relative error of the vertical dipole radiation pattern (30), on the dimensionless distance κ.

V. Conclusions

Integral representations of the fields of a Hertz point radiator in the form of the Hankel transformation (8), (9) as well as integrals with infinite limits (10), (11) are obtained.

In order to assess the accuracy of the method for calculating the Sommerfeld integrals, the exact analytical expressions for the integrals in cylindrical (13), (16) and spherical coordinate systems (17) are given.

In this paper, the use of the auxiliary integral (36) forms the basis of the method for calculating the asymptotics of the Sommerfeld integrals.

Qualitative and quantitative comparisons of the final results with exact expressions for the Hertz radiator (17) obtained directly from integral representations in cylindrical and spherical coordinate systems are presented.

Fig. 3 shows the dependence of the relative error of the radiation pattern on the distance κ, which does not exceed one percent. As an example of the reliability of the technique, it is shown that the expansion of the Hankel functions in an infinite power series leads to the well-known formula (41).

This work can be continued in solving the Sommerfeld problem.

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Appendix A
Replacing the Bessel Function by the Hankel Function in the Hankel Transforms

Let an arbitrary analytic function $f(k_\rho)$ satisfy the condition

$$f(k_\rho) = e^{i\pi\nu} f(e^{-i\pi} k_\rho), \tag{32}$$

then the representation of the inverse Hankel transform of order ν of the function $f(k_\rho)$ is

$$\int\limits_0^\infty f(k_\rho) \mathrm{J}_\nu(k_\rho \rho) k_\rho dk_\rho = \frac{1}{2} \int\limits_{-\infty+i0}^\infty f(k_\rho) \mathrm{H}_\nu^{(1)}(k_\rho \rho) k_\rho dk_\rho. \tag{33}$$

Using the representation of the Bessel function

$$\mathrm{J}_\nu(k_\rho \rho) = \frac{1}{2}\big(\mathrm{H}_\nu^{(1)}(k_\rho \rho) + \mathrm{H}_\nu^{(2)}(k_\rho \rho)\big), \tag{34}$$

and the analytic continuation of the Hankel function of the second kind

$$\mathrm{H}_{\nu}^{(2)}(k_\rho \rho) = -e^{i\pi\nu}\mathrm{H}_{\nu}^{(1)}\big(k_\rho \rho\, e^{i\pi}\big), \tag{35}$$

the integral along the negative real semi-axis can be expressed in terms of the Hankel function of the first kind. Then, based on the equality of the integrands and the condition (33), the integrals are combined into one integral, which is contained in the right side of the expression (32).

In this case, it should be borne in mind that the Hankel function has a cut along the negative real semi-axis. Therefore, the path of integration must pass parallel to the cut, above it, at a distance equal to a vanishingly small imaginary value $+i0$, which we will omit everywhere below.

In particular, the function $f(k_\rho)$ must be even with respect to the function $\mathrm{J}_0(k_\rho \rho)$ or $\mathrm{H}_0^{(1)}(k_\rho \rho)$, according to condition.

Appendix B
An auxiliary integral for calculating Sommerfeld integrals

Asymptotic calculations of the Sommerfeld integrals in the form of an infinite series can be optimally performed using the auxiliary integral

$$\int\limits_{S_z} \theta^{2m} e^{-i\kappa\theta^2/2} d\theta = \Big(\frac{2}{i\kappa}\Big)^{m+\frac{1}{2}} \Gamma\big(m+\tfrac{1}{2}\big). \tag{36}$$

In order to calculate the auxiliary integral, we first deform the contour S_z to the imaginary axis, which passes from top to bottom and slightly deviates from it by an infinitesimal real value to ensure the convergence of the integral. Then the integral can be sequentially represented as

$$\int\limits_{i\infty-0}^{-i\infty+0} \theta^{2m} e^{-i\kappa\theta^2/2} d\theta = -2 \int\limits_{0}^{i\infty-0} \theta^{2m} e^{-i\kappa\theta^2/2} d\theta = 2\sqrt{\pi}\,(2i)^{m-\frac{1}{2}} \frac{d^m}{d\kappa^m} \kappa^{-\frac{1}{2}}, \tag{37}$$

where the last integral can be found by m-fold calculation of the derivative with respect to the parameter κ on both sides of the equality sign in the expression for the integral

$$\int\limits_{0}^{i\infty-0} e^{-i\kappa\theta^2/2} d\theta = -\sqrt{\frac{\pi}{2\kappa}} e^{-i\pi/4}, \tag{38}$$

taking into account the representation of the gamma function

$$\Gamma(m+\tfrac{1}{2}) = \sqrt{\pi}\,\frac{1\cdot 3 \cdot \dots \cdot (2m-1)}{2^m}.$$

The last integral (38) can easily be obtained

$$\mathrm{C}(\infty) + i\,\mathrm{S}(\infty) = \int\limits_{0}^{\infty} e^{i\pi t^2/2} dt$$

using the Fresnel integrals [12] and the substitution $t = -i\sqrt{\kappa/\pi}\,\theta$.

A. Calculating asymptotics of the Hankel function

Another illustrative example of demonstrating the technique for calculating the Sommerfeld integral using the auxiliary integral (36) is the asymptotic expansion of the Hankel function in a power series for large values of the argument.

Let us use the Sommerfeld integral representation for the Hankel function

$$\mathrm{H}_{\nu}^{(1)}(z) = \frac{1}{\pi} \int\limits_{S_z} e^{i\nu(\theta-\frac{\pi}{2})} e^{iz\cos\theta} d\theta. \tag{39}$$

In order to calculate the asymptotic formula of the Hankel function in the form of a power series, as a rule, we expand the integrand in the Maclaurin series

$$e^{i\nu(\theta-\pi/2)} e^{iz\cos\theta}\, e^{-iz(1-\theta^2/2)} = \sum_{m=0}^{\infty} \frac{h_m}{m!}\theta^m$$

with coefficients

$$h_m = e^{-i(z+\nu\pi/2)} \frac{d^m}{d\theta^m} \big(e^{iz(\cos\theta+\theta^2/2)} e^{i\nu\theta}\big)_{\theta=0}$$

and express the Sommerfeld integral (39) in terms of the auxiliary integral (36)

$$\mathrm{H}_{\nu}^{(1)}(z) = \frac{e^{iz}}{\pi} \sum_{m=0}^{\infty} \frac{h_m}{m!} \int\limits_{S} \theta^m e^{-iz\theta^2/2} d\theta.$$

Here one should bear in mind that odd integrands (with odd values of m) are omitted as the corresponding integral vanishes.

Thus, using formula (36)and calculating the coefficients

$$\{h_{2m}\} = e^{-i\nu\pi/2}\Big(1; -\nu^2; \nu^4 + iz; -(\nu^6 + i15\nu^2 z + iz); \nu^8 + i(70\nu^4 + 28\nu^2 + 1)z - 35z^2); \dots\Big), \tag{40}$$

finally we get the asymptotic expansion of Hankel functions of the first kind for $z \to \infty$ (see [12])

$$\mathrm{H}_{\nu}^{(1)}(z) = \frac{e^{iz}}{\pi} \sum_{m=0}^{\infty} \frac{h_{2m}}{(2m)!} \Big(\frac{2}{iz}\Big)^{m+\frac{1}{2}} \Gamma\big(m+\tfrac{1}{2}\big) = \sqrt{\frac{2}{i\pi z}}\, e^{i(z-\nu\frac{\pi}{2})}\Big(1 + i\frac{(4\nu^2-1)}{8z} - \frac{(4\nu^2-1)(4\nu^2-9)}{2!\,(8z)^2} + \dots\Big). \tag{41}$$

The main term of the series is ($m = 0$)

$$\mathrm{H}_{\nu}^{(1)}(z) \simeq \sqrt{\frac{2}{\pi z}} e^{i(z-\nu\frac{\pi}{2}-\frac{\pi}{4})}. \tag{42}$$