Analytical Solution of the Differential Equation of Secular Perturbations in the Three-Body Problem with Variable Masses [1]

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The system of three mutually gravitating bodies T_0, T_1 and T_2 with variable masses changing isotropically in different rates

$$\frac{\dot{m}_0(t)}{m_0(t)} \neq \frac{\dot{m}_1(t)}{m_1(t)}, \quad \frac{\dot{m}(t)_0}{m_0(t)} \neq \frac{\dot{m}_2(t)}{m_2(t)}, \quad \frac{\dot{m}_1(t)}{m_1(t)} \neq \frac{\dot{m}_2(t)}{m_2(t)} \tag{1}$$

is investigated. With the use of the analytical calculation system *Mathematica* [1] the perturbing function written in terms of analogues of the second system of the Poincare elements is obtained in the form of series in powers of small quantities e_i, i_i $(i = 1, 2)$ [2], [3] up to the second order inclusively.

Equations for the secular perturbations have the form

$$\begin{array}{lll} \dot{\Lambda}_i = 0, & \dot{\xi}_i = \dfrac{\partial R_i^*}{\partial \eta_i}, & \dot{p}_i = \dfrac{\partial R_i^*}{\partial q_i}, \\ \dot{\lambda}_i = -\dfrac{\partial R_i^*}{\partial \Lambda_i}, & \dot{\eta}_i = -\dfrac{\partial R_i^*}{\partial \xi_i}, & \dot{q}_i = -\dfrac{\partial R_i^*}{\partial p_i}, \quad (i = 1, 2) \end{array} \tag{2}$$

$$R_1^* = \frac{1}{\gamma_1^2(t)} \cdot \frac{\tilde{\beta}_1^4}{2\mu_{10}\Lambda_1^2} + F_1^*(\xi_i, \eta_i, t) + F_1^*(p_i, q_i, t),$$

$$R_2^* = \frac{1}{\gamma_2^2(t)} \cdot \frac{\tilde{\beta}_2^4}{2\mu_{20}\Lambda_2^2} + F_2^*(\xi_i, \eta_i, t) + F_2^*(p_i, q_i, t),$$

$$F_1^*(\xi_i, \eta_i, t) = K_0 + K_1(\xi_1^2 + \eta_1^2) + K_2(\xi_2^2 + \eta_2^2) + K_3(\xi_1\xi_2 + \eta_1\eta_2),$$

$$F_2^*(\xi_i, \eta_i, t) = K_0' + K_1'(\xi_1^2 + \eta_1^2) + K_2'(\xi_2^2 + \eta_2^2) + K_3'(\xi_1\xi_2 + \eta_1\eta_2),$$

$$F_1^*(p_i, q_i, t) = -\frac{f m_1 m_2 \nu_0 B_1}{8\psi_1} F(p_i, q_i), \quad F_2^*(p_i, q_i, t) = -\frac{f m_1 m_2 \nu_0 B_1}{8\psi_2} F(p_i, q_i),$$

$$F(p_i, q_i) = (p_1^2 + q_1^2)/\Lambda_1 + (p_2^2 + q_2^2)/\Lambda_2 - 2(p_1 p_2 + q_1 q_2)/\sqrt{\Lambda_1 \Lambda_2}.$$

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$$\tilde{\beta}_1^2 = f \cdot \mu_1(t_0) m_1(t_0) m_0(t_0),\ \tilde{\beta}_2^2 = f \cdot \mu_2(t_0) m_2(t_0)[m_0(t_0) + m_1(t_0)],\ \mu_i(t_0) = \mu_{i0},$$

$$\mu_1(t) = \frac{m_1 m_0}{m_0 + m_1} \neq const,\ \mu_2(t) = \frac{m_2(m_1 + m_0)}{m_0 + m_1 + m_2} \neq const,\ \psi_i = \mu_i(t)/\mu_i(t_0),$$

$$\gamma_1 = \gamma_1(t) = \frac{m_0(t_0) + m_1(t_0)}{m_0(t) + m_1(t)},\quad \gamma_2 = \gamma_2(t) = \frac{m_0(t_0) + m_1(t_0) + m_2(t_0)}{m_0(t) + m_1(t) + m_2(t)},\quad \nu_0 = \frac{m_0(t)}{m_0(t) + m_1(t)},$$

where f is a gravitational constant, K_j, K_j', $j = 0, 1, 2, 3$ are some given function of time, B_1 is the Laplace coefficient. The analytical solutions of the equations (2) are found in the form

$$\xi_1 = \frac{\gamma_{22}}{\Delta} M_1 \cos(\tau_1(t) + \beta_1) - \frac{\gamma_{21}}{\Delta} E\left\{[M_2 + J_2] \cos\tau_2(t) - [N_2 - I_2] \sin\tau_2(t)\right\},$$

$$\eta_1 = -\frac{\gamma_{22}}{\Delta} M_1 \sin(\tau_1(t) + \beta_1) + \frac{\gamma_{21}}{\Delta} E\left\{[M_2 + J_2] \sin\tau_2(t) + [N_2 - I_2] \cos\tau_2(t)\right\},$$

$$\xi_2 = -\frac{\gamma_{12}}{\Delta} M_1 \cos(\tau_1(t) + \beta_1) + \frac{\gamma_{11}}{\Delta} E\left\{[M_2 + J_2] \cos\tau_2(t) - [N_2 - I_2] \sin\tau_2(t)\right\},$$

$$\eta_2 = -\frac{\gamma_{12}}{\Delta} M_1 \sin(\tau_1(t) + \beta_1) - \frac{\gamma_{11}}{\Delta} E\left\{[M_2 + J_2] \sin\tau_2(t) + [N_2 - I_2] \cos\tau_2(t)\right\},$$

$$I_2 = I_2(\tau_2(t)) = \int\left(\tilde{B}_2 \cos\tau_2 + \tilde{A}_2 \sin\tau_2\right) d\tau_2,\quad J_2 = J_2(\tau_2(t)) = \int\left(\tilde{A}_2 \cos\tau_2 - \tilde{B}_2 \sin\tau_2\right) d\tau_2,$$

$$\Delta = \Delta(t) = \gamma_{11}\gamma_{22} - \gamma_{21}\gamma_{12} \neq 0,\quad E = E(t) = exp\left(\int \frac{\gamma_{11}\dot{\gamma}_{21} - \gamma_{21}\dot{\gamma}_{12}}{\Delta} dt\right),$$

$$p_1 = \gamma_{11}^* M_1^* \cos\beta_1^* + \gamma_{12}^* M_2^* \cos\left(\sigma_2 \int \psi_2^*(t) dt + \beta_2^*\right),$$
$$q_1 = \gamma_{11}^* M_1^* \sin\beta_1^* + \gamma_{12}^* M_2^* \sin\left(\sigma_2 \int \psi_2^*(t) dt + \beta_2^*\right),\quad \sigma_2 = 2(\Lambda_1 + \Lambda_2)/\Lambda_1\Lambda_2,$$

$$p_2 = \gamma_{21}^* M_1^* \cos\beta_1^* + \gamma_{22}^* M_2^* \cos\left(\sigma_2 \int \psi_2^*(t) dt + \beta_2^*\right),$$
$$q_2 = \gamma_{21}^* M_1^* \sin\beta_1^* + \gamma_{22}^* M_2^* \sin\left(\sigma_2 \int \psi_2^*(t) dt + \beta_2^*\right),\quad \psi_2^*(t) = -f m_1 m_2 \nu_0 B_1 / 8\psi_2,$$

where M_1, β_1, M_2, N_2, M_1^*, β_1^*, M_2^*, β_2^* are constants of integration, γ_{ij}^* are constant, $\tau_1(t)$, $\tau_2(t)$, $\tilde{A}_2(t)$, $\tilde{B}_2(t)$, $\gamma_{ij}(t)$ is a known function of time.

The solutions which are studied in the paper can be effectively used in the analysis of dynamic evolution of non-stationary triple gravitating hierarchical systems, primarily in two planetary (two protoplanetary) three-body problem with variable masses.

References

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