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Contents

Explicit Model for Surface Waves on an Elastic Half-Space Coated by a Thin Vertically Inhomogeneous Layer

Ali Mubaraki, Danila Prikazchikov, and Askar Kudaibergenov

Abstract The study is focussed on surface waves propagating in an isotropic elastic half-space coated with a thin, vertically inhomogeneous layer, subject to action of a prescribed normal surface stress. The effective boundary conditions modelling an inhomogeneous coating are derived in the long-wave limit, generalising the those for a thin homogeneous isotropic layer. A singularly perturbed hyperbolic equation on the interface is then deduced, governing surface wave propagation. The effect of the perturbative pseudo-differential operator including the structure of the quasi-front emerging for a point impulse loading, is analysed.

Keywords Surface waves · Thin coating · Inhomogeneous

1 Introduction

Thin films and coatings have numerous applications in engineering and biological sciences, see e.g. [1–6], to name a few. In addition, a number of technological developments are associated with related multi-layered structures, see e.g. [7] and references therein.

Often the effect of a thin coating on the half-space is modelled by means of the so-called effective boundary conditions, starting from the original work [8], and still popular, see e.g. [9, 10] and references therein.

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The method of effective boundary conditions was also implemented for analysis of surface wave field in a coated half-space, within the framework of hyperbolic-elliptic models for the Rayleigh wave induced by a prescribed surface load, see [11, 12] for more detail. As a result, the contribution of surface wave to the overall dynamic response in the long wave limit is described by elliptic equations over the interior associated with decay away from the surface, and a singularly perturbed wave equation on the boundary governing surface wave propagation.

In this paper, we extend these results for a thin vertically inhomogeneous coating layer, with density and material parameters being depth-dependent. First, we derive the effective boundary conditions by employing a standard long wave asymptotic procedure, well established for thin structures, see e.g. [13, 14]. Then, we follow a slow-time perturbation scheme proposed in [11], with the small parameter corresponding to the proximity of the wave phase velocity to that of the Rayleigh wave. As a result, we obtain a wave equation for the longitudinal elastic potential, which is singularly perturbed by a pseudo-differential operator. The amplitude of the perturbation depends on the combination of the material parameters of both coating and the substrate. As observed earlier in [11] for the case of a homogeneous coating layer, the sign of this coefficient plays a crucial role, distinguishing between the case of a local maximum/minimum of the phase speed at the Rayleigh wave speed in the long wave limit. Finally, we illustrate the developments by considering a model example of a concentrated vertical impulse loading applied on the surface of a two-layered coating.

2 Basic Equations

Consider an elastic layer of thickness h, occupying the domain $0 \leq x_3 \leq h$, coating a homogeneous half-space $x_3 \geq h$, see Fig. 1.

The layer is assumed to be vertically inhomogeneous, with the constitutive relations given by

$$\sigma_{ij} = \lambda_c \left(u_{1,1} + u_{2,2} + u_{3,3}\right) \delta_{ij} + \mu_c \left(u_{i,j} + u_{j,i}\right), \tag{1}$$

where σ_{ij}, $i, j = 1, 2, 3$, are the Cauchy stress tensor components, u_i are displacement components, $\lambda_c = \lambda(x_3)$ and $\mu_c = \mu(x_3)$ are the Lamé elastic moduli, and

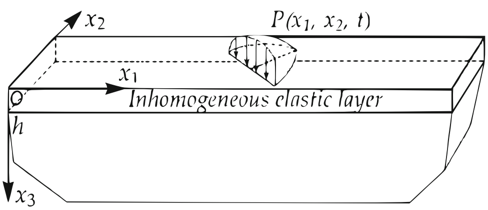

Fig. 1 An inhomogeneous layer by a coated half-space

δ_{ij} is the Kronecker delta. Here and below a comma denotes differentiation with respect to the corresponding variable. The governing equations of motion in the 3D elasticity are taken as (see e.g. [15])

$$\sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = \rho_c\, u_{i,tt}, \tag{2}$$

where $\rho_c = \rho\,(x_3)$ is volume mass density. The longitudinal and transverse wave speeds are introduced as

$$c_1\,(x_3) = \sqrt{\frac{\lambda_c + 2\mu_c}{\rho_c}}, \qquad \text{and} \qquad c_2\,(x_3) = \sqrt{\frac{\mu_c}{\rho_c}}, \tag{3}$$

respectively. The boundary conditions at the surface $x_3 = 0$ are taken in the form

$$\sigma_{3m} = 0, \qquad \text{and} \qquad \sigma_{33} = -P, \qquad m = 1, 2, \tag{4}$$

where $P = P(x_1, x_2, t)$ is a prescribed vertical load, with the continuity conditions at the interface assumed as

$$u_i = v_i \qquad \text{at} \qquad x_3 = h, \tag{5}$$

where $v_i = v_i(x_1, x_2, t)$, $i = 1, 2, 3$ are displacements on the surface of the substrate.

3 Effective Boundary Conditions

First, we derive the effective boundary conditions, accounting for the effect of the thin coating layer. Below we implement the direct asymptotic integration of the equations in elasticity, see e.g. [11]. A small parameter ϵ, associated with the long-wave limit, is specified as

$$\epsilon = \frac{h}{L} \ll 1, \tag{6}$$

where L is the typical wave length. We introduce the scaling

$$\xi_m = \frac{x_m}{L}, \qquad \eta = \frac{x_3}{h}, \qquad \tau = \frac{t\, c_h}{L}, \tag{7}$$

with

$$u_i^* = \frac{u_i}{L}, \quad v_i^* = \frac{v_i}{L}, \quad \sigma_{mn}^* = \frac{\sigma_{mn}}{\mu_h}, \quad \sigma_{3i}^* = \frac{\sigma_{3i}}{\epsilon\, \mu_h}, \quad p^* = \frac{P}{\epsilon\, \mu_h}, \tag{8}$$

where $c_h = c_2(h)$, $\mu_h = \mu_c(h)$, $\rho_h = \rho_c(h)$, $m, n = 1, 2$ and all quantities with the asterisk are assumed to be of the same asymptotic order. Then the equation of motion (2) and the constitutive relations (1) can be written explicitly as

$$\begin{aligned} \sigma^*_{mm,\xi_m} + \sigma^*_{mn,\xi_n} + \sigma^*_{m3,\eta} &= \rho_*\, u^*_{m,\tau\tau}, \\ \sigma^*_{33,\eta} + \epsilon\left(\sigma^*_{3m,\xi_m} + \sigma^*_{3n,\xi_n}\right) &= \rho_*\, u^*_{3,\tau\tau}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \sigma^*_{mn} &= \kappa_2^2\left(u^*_{m,\xi_n} + u^*_{n,\xi_m}\right), \\ \epsilon\,\sigma^*_{mm} &= \left(\kappa_1^2 - 2\kappa_2^2\right) u^*_{3,\eta} + \epsilon\left(\kappa_1^2\, u^*_{m,\xi_m} + \left(\kappa_1^2 - 2\kappa_2^2\right) u^*_{n,\xi_n}\right), \\ \epsilon^2\,\sigma^*_{m3} &= \kappa_2^2\left(u^*_{m,\eta} + \epsilon\, u^*_{3,\xi_m}\right), \\ \epsilon^2\,\sigma^*_{33} &= \kappa_1^2\, u^*_{3,\eta} + \epsilon\,(\kappa_1^2 - 2\kappa_2^2)\left(u^*_{m,\xi_m} + u^*_{n,\xi_n}\right), \end{aligned} \tag{10}$$

where $\rho_*(\eta) = \rho_c/\rho_h$, $\kappa_1^2 = (\lambda_c + 2\mu_c)/\mu_h$, $\kappa_2^2 = \mu_c/\mu_h$ and $\kappa_c^2 = \kappa_1^2/\kappa_2^2$, with $1 \le m \ne n \le 2$. On substituting $u^*_{3,\eta}$ from $(10)_4$ into $(10)_2$, we get

$$\sigma^*_{mm} = 4\kappa_2^2\left(1 - \kappa_c^{-2}\right) u^*_{m,\xi_m} + \left(1 - 2\kappa_c^{-2}\right)\left(2\kappa_2^2\, u^*_{n,\xi_n} + \epsilon\,\sigma^*_{33}\right). \tag{11}$$

The conditions (4) and (5) become

$$\begin{aligned} \sigma^*_{3m} = 0, \qquad \sigma^*_{33} &= -p^* \qquad \text{at} \qquad \eta = 0, \\ \text{and} \qquad u^*_i &= v^*_i, \qquad \text{at} \qquad \eta = 1. \end{aligned} \tag{12}$$

Next, expand the displacements and stresses as asymptotic series

$$\begin{pmatrix} u^*_i \\ \sigma^*_{mm} \\ \sigma^*_{mn} \\ \sigma^*_{3i} \end{pmatrix} = \begin{pmatrix} u^{(0)}_i \\ \sigma^{(0)}_{mm} \\ \sigma^{(0)}_{mn} \\ \sigma^{(0)}_{3i} \end{pmatrix} + \epsilon \begin{pmatrix} u^{(1)}_i \\ \sigma^{(1)}_{mm} \\ \sigma^{(1)}_{mn} \\ \sigma^{(1)}_{3i} \end{pmatrix} + \dots . \tag{13}$$

Then, at leading order, we have

$$\begin{aligned} \sigma^{(0)}_{mm,\xi_m} + \sigma^{(0)}_{mn,\xi_n} + \sigma^{(0)}_{m3,\eta} &= \rho_*\, u^{(0)}_{m,\tau\tau}, \\ \sigma^{(0)}_{33,\eta} &= \rho_*\, u^{(0)}_{3,\tau\tau}, \\ \sigma^{(0)}_{mn} &= \kappa_2^2\left(u^{(0)}_{m,\xi_n} + u^{(0)}_{n,\xi_m}\right), \\ \sigma^{(0)}_{mm} &= 4\kappa_2^2\left(1 - \kappa_c^{-2}\right) u^{(0)}_{m,\xi_m} + 2\kappa_2^2\left(1 - 2\kappa_c^{-2}\right) u^{(0)}_{n,\xi_n}, \\ u^{(0)}_{i,\eta} &= 0, \end{aligned} \tag{14}$$

subject to

$$\sigma_{3m}^{(0)} = 0\,, \qquad \sigma_{33}^{(0)} = -p^* \qquad \text{at} \qquad \eta = 0, \\ \text{and} \qquad u_i^{(0)} = v_i^*, \qquad \text{at} \qquad \eta = 1. \tag{15}$$

Equations (14)$_5$ with boundary conditions (15)$_2$ imply

$$u_i^{(0)} = v_i^*, \quad i = 1, 2, 3. \tag{16}$$

Therefore, from (14)$_2$ and (15)$_1$ we have

$$\sigma_{33}^{(0)} = v_{3,\tau\tau}^* \int_0^\eta \rho_* (z)\, dz - p^*. \tag{17}$$

Hence, (14)$_1$, (14)$_4$, (16) and (15)$_1$ yield

$$\begin{aligned} \sigma_{3m}^{(0)} = v_{m,\tau\tau}^* \left(\int_0^\eta \rho_* (z)\, dz \right) - 4 v_{m,\xi_m\xi_m}^* \left(\int_0^\eta \kappa_2^2 (z) \left(1 - \kappa_c^{-2} (z) \right) dz \right) \\ - v_{m,\xi_n\xi_n}^* \left(\int_0^\eta \kappa_2^2 (z)\, dz \right) - v_{n,\xi_m\xi_n}^* \left(\int_0^\eta \kappa_2^2 (z) \left(3 - 4\kappa_c^{-2} (z) \right) dz \right). \end{aligned} \tag{18}$$

Finally, the effective boundary conditions on the interface $x_3 = h$ may be expressed in terms of the original variables as

$$\begin{aligned} \sigma_{3m} &= h \left(\tilde{\rho}\, u_{m,tt} - \tilde{\gamma}\, u_{m,mm} - \tilde{\mu}\, u_{m,nn} - (\tilde{\gamma} - \tilde{\mu})\, u_{n,mn} \right), \\ \sigma_{33} &= h \tilde{\rho}\, u_{3,tt} - P, \end{aligned} \tag{19}$$

where $\gamma(x_3) = 4\mu_c (x_3) \left(1 - \kappa_c^{-2} (x_3) \right)$ and a tilde over a quantity denotes its mean value over the thickness of the layer

$$\tilde{f} = \frac{1}{h} \int_0^h f(x_3) dx_3.$$

Note that in case of a homogeneous isotropic layer the derived effective boundary conditions (19) reduce to the well-known ones first obtained in [8], see also [11], cf. (3.17).

4 Asymptotic Model for Surface Wave

With the effective boundary conditions (19) derived, an asymptotic model for surface wave may now be constructed, generalising the previous results in [11] to a coating with vertically inhomogeneous material properties. We arrive at the following boundary value problem for a homogeneous isotropic substrate, containing the conventional Navier equations of motion

$$(\lambda + \mu)\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \mu\Delta\mathbf{u} = \rho\mathbf{u}_{,tt}, \tag{20}$$

subject to ($x_3 = h$)

$$\begin{aligned} &\mu\left(u_{1,3} + u_{3,1}\right) = h\left(\tilde{\rho}\,u_{1,tt} - \tilde{\gamma}\,u_{1,11} - \tilde{\mu}\,u_{1,22} - (\tilde{\gamma} - \tilde{\mu})\,u_{2,12}\right),\\ &\mu\left(u_{2,3} + u_{3,2}\right) = h\left(\tilde{\rho}\,u_{2,tt} - \tilde{\gamma}\,u_{2,22} - \tilde{\mu}\,u_{2,11} - (\tilde{\gamma} - \tilde{\mu})\,u_{1,12}\right),\\ &\lambda(u_{1,1} + u_{2,2}) + (\lambda + 2\mu)u_{3,3} = h\tilde{\rho}\,u_{3,tt} - P. \end{aligned} \tag{21}$$

In above $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, Δ is a 3D Laplace operator in spatial coordinates, λ and μ are the constant Lamé parameters of the substrate, and ρ is its volume mass density.

Following the procedure in [11], the Radon integral transform is applied to (20) and (21), resulting in a reduction to a 2D formulation. Then, a slow-time perturbation scheme may be established, revealing the free Rayleigh wave at leading order, with the perturbed wave equation following from the analysis of correction terms. The resulting explicit formulation for surface wave field is expressed in terms of for the longitudinal Lamé potential ϕ, and two non-zero components of the vector shear potential, ψ_1 and ψ_2, with the displacement field expressed using the Helmholtz theorem

$$\mathbf{u} = \mathrm{grad}\,\phi + \mathrm{curl}\,\boldsymbol{\psi}, \tag{22}$$

with $\boldsymbol{\psi} = (-\psi_2, \psi_1, 0)$, for more details see [12]. The behaviour over the interior of the half-space is governed by pseudo-static elliptic equations

$$\phi_{,33} + \alpha_R^2\,\Delta_2\phi = 0, \qquad \psi_{m,33} + \beta_R^2\,\Delta_2\psi_m = 0, \quad m = 1, 2, \tag{23}$$

where $\Delta_2 = \partial_{11} + \partial_{22}$ is the 2D Laplacian in x_1 and x_2 and

$$\alpha_R = \sqrt{1 - \frac{c_R^2}{c_1^2}}, \qquad \beta_R = \sqrt{1 - \frac{c_R^2}{c_2^2}}, \qquad c_1^2 = \frac{\lambda + 2\mu}{\rho}, \qquad c_2^2 = \frac{\mu}{\rho},$$

with c_1, c_2, and c_R conventionally denoting the longitudinal, transverse, and Rayleigh wave speeds. The boundary condition for $(23)_1$ is given by a singularly perturbed wave equation

$$\Delta_2\phi - \frac{1}{c_R^2}\phi_{,tt} - bh\sqrt{-\Delta_2}\,(\Delta_2\theta) = -\frac{1 + \beta_R^2}{2\mu B}\,P, \tag{24}$$

with

$$B = \frac{1-\alpha_R^2}{\alpha_R}\,\beta_R + \frac{1-\beta_R^2}{\beta_R}\,\alpha_R - 1 + \beta_R^4,$$

and the constant b inheriting properties of both coating and substrate

$$b = \frac{1-\beta_R^2}{2\mu B}\left(\tilde{\rho} c_R^2\,(\alpha_R+\beta_R) - \tilde{\gamma}\beta_R\right). \tag{25}$$

It can be easily verified that in case of a homogeneous isotropic coating layer the latter reduces to earlier results (cf. (4.23) in [11]). The differential relations between the potentials on the boundary $x_3 = h$ are

$$\phi_{,3} = -\frac{1+\beta_R^2}{2}\left(\psi_{1,1}+\psi_{2,2}\right), \qquad \phi_{,m} = \frac{2}{1+\beta_R^2}\,\psi_{m,3}, \quad m = 1, 2. \tag{26}$$

5 Illustrative Example

In order to illustrate the derived formulation, let us restrict ourselves to a the plane-strain problem for a concentrated impact force $P(x_1, t) = P_0\delta(x_1)\delta(t)$, acting on the surface of a two-layered coating, with the material and geometrical parameters of the layers denoted with subscripts 1 and 2. The wave equation (24) may be rewritten in the form

$$\theta_{,ss} - \frac{1}{c_R^2}\theta_{,\tau_R\tau_R} - h_L\,\mathrm{sgn}\,b\,\sqrt{-\partial_{ss}}\left(\theta_{,ss}\right) = -\delta(s)\delta(\tau_R), \tag{27}$$

where $s = x_1/L$, $\tau_R = tc_R/L$ are the dimensionless coordinates, and

$$\theta = -\frac{4\mu B}{(1+\beta_R^2)c_R P_0}\,\phi\big|_{x_2=h_1+h_2}, \qquad h_L = \frac{(h_1+h_2)|b|}{L} \ll 1, \tag{28}$$

with the constant b defined according to (25) with

$$\tilde{\rho} = \frac{\rho_1 h_1 + \rho_2 h_2}{h_1+h_2}, \qquad \tilde{\gamma} = \frac{4\mu_1 h_1(1-\kappa_{c1}^{-2}) + 4\mu_2 h_2(1-\kappa_{c2}^{-2})}{h_1+h_2}. \tag{29}$$

Equation (27) may be solved by asymptotic matching, see [11], resulting in

$$\theta = \frac{1}{2}\left[1 - \mathrm{sgn}\,(b)\left(\frac{1}{2} + \mathrm{sgn}\,(\chi)\big(C(\chi)+S(\chi)\big) - C^2(\chi) - S^2(\chi)\right)\right], \tag{30}$$

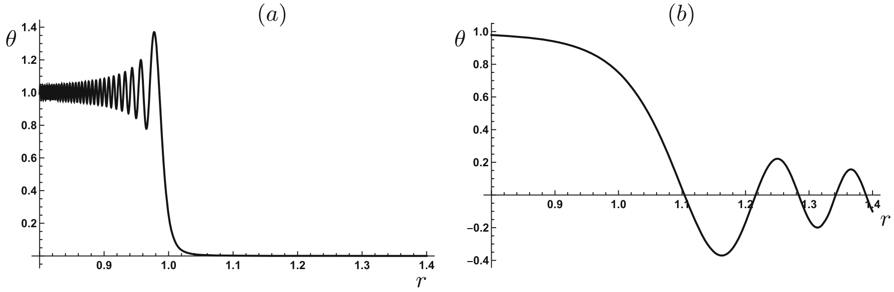

Fig. 2 Quasi-front type behaviour for a two-layered coating: (**a**) rubber-nylon coating on polysterene substrate; (**b**) nylon-polysterene coating on a rubber substrate

where $\chi = (s - \tau_R) \mathrm{sgn}\, b / \sqrt{2 h_L \tau_R}$ and $C(x)$ and $S(x)$ denote the Fresnel integrals. Illustrations of the solution (30) is presented below in Fig. 2, showing dependence of θ on s, with $t_R = 1$, $h_1 = 0.1$, $h_2 = 0.2$. The material properties are taken as follows: for rubber the Young's modulus $E = 0.1\, GPa$, volume mass density $\rho = 930\,\mathrm{kg/m^3}$, Poisson ratio $\nu = 0.49$, for nylon $E = 2.95\,\mathrm{GPa}$, $\rho = 1130\,\mathrm{kg/m^3}$, $\nu = 0.39$, for polystyrene $E = 3.1\,\mathrm{GPa}$, $\rho = 1040\,\mathrm{kg/m^3}$, $\nu = 0.35$. As may be seen from the graphs, there are possibilities of receding and advancing quasi-fronts, as noticed previously in [11], associated with the local min/max of the phase velocity at the Rayleigh wave speed in the long-wave limit. Moreover, the velocity of oscillations could also differ on the material parameters. In case of the coating involving soft rubber layer (with contrast in stiffness between rubber and polystyrene exceeding 30), the oscillations of the quasi-front are rapid, whereas in case of a soft rubber substrate, the oscillations are relatively slow.

6 Concluding Remarks

The methodology of hyperbolic-elliptic models for surface wave field has been extended to the case of a half-space coated by a vertically inhomogeneous layer. Further developments may include analysis of other types of boundary conditions [16], near-resonant regimes of moving loads [17], anisotropy [18], as well as a more general treatment of a vertically inhomogeneous half-space, see [19].

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