Stabilizing Regulator in One Class of Continuous System with Control Constraints

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***Abstract*—The article considers the problem of constructing a stabilizing control for one class of nonlinear continuous systems with constraints on controls, in which the right-hand sides are linear in state and formally linear in control, but the coefficients of the matrix under control can be dependent on the coordinates of the state vector. Feedback in the work is built by introducing a quadratic quality criterion with a weight matrix for a state with coefficients depending on the coordinates of the state vector, i.e. by solving an auxiliary optimal control problem, where the SDRE approach can be applied in which the corresponding matrix Riccati equation is solved, the matrices in which can be state-dependent. But for this class of control problems, the article proposes an approach to constructing a stabilizing control by the extension method of V.F. Krotov, which makes it possible not to recalculate the matrix of gains in the feedback loop at each step of calculating the trajectories of the closed-loop system. In this work, it is possible to prove the asymptotic stability of the zero equilibrium position of a closed-loop system by constructing the Lyapunov function, which is the sum of the classical positive definite quadratic form for a linear control system. At the end of the work, a numerical experiment is presented that illustrates the statements using the example of stabilization in a three-sector economic system.**

***Index Terms*—nonlinear control systems, SDRE approach, Lagrange multiplier method, stabilizing controller.**

I. INTRODUCTION

In control theory, much attention is paid to the problem of studying stability and the problems of the stabilization in nonlinear control systems. In applied problems, there are various approaches to the construction of control laws in nonlinear feedback system s. In mathematical economics, there are models of control problems for economic systems, the dynamics of state variables in which is described by nonlinear systems of ordinary differential equations [1]. Recently, new control algorithms for nonlinear systems have appeared using the formal linear structure by state and control on the base of so called SDRE approach([2]- [5]). Here the application of such approach is applied to the three-sector economic system and we consider the problem of constructing a stabilizing control in three-sector economic system with constraints on controls, in which the right-hand sides are linear in state, and the matrix of coefficients is constant and formally linear in control, but the coefficients of the matrix under control can be dependent on the coordinates of the vector states.

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II. STATEMENT OF THE PROBLEM.

Let the nonlinear controlled system have the following form

$$\dot{y} = Ay + B(y)u, \quad y(t_0) = y_0 \tag{1}$$

where $y(t) \in R^n$, $u(t) \in R^m$ the state and control vectors for $t \in [t_0, \infty)$, respectively, A is the constant matrix. The matrix coefficients $B(y)$ are bounded and continuously differentiable with respect to y, y_0 - the given initial value; t_0 - the given initial moment of time, the control $u(t)$ is a piecewise continuous vector function and at each moment of time, it satisfies the constraints

$$u(t) \in U(T) = \{u : \gamma_1(t) \leq u \leq \gamma_2(t),\, t \in [t_0, \infty)\} \tag{2}$$

The problem: it is required to find a stabilizing control $u(y, t)$.

To find such a control, we will use the auxiliary optimal control problem wits the quadratic functional

$$J(u) = \frac{1}{2} \int_{t_0}^{\infty} [y^T Q(y) y + u^T R u]\, dt, \tag{3}$$

where $R > 0$ is a given constant positive definite matrix, $Q(y)$ is a positive semi definite matrix at $y \in R^n$, the specific form of which is defined below and, in this case, its coefficients can be functions of the state.

III. STABILIZING CONTROL.

Problem (1) - (3) will be solved using the SDRE technique [3], in which the key place is occupied by the solution of the algebraic matrix Riccati equation with state-dependent coefficients. Let us show that due to the specifics of the problem, the matrix $Q(y)$ can be chosen so that the corresponding Riccati equation for calculating the matrix of feedback gains will have constant coefficients.

To solve the stabilization problem, we use the principle of liberation from differential constraints when solving extremal problems, according to the scheme of the Lagrange multiplier method. Let us apply the extension principle [6],[7], the

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essence of which is to replace the original optimal control problem with another, where the latter is determined on a wider set of admissible controls.

As a result, to solve the problem, we add to the expression for the functional (3) the system of differential equations (1) with the Lagrange multiplier $\lambda = Ky$, as well as the expression $\lambda_1^T(t)[\gamma_1 - u(t)] + \lambda_2^T(t)[u(t) - \gamma_2]$, where $\lambda_1(t) \geq 0$, $\lambda_2(t) \geq 0$. Those the factor Ky removes the restrictions imposed on, in the form of a system of differential equations (1), and the functions $\{\lambda_1(t), \quad \lambda_2(t)\}$ - the corresponding restrictions imposed on control (2). The coefficients of the matrix $Q(y)$ in the quality criteria $J(u)$, are selected so that the pair (y, u) minimizes the sum of the weighted deviations of the vector values y from the zero equilibrium position and the weighted control costs. So, assuming that the pair (y, u) satisfies (1), (2), instead of the criterion, we introduce

$$L(y,u) = \int_{t_0}^{\infty} \{ \frac{1}{2} y^T Q(y) y + \frac{1}{2} u^T R u + (Ky)^T \times \times (Ay + B(y)u - \dot{y}) + \lambda_1^T[\gamma_1 - u] + \lambda_2^T[u - \gamma_2]\}\, dt \tag{4}$$

where $K > 0$ is a constant matrix.

We denote by $\Delta(t_0, y_0)$ the set of all admissible pais, i.e.

$$\Delta(t_0,y_0) = \{(y,u) : u(t) \in U(t), \dot{y} = Ay + B(y)u,\ t_0 < t < \infty, \quad y(t_0) = y_0\}. \tag{5}$$

We define the factors $\lambda_1(t) \geq 0$, $\lambda_2(t) \geq 0$, in such a way that the following conditions

$$\lambda_1^T(\gamma_1 - \tilde{u}) = 0, \quad \lambda_2^T(\tilde{u} - \gamma_2) = 0 \tag{6}$$

are fulfilled on the optimal pair $(\tilde{y}(t), \tilde{u}(t)) \in \Delta(t_0, y_0)$. We denote

$$M(y,u,t) = \frac{1}{2} y^T Q(y) y + \frac{1}{2} u^T R u + (Ky)^T (Ay + + B(y)u) + \lambda_1^T[\gamma_1 - u] + \lambda_2^T[u - \gamma_2]. \tag{7}$$

From the condition $\frac{\partial M}{\partial u} = 0$ we find the control $u(y,t)$

$$u(y,t) = -R^{-1}[B^T(y)Ky - \lambda_1 + \lambda_2] = \omega(y,t) + \varphi(y,t). \tag{8}$$

where $\omega(y,t) = -R^{-1}[B^T(y)Ky(t)$,
$\varphi(y,t) = -R^{-1}[-\lambda_1(y,t) + \lambda_2(y,t)]$.

We make the choice as follows

$$\lambda_1(y,t) = -R \cdot inf(0, \omega(y,t) - \gamma_1(t)), \lambda_2(y,t) = -R \cdot inf(0, \gamma_2(t) - \omega(y,t)), \tag{9}$$

$$\varphi(y,t) = -R^{-1}[-\lambda_1(y,t) + \lambda_2(y,t)], \tag{10}$$

Now we will define the matrix K as a solution to the equation

$$KA + A^T K - KB(y)R^{-1}B^T(y)K + Q(y) = 0.$$

Selecting $Q(y)$ in the last equation in the form of the following matrix

$$Q(y) = KB(y)R^{-1}B^T(y)K - KB^sR^{-1}(B^s)^TK + Q_1, \tag{11}$$

where $B^s = B(0)$ and $Q_1 > 0$ is a constant matrix such that $Q(y) > 0$ it holds for all $y \in R^n$ and we obtain for K the algebraic equation with constant coefficients

$$-KA - A^TK + KB^sR^{-1}(B^s)^TK + Q_1 = 0, \tag{12}$$

The use of the extension method is illustrated here in the following statement.

Lemma. Let the conditions be satisfied:

1. Pair $(y(t), u(t)) \in \Delta(t_0, y_0)$.

2.There are functions $\lambda_1(y,t) \geq 0$, $\lambda_2(y,t) \geq 0$ satisfying conditions (6), (9).

3. There is a constant matrix $K > 0$ such that the solution of equation (1) along control (8) tends to zero at $t \longrightarrow \infty$.

Then, for admissible and optimal pairs $(y(t), u(t)) \in \Delta(t_0, y_0)$, $(\tilde{y}(t), \tilde{u}(t)) \in \Delta(t_0, y_0)$ the following relations hold

$$L(y,u) = \int_{t_0}^{\infty} M(y(t), u(t), t)\, dt + + \frac{1}{2} y^T(t_0) K y(t_0) \leq J(u), \quad L(\tilde{y}, \tilde{u}) = J(\tilde{u})$$

Proof. Let us introduce the function

$$\nu(y,t) = \frac{1}{2} y^T K y \tag{13}$$

and calculate here the total derivative with respect to the independent variable $\dot{\nu} = (Ky)^T \dot{y}$. Since it is continuous in t, integrating (13) and summing the resulting expression with (4), we obtain

$$L(y,u) = \int_{t_0}^{\infty} \{ \frac{1}{2} y^T Q(y) y + \frac{1}{2} u^T R u + (Ky)^T (Ay + + B(y)u) - \frac{d\nu}{dt} + \lambda_1^T[\gamma_1 - u] + \lambda_2^T[u - \gamma_2]\}\, dt = = \int_{t_0}^{\infty} \{ \frac{1}{2} y^T Q(y) y + \frac{1}{2} u^T R u + (Ky)^T (Ay + + B(y)u) + \lambda_1^T[\gamma_1 - u] + \lambda_2^T[u - \gamma_2]\}\, dt + + \frac{1}{2} y(t_0)^T K y(t_0).$$

Taking into account (7), from this we have

$$L(y,u) = \int_{t_0}^{\infty} M(y(t), u(t), t) dt + \frac{1}{2} y(t_0)^T K y(t_0) \tag{14}$$

Criterion (15) is defined on a pair $(y(t), u(t)) \in \Delta(t_0, y_0)$. Now let us show that this pair satisfies the inequality $L(y,u) \leq J(u)$. Indeed, let $(y(t), u(t)) \in \Delta(t_0, y_0)$ then, by virtue of (1) and taking into account $\lambda_1^T(\gamma_1 - u) \leq 0$, $\lambda_2^T(u - \gamma_2) \leq 0$ from (4), we have

$$L(y,u)=\int_{t_0}^{\infty}\{\frac{1}{2}y^TQ(y)y+\frac{1}{2}u^TRu+(Ky)^T(Ay+$$
$$+B(y)u-\dot{y})+\lambda_1^T[\gamma_1-u]+\lambda_2^T[u-\gamma_2]\}\,dt= \quad (15)$$
$$=J(u)+\int_{t_0}^{\infty}\{\lambda_1^T[\gamma_1-u]+\lambda_2^T[u-\gamma_2]\}\,dt\le J(u)$$

Suppose $(\tilde{y}(t),\tilde{u}(t))\in\Delta(t_0,y_0)$ that it delivers the minimum value to criterion (4). Then, taking into account the value of the criterion (15) and condition (6), we obtain

$$L(\tilde{y},\tilde{u})=\int_{t_0}^{\infty}[\lambda_1^T(\gamma_1-\tilde{u})+\lambda_2^T(\tilde{u}-\gamma_2)]\,dt+J(\tilde{u})=J(\tilde{u})$$

that is the proof of the lemma is complete.

We introduce the conditions

I. Let there exist a constant positive definite matrix $Q_1>0$ such that for all $y\in R^n$ the matrix $Q(y)>0$.

II. The triplet of constant matrices $(A,B^s,Q_1^{1/2})$ is controllable and observable, where $B^s=B(0)$.

Theorem. Under the conditions for the existence of an admissible pair $(y(t),u(t))\in\Delta(t_0,y_0)$ and functions $\lambda_1(y,t)\ge 0$, $\lambda_2(y,t)\ge 0$ satisfying conditions (6), (9), as well as under conditions I and II, the zero equilibrium position in system (1) along control (8) is asymptotically stable if the constant matrix $K>0$ is the solution of the (12).

Proof. Condition II implies the existence of the constant root $K>0$ in (12). Closed-loop system (1) along the control (8) has the form

$$\frac{dy}{dt}=A_1(y)y(t)+B(y)\varphi(y,t),\qquad y(t_0)=y_0 \quad (16)$$

with a zero equilibrium position, because $A_1(y)=A-B(y)R^{-1}(B(y))^TK$. We introduce the Lyapunov function

$$V(y,t)=\frac{1}{2}y^T(t)Ky(t)+\frac{1}{2}\int_{t_0}^{\infty}\varphi^T(y,\tau)R\varphi(y,\tau)\,d\tau \quad (17)$$

and its total derivative along the trajectories of system (16) has the following expression

$$\begin{aligned}\frac{dV}{dt}&=y^T(t)K(A_1(y)y(t)+B(y)\varphi(y,t))-\\&-\frac{1}{2}\varphi^T(y,t)R\varphi(y,t)\,d\tau\end{aligned} \quad (18)$$

Taking into account the forms and the properties of quadratic forms we obtain from (18) that

$$\frac{dV}{dt}=\frac{1}{2}y^T(t)[KA_1(y)+A_1^T(y)K]y(t)+$$
$$+y^T(t)KB(y)\varphi(y,t)-\frac{1}{2}\varphi^T(y,t)R\varphi(y,t)=$$
$$=-\frac{1}{2}y^T(t)[Q(y)+KB(y)R^{-1}B^T(y)K]y(t)+$$
$$+y^T(t)KB(y)\varphi(y,t)-\frac{1}{2}\varphi^T(y,t)R\varphi(y,t)$$

Transforming the last expression, as a consequence $Q(y)>0$, $R>0$, it can be established that

$$\frac{dV}{dt}=-\frac{1}{2}y^T(t)Q(y)y(t)-\frac{1}{2}(\varphi(y,t)-R^{-1}B^T(y)\times$$
$$\times Ky(t))^TR(\varphi(y,t)-R^{-1}B^T(y)Ky(t))<0,\ \forall y\in R^n$$

Now, it is obvious that there exist scalar continuous non-decreasing functions $\omega_i(y)>0$, $\omega_i(0)=0$, $i=1,2,3$, satisfying the inequalities $\omega_1(\|y\|)\le V(y)\le\omega_2(\|y\|)$, $\dot{V}(y)\le-\omega_3(\|y\|)$. Therefore, according to the second Lyapunov method, the statement about the asymptotic stability of the zero equilibrium in system (16) holds.

IV. Algorithm for constructing a stabilizing feedback in problem (1), (2).

2.1. We select the weight matrices $Q(y)>0$, $R>0$ in the criterion (3).

2.2. We find a positive definite constant matrix Q_1 such that the triplet of constant matrices $\{A,B^s,Q_1^{1/2}\}$ is controllable and observable and, at the same time, for all $y\in R^n$ the matrix $Q(y)$ in (11) remains positive definite.

2.3. Find the constant matrix $K>0$ as a solution to the Riccati equation (12).

2.4. We calculate the feedback control by formulas (8), (9).

So, the stabilizing control is found by solving the inverse problem of the weight matrix finding for the functional (3) with the help of the auxiliary optimal control problem solution in order to obtain a matrix of feedback gains in $u(y,t)$. A similar technique for a finding of weight matrix for state variables in the performance criterion when a construction a stabilizing controller according to the algorithm for a solution of linear quadratic optimal control problems was proposed in [5] and was also used in [8] - [9].

V. Numerical experiments

Let us consider a control problem for an economic three-sector structure, the equations of dynamics in which there is a system of three ordinary differential equations with coefficients depending on the state.

$$\frac{dy}{dt}=Ay+B(y)u,\quad y(t_0)=y_0,\quad t\in[t_0,\infty),$$

where

$$B(y)=\begin{pmatrix}A_1(y_1+k_1^s)^{\alpha_1} & 0 & 0\\ 0 & A_1(y_1+k_1^s)^{\alpha_1} & 0\\ 0 & 0 & A_1(y_1+k_1^s)^{\alpha_1}\end{pmatrix}$$

$$A=\begin{pmatrix}-\lambda_1 & 0 & 0\\ 0 & -\lambda_2 & 0\\ 0 & 0 & -\lambda_0\end{pmatrix},$$

we obtain the form of equations.

Here are the results of calculations for one example in the framework of the conditions of the theorem.

Let there be the following initial data given in

Let the vector of initial states $y(t_0)=(-700,\,-300,\,300)^T$ be also given, and the matrices R, Q_1, B^s have the following form

Table I
PARAMETER VALUES FOR A THREE-SECTOR ECONOMIC MODEL

i	α_i	β_i	λ_i	A_i	k_i^s
0	0.46	0.39	0.05	6.19	966.4430
1	0.68	0.29	0.05	1.35	2410.1455
2	0.49	0.52	0.05	2.71	1090.1238

$$R = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 16 \cdot 10^{-4} & 0 & 0 \\ 0 & 8 \cdot 10^{-4} & 0 \\ 0 & 0 & 8 \cdot 10^{-4} \end{pmatrix},$$

$$B^s = B(0) = \begin{pmatrix} A_1 (k_1^s)^{\alpha_1} & 0 & 0 \\ 0 & A_1 (k_1^s)^{\alpha_1} & 0 \\ 0 & 0 & A_1 (k_1^s)^{\alpha_1} \end{pmatrix},$$

Using the matrix algebraic Riccati equation with constant coefficients (12), we define the matrix

$$K = \begin{pmatrix} 0.2033 \cdot 10^{-2} & 0 & 0 \\ 0 & 0.1094 \cdot 10^{-2} & 0 \\ 0 & 0 & 0.1090 \cdot 10^{-2} \end{pmatrix}$$

According to the theorem, we have that the matrix K is positive definite, the Lyapunov function $V > 0$ of the form (17), and its total derivative $\frac{dV}{dt} < 0$ along the trajectories of system (16). Consequently, the zero equilibrium position of system (16) along control (8) is asymptotically stable.

Let us introduce constraints on the controls:

$$-0.41 \le u_1 \le 0.4524,$$
$$-0.41 \le u_2 \le 0.49,$$
$$-0.41 \le u_3 \le 0.48$$

Using the values of the parameters for the three-sector economic model, we construct graphs of trajectories of system (16) and graphs of optimal control with constraints (8).

From Fig. 2 one can see that the coordinates of the control vector take in values on the boundary and inside the admissible point set.

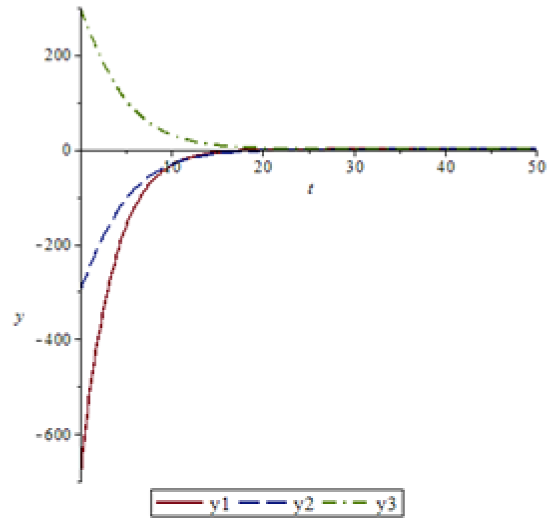

Fig. 1. Graphs of trajectories.

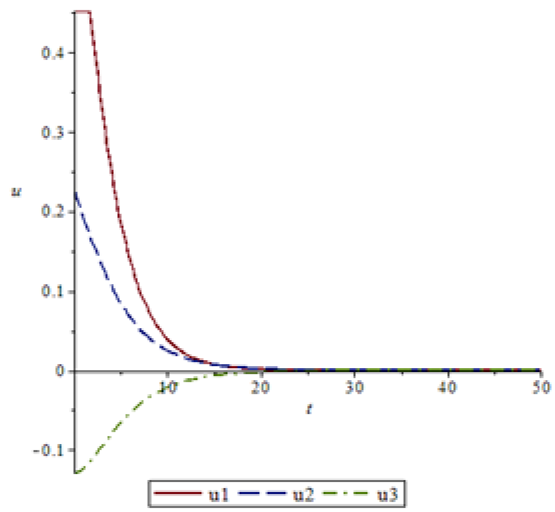

Fig. 2. Graphs of stabilizing controls.

CONCLUSION.

The problem of constructing a stabilizing control for a nonlinear control system with control constraints is considered, where the right-hand sides in the system are linear in the state and formally linear in the control, but the coefficients of the matrix under control can be dependent on the coordinates of the state vector. The stabilizing control for the considered nonlinear system is based on the solution of the inverse problem, which consists in finding the coefficients of the weight matrix in quality criterion and then solving the auxiliary optimal control problem in the form of feedback. Using the second Lyapunov method, the asymptotic stability of the closed-loop system is established. For the considered class of problems, a new approach to constructing of a stabilizing control is proposed, where used by the extension method in the presence of control constraints and allows not to recalculate, in contrast to the SDRE approach, at each step of calculating the trajectories, the matrix of gains.

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