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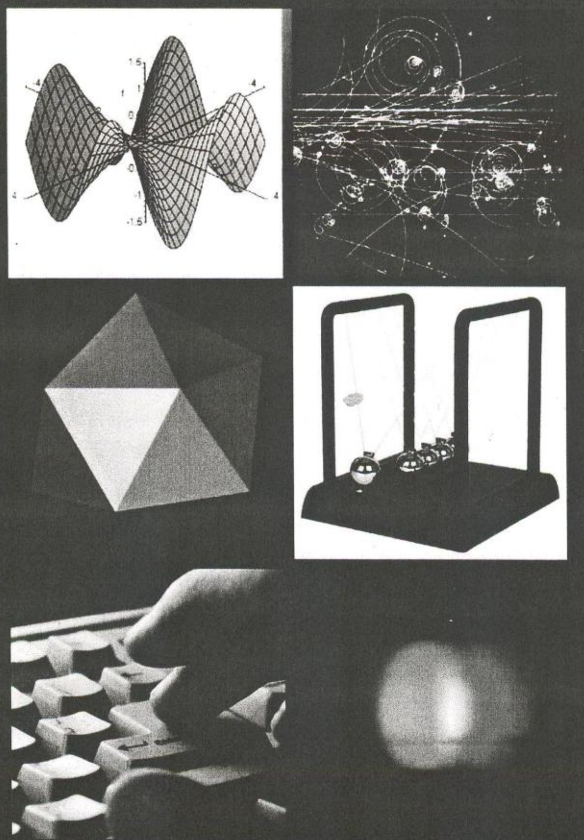

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Multivariate analogues of the Cauchy-Riemann system and representations of a solution through harmonic functions

Abstract. At present there exist a large variety of multivariate generalizations of holomorphic functions. The most general is a four-dimensional generalization of the Cauchy-Riemann system. In this paper a five-dimensional generalization of holomorphic vector has been obtained for the first time by introducing the two quaternion function and quaternion differentiation. Riemann-Hilbert problem and one problem in the layer are studied considering a holomorphic vector quaternion through harmonic function and its derivatives. Till now the five-dimensional generalization of holomorphic vector was unknown. Now, we will build the five-dimensional analogue of the Cauchy-Riemann system. The solution of Riemann-Hilbert problem is obtained in five dimensional half-space.
Keywords: Cauchy-Riemann system, Moisil-Teodorescu system, generalization of the Cauchy-Riemann system, quaternion, problem of Riemann-Hilbert.

Introduction

In order for the single-valued function $w = u + iv$, defined in a domain $D \in R^2$, to be holomorphic it is necessary and sufficient that its real and imaginary parts satisfy equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0. \qquad (1)$$

System from [1]

$$\begin{cases} u_x + v_y + w_z = 0, & s_x - v_z + w_y = 0, \\ s_y + u_z - w_x = 0, & s_z - u_y + v_x = 0, \end{cases} \qquad (2)$$

is a three-dimensional generalization of the system (1). Since every function s, u, w, v is a harmonic function, in literature solutions of this system are called holomorphic vectors, whereas systems (1) and (2) are called Cauchy-Riemann system and Moisil-Theodorescu system respectively. Till now the five-dimensional generalization of holomorphic vector was unknown. Now, we will build the five-dimensional analogue of the Cauchy-Riemann system.

In [3] system (2) was obtained by the following: it is known that the gradient of the solution $U(x, y, z)$ of the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \qquad (3)$$

$\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) = (u, v, w,)$ satisfies the system

$$\begin{cases} u_x + v_y + w_z = 0, & -v_z + w_y = 0, \\ u_z - w_x = 0, & -u_y + v_x = 0. \end{cases} \qquad (4)$$

$$\begin{cases} u_x + v_y + w_z = 0, & -v_z + w_y = 0, \\ u_z - w_x = 0, & -u_y + v_x = 0. \end{cases} \tag{4}$$

Let us introduce the fourth unknown function s as following:

$$\begin{cases} u_x + v_y + w_z = 0, & s_x - v_z + w_y = 0, \\ s_y + u_z - w_x = 0, & s_z - u_y + v_x = 0, \end{cases} \tag{5}$$

and Moisil-Theodorescu system is obtained (2).

If to the left side of each equation of the system (5) s_t, u_t, v_t, w_t are added, then obtained system will be a system which might be met in the theory of holomorphic quaternion.

$$\begin{cases} s_t - u_x - v_y - w_z = 0, & u_t + s_x - v_z + w_y = 0, \\ v_t + s_y + u_z - w_x = 0, & w_t + s_z - u_y + v_x = 0, \end{cases} \tag{6}$$

System (6) is also called a system of Moisil-Theodorescu and is a four-dimensional generalization of the Cauchy-Riemann system.

One more method of building this function is described in [4]. In this work a system of the first order differential equations in R^4 is considered.

$$\sum_{j=1}^{4} B_j \frac{\partial U}{\partial x_j} = 0, \tag{7}$$

where B_j are constant complex matrices of the size 2×2, unknown vector $U = (u, v)$ – column taken from the complex functions $u = u(x), v = v(x), x = (x_1, x_2, x_3, x_4) \in R^4$. This system is also called a four-dimensional generalization of the Cauchy-Riemann system, if the components u, v for each U are harmonic functions. Now, let us denote through $u = (S, U, V, W)$ a vector-column composed from real and imaginary parts of the components of the solution U of the original system (1):

$$S = Re u, U = Im u, V = Re v, W = Im v$$

and after homotopic classification system (7) is written in a form

$$\begin{cases} s_t + u_x + \rho b_1 v_y - \rho b_2 w_y - \rho b_2 v_z - \rho b_1 w_z = 0, \\ u_t - s_x + \rho b_2 v_y + \rho b_1 w_y + \rho b_1 v_z - \rho b_2 w_z = 0, \\ v_t - w_x - \rho b_1 s_y - \rho b_2 u_y + \rho b_2 s_z - \rho b_1 u_z = 0, \\ w_t + v_x + \rho b_2 s_y - \rho b_1 u_y + \rho b_1 s_z + \rho b_2 u_z = 0, \end{cases} \tag{8}$$

where $b = b_1 + ib_2$ is an arbitrary complex number, $\rho = (b_1^2 + b_2^2)^{-1}$.

In this system assumed that $b_1 = 1,\ b_2 = 0$ and that S, U, V, W are independent from t, then system (5) of Moisil-Theodorescu is obtained, which is the unique three-dimensional generalization of Cauchy-Riemann system. Assumed $b_1 = 0,\ b_2 = 1$ and S, U, V, W are functions of all four arguments t, x, y, z, then the system which is a four-dimensional generalization of the Cauchy-Riemann system is obtained. It might be met in a theory of holomorphic quaternion [5] [6].

From the system (8) it is easy to obtain the three-dimensional system of Moisil-Theodorescu [7] given that S, U, V, W are independent from t.

$$\begin{cases} u_x + v_y + b_1 w_z - b_2 s_z = 0, \\ v_x - v_y + b_1 s_z + b_2 w_z = 0, \\ w_x - s_y - \rho b_1 u_z - \rho b_2 v_z = 0, \\ s_x + w_y - \rho b_1 v_z + \rho b_2 u_z = 0, \end{cases} \tag{9}$$

Hereof given that $b_1 = 1,\ b_2 = 0$ Moisil-Theodorescu system (5) is obtained. Let us rewrite the system (9) in a form

$$\begin{pmatrix} 2\frac{\partial}{\partial \xi} & b\frac{\partial}{\partial z} \\ -\bar{b}\frac{\partial}{\partial \bar{z}} & 2\frac{\partial}{\partial \bar{\xi}} \end{pmatrix}\begin{pmatrix} \rho \\ q \end{pmatrix} = 0\,, \qquad (9^*)$$

where ξ=x+iy, $\frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{\xi}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, $b = b_1 + ib_2$, $\rho = (b_1^2 + b_2^2)^{-1}$, $\rho = s + iu,\ q = w + iv,$ and the following representation from [8] through the derivatives of the harmonic functions $G(x,y,z)$ and $\tau(x,y,z)$ is obtained:

$$u = b_1 G_z - b_2 \tau_z, \qquad v = b_1 G_z + b_2 G_z,$$
$$w = -G_x - \tau_y, \qquad s = -\tau_x + G_y. \qquad (10)$$

Now let us build the general representation of the solution of the system (8) through derivatives of two arbitrary harmonic functions. For that purpose let us consider two independent complex variables $\zeta = t + ix$, $\eta = y + iz$ and two complex unknown functions U=s+iu, V=v+iw. Then, system (8) might be written in a form

$$\frac{\partial U}{\partial \zeta} + \rho b\frac{\partial V}{\partial \bar{\zeta}} = 0, \quad -\rho\bar{b}\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \bar{\zeta}} = 0, \qquad (11)$$

where

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial t} - i\frac{\partial}{\partial x}\right), \quad \frac{\partial}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial x}\right)$$

$$\frac{\partial}{\partial \bar{\eta}} = \frac{1}{2}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right) \quad \frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial y} - i\frac{\partial}{\partial z}\right).$$

Considering complex harmonic function of four variables t,x,y,z :

$$\varphi(t,x,y,z) = G(t,x,y,z) + i\,\tau(t,x,y,z),$$

then general solution of the system (11) is given by the formula

$$U = b\frac{\partial \varphi}{\partial \bar{\zeta}}, \quad V = \frac{\partial \varphi}{\partial \eta}.$$

Separating real and imaginary parts, the representation of the solution of the system (8) through the derivatives of harmonic functions G and τ might be obtained:

$$\begin{aligned} S &= b_1(G_t - \tau_x) - b_2(\tau_t + G_x), \\ u &= b_1(\tau_t + G_x) + b_2(G_t - \tau_x), \\ v &= G_y + \tau_z, \quad w = \tau_y - G_z. \end{aligned} \tag{12}$$

In a three-dimensional case for a system (9) the famous problem of Riemann-Hilbert is solved in [8]. Let us consider the problem in a domain $D \subset R^3$: it is necessary to find the area in a domain $D \subset R^2$ so that the regular solution of the system (9) has boundary conditions:

$$\alpha_j u + \beta_j v + \gamma_j w + \delta_j s = f_j, \quad j = 1,2, \tag{13}$$

here $\alpha_j, \beta_j, \gamma_j, \delta_j, f_j$ are continuous functions which satisfy the condition of Hölder. It is known that the property of the boundary-value problem being regularized is necessary and sufficient condition of the Noetherian property of the operator. For the system of type (8) there is no single boundary-value problem which might be regularized in none of the bounded domains [5] [6].

In the work [9] the correctness of one boundary-value problem for the system of the type (8) is shown with the smaller members in an infinite domain $D \equiv \{0 < t < h, \ -\infty < x, y, z < +\infty\}$. Let us denote through T the operator which displays the vector $u \equiv (s, u, v, w)$ onto the left sides of the equation (8). It is needed to find the solution of the non-uniform system in a domain D

$$TU + A(x)U = F(x), \tag{14}$$

$U(x) \in C^{\infty}(\overline{D}) \cap W_2^2(D)$, a $A(x)$ is given fourth order quadratic matrix and $F(x)$ is given four-dimensional vector-function. Components $U(x)$ satisfy the following conditions on the boundaries:

$$s(\Gamma) = U(\Gamma) = v(t = 0) = w(t = h) = 0. \tag{15}$$

Theorem. If a matrix $A(x) \in C(\overline{D})$ and if there exists some positive number $\delta < \frac{\sqrt{2}}{h}$ such that $\|AU\|_0 \le \delta \|U\|_0$, then for any vector-function $F(x) \in L_2(D)$ problem (14) - (15) has unique solution $U(x) \in W_2^1(D)$.

Till now the five-dimensional generalization of holomorphic vector was unknown. Now, we will build the five-dimensional analogue of the Cauchy-Riemann system.

Let

$b = b_1 + ib_2 + jb_3 + kb_4, \quad \overline{b} = b_1 - ib_2 - jb_3 - kb_4$ be a quaternion and conjugate quaternion numbers, whereas $\partial = \partial_{x1} + i\partial_{x2} + j\partial_{x3} + k\partial_{x4}$ $\overline{\partial} = \partial_{x1} - i\partial_{x2} - j\partial_{x3} - k\partial_{x4}$ are quaternions of the differentiation, $b_l (l = 1,2,3,4)$ - real constants, $\rho = (b_1^2 + b_2^2 + b_3^2 + b_4^2)^{-1}$, moreover, i, j, k - quaternion units which has properties as $ij = k, \ ki = j, \ jk = i, \ i^2 = j^2 = k^2 = -1$.

In a space R^5 of the variables $(x_1, x_2, x_3, x_4, x_5)$ let us consider the system of equations of the first order

$U = u_1 + iu_2 + ju_3 + ku_4, \quad V = u_5 + iu_6 + ju_7 + ku_8,$

$$\begin{cases} \dfrac{\partial U}{\partial x_5} + b\partial V = 0, \\ -\rho \bar{b}\bar{\partial} U + \dfrac{\partial V}{\partial x_5} = 0. \end{cases} \tag{16}$$

In this system each of the functions $u_1, u_2, u_3, ..., u_8$ is a harmonic function. We introduce a quaternion harmonic function from $x = (x_1, x_2, x_3, x_4, x_5)$:

$$u_l = \frac{\partial \varphi_l}{\partial x_5}, \ l = 1,2,3,4, \quad u_m = \rho(B_m, M_m), \ m = 5,6,7,8 \tag{17}$$

where

$$B_5 = (b_1, b_2, b_3, b_4), \ M_5 = (m_1, m_2, m_3, m_4), \ B_6 = (b_1, -b_2, -b_3, -b_4),$$
$$M_6 = (m_2, m_1, m_3, m_4), \ B_7 = (b_1, b_2, -b_3, -b_4), \ M_7 = (m_3, m_4, m_1, m_2),$$
$$B_8 = (b_1, -b_2, b_3, -b_4), \ M_8 = (m_4, m_3, m_2, m_1),$$
$$m_1 = \frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} + \frac{\partial \varphi_3}{\partial x_3} + \frac{\partial \varphi_4}{\partial x_4}, \quad m_2 = \frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2} + \frac{\partial \varphi_4}{\partial x_3} - \frac{\partial \varphi_3}{\partial x_4},$$
$$m_3 = \frac{\partial \varphi_3}{\partial x_1} + \frac{\partial \varphi_4}{\partial x_2} - \frac{\partial \varphi_1}{\partial x_3} + \frac{\partial \varphi_2}{\partial x_4}, \quad m_4 = \frac{\partial \varphi_4}{\partial x_1} - \frac{\partial \varphi_3}{\partial x_2} + \frac{\partial \varphi_2}{\partial x_3} - \frac{\partial \varphi_1}{\partial x_4}.$$

Using representation (17) Riemann-Hilbert problem about the finding of the regular in the half-space $R_t^5 \equiv \{x_5 > 0\}$ of the solution $W = (U, V) = (u_1, u_2, ..., u_8)$ of the system (16) which satisfies the following conditions on a boundary $\Gamma \equiv \{x_5 = 0\}$:

$$\sum_{l=1}^{8} a_{lm} u_l = f_m, \ m = 1,2,3,4, \tag{18}$$

where a_{lm}, f_m $(l = 1,2,...8, \ m = 1,2,3,4)$ are given functions on boundary, comes down to the problem about the inclined derivative

$$\sum_{s=1}^{4} (\alpha_{sr}, \nabla \varphi_r) = f_r, \ r = 1,2,3,4, \tag{19}$$

where a_{sr} are defined vectors. Thus, we came to the problem of finding in a half-space $x_5 > 0$ a harmonic function $\varphi_l(x)$, $l = 1,2,3,4$ which satisfies the following conditions on a boundary

$$\frac{\partial \varphi_l}{\partial x_5} = f_l, \ l = 1,2,3,4. \tag{20}$$

Solution of the problem (20) which tends to 0 at infinity could be found by the formula from [10]:

$$\varphi_l(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{f_l(y)dy_1dy_2dy_3dy_4}{\sqrt{\sum_{i=1}^{4}(x_i-y_i)^2+x_5^2}}, \quad l=1,2,3,4. \tag{21}$$

Using formula (17) the unique solution of the problem can be obtained.

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