

ISSN 2518-1726 (Online),
ISSN 1991-346X (Print)

ҚАЗАҚСТАН РЕСПУБЛИКАСЫ
ҰЛТТЫҚ ҒЫЛЫМ АКАДЕМИЯСЫНЫҢ
Әль-фараби атындағы Қазақ ұлттық университетінің

Х А Б А Р Л А Р Ы

ИЗВЕСТИЯ

НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК
РЕСПУБЛИКИ КАЗАХСТАН
Казахский национальный университет
имени Аль-фараби

NEWS

OF THE NATIONAL ACADEMY OF SCIENCES
OF THE REPUBLIC OF KAZAKHSTAN
Al-farabi kazakh
national university

SERIES
PHYSICO-MATHEMATICAL

4 (326)

JULY-AUGUST 2019

PUBLISHED SINCE JANUARY 1963

PUBLISHED 6 TIMES A YEAR

ALMATY, KAZAKHSTAN

Б а с р е д а к т о р ы
ф.-м.ғ.д., проф., ҚР ҰҒА академигі **Ғ.М. Мұтанов**

Р е д а к ц и я а л қ а с ы:

Жұмаділдаев А.С. проф., академик (Қазақстан)
Кальменов Т.Ш. проф., академик (Қазақстан)
Жантаев Ж.Ш. проф., корр.-мүшесі (Қазақстан)
Өмірбаев У.У. проф. корр.-мүшесі (Қазақстан)
Жүсіпов М.А. проф. (Қазақстан)
Жұмабаев Д.С. проф. (Қазақстан)
Асанова А.Т. проф. (Қазақстан)
Бошқаев К.А. PhD докторы (Қазақстан)
Сұраған Д. корр.-мүшесі (Қазақстан)
Quevedo Hernando проф. (Мексика),
Джунушалиев В.Д. проф. (Қырғыстан)
Вишневский И.Н. проф., академик (Украина)
Ковалев А.М. проф., академик (Украина)
Михалевич А.А. проф., академик (Белорус)
Пашаев А. проф., академик (Әзірбайжан)
Такибаев Н.Ж. проф., академик (Қазақстан), бас ред. орынбасары
Тигиняну И. проф., академик (Молдова)

«ҚР ҰҒА Хабарлары. Физика-математикалық сериясы».

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Меншіктенуші: «Қазақстан Республикасының Ұлттық ғылым академиясы» РҚБ (Алматы қ.)
Қазақстан республикасының Мәдениет пен ақпарат министрлігінің Ақпарат және мұрағат комитетінде
01.06.2006 ж. берілген №5543-Ж мерзімдік басылым тіркеуіне қойылу туралы куәлік

Мерзімділігі: жылына 6 рет.
Тиражы: 300 дана.

Редакцияның мекенжайы: 050010, Алматы қ., Шевченко көш., 28, 219 бөл., 220, тел.: 272-13-19, 272-13-18,
<http://physics-mathematics.kz/index.php/en/archive>

© Қазақстан Республикасының Ұлттық ғылым академиясы, 2019

Типографияның мекенжайы: «Аруна» ЖК, Алматы қ., Муратбаева көш., 75.

Главный редактор
д.ф.-м.н., проф. академик НАН РК **Г.М. Мутанов**

Редакционная коллегия:

Джумадильдаев А.С. проф., академик (Казахстан)
Кальменов Т.Ш. проф., академик (Казахстан)
Жантаев Ж.Ш. проф., чл.-корр. (Казахстан)
Умирбаев У.У. проф. чл.-корр. (Казахстан)
Жусупов М.А. проф. (Казахстан)
Джумабаев Д.С. проф. (Казахстан)
Асанова А.Т. проф. (Казахстан)
Бошкаев К.А. доктор PhD (Казахстан)
Сураган Д. чл.-корр. (Казахстан)
Quevedo Hernando проф. (Мексика),
Джунушалиев В.Д. проф. (Кыргызстан)
Вишневский И.Н. проф., академик (Украина)
Ковалев А.М. проф., академик (Украина)
Михалевич А.А. проф., академик (Беларусь)
Пашаев А. проф., академик (Азербайджан)
Такибаев Н.Ж. проф., академик (Казахстан), зам. гл. ред.
Тигиняну И. проф., академик (Молдова)

«Известия НАН РК. Серия физико-математическая».

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Собственник: РОО «Национальная академия наук Республики Казахстан» (г. Алматы)

Свидетельство о постановке на учет периодического печатного издания в Комитете информации и архивов
Министерства культуры и информации Республики Казахстан №5543-Ж, выданное 01.06.2006 г.

Периодичность: 6 раз в год.

Тираж: 300 экземпляров.

Адрес редакции: 050010, г. Алматы, ул. Шевченко, 28, ком. 219, 220, тел.: 272-13-19, 272-13-18,
<http://physics-mathematics.kz/index.php/en/archive>

© Национальная академия наук Республики Казахстан, 2019

Адрес типографии: ИП «Аруна», г. Алматы, ул. Муратбаева, 75.

E d i t o r i n c h i e f
doctor of physics and mathematics, professor, academician of NAS RK **G.M. Mutanov**

E d i t o r i a l b o a r d :

Dzhumadildayev A.S. prof., academician (Kazakhstan)
Kalmenov T.Sh. prof., academician (Kazakhstan)
Zhantayev Zh.Sh. prof., corr. member. (Kazakhstan)
Umirbayev U.U. prof. corr. member. (Kazakhstan)
Zhusupov M.A. prof. (Kazakhstan)
Dzhumabayev D.S. prof. (Kazakhstan)
Asanova A.T. prof. (Kazakhstan)
Boshkayev K.A. PhD (Kazakhstan)
Suragan D. corr. member. (Kazakhstan)
Quevedo Hernando prof. (Mexico),
Dzhunushaliyev V.D. prof. (Kyrgyzstan)
Vishnevskiy I.N. prof., academician (Ukraine)
Kovalev A.M. prof., academician (Ukraine)
Mikhalevich A.A. prof., academician (Belarus)
Pashayev A. prof., academician (Azerbaijan)
Takibayev N.Zh. prof., academician (Kazakhstan), deputy editor in chief.
Tiginyanu I. prof., academician (Moldova)

News of the National Academy of Sciences of the Republic of Kazakhstan. Physical-mathematical series.

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Owner: RPA "National Academy of Sciences of the Republic of Kazakhstan" (Almaty)

The certificate of registration of a periodic printed publication in the Committee of information and archives of the Ministry of culture and information of the Republic of Kazakhstan N 5543-Ж, issued 01.06.2006

Periodicity: 6 times a year

Circulation: 300 copies

Editorial address: 28, Shevchenko str., of. 219, 220, Almaty, 050010, tel. 272-13-19, 272-13-18,
<http://physics-mathematics.kz/index.php/en/archive>

© National Academy of Sciences of the Republic of Kazakhstan, 2019

Address of printing house: ST "Aruna", 75, Muratbayev str, Almaty

МАЗМҰНЫ

Зазулин Д.М., Буртебаев Н., Петерсон Р.Ж., Артемов С.В., Игамов С., Керимкулов Ж.К., Алимов Д.К., Мухамеджанов Е.С., Насурлла Маулен, Сабидолда А., Насурлла Маржан, Ходжаев Р. Төмен энергиялардағы $P-^{12}C$ радиациялық карпуының жаңа нәтижелері.....	5
Жадыранова А.А. $n = 3$ және $N = 2$ жағдайлары үшін енгізілген жаңа жүйе $a_i, b_i, c_i, V_0 = 0$ болғандағы WDVV ассоциативтілік теңдеуінің иерархиясы.....	14
Шалданбаев А.Ш., Ақылбаев М.И., Шалданбаева А.А., Бейсебаева А.Ж. Кіші мүшелі толқын операторының спектралдік қасиеттері.....	22
Шалданбаев А.Ш., Шалданбаева А.А., Шалданбай Б.А. Потенциалы симметриялы, ал шекаралық шарттары ажырамайтын Штурм-Лиувилл операторының кері есебі туралы.....	30
Дасибеков А., Юнусов А.А., Юнусова А.А., Корганбаев Б.Н., Қуатбеков Н.А., Такибаева Г.А., Еспембетова Д.Н. Серпімділік қасиеттері бар және шектестік жағдайлары әртекті топрақ қабатың тығыздау.....	42
Мамырбаев Ө.Ж., Шаяхметова Ә.С., Сейсенбекова П.Б. Байес тәсіліне негізделген ақпараттық технологиялар бағытында білім алушылардың құзыреттілігін арттыру үшін зияткерлік ортаны құру әдістемесі.....	50
Калимолдаев М., Ахметжанов М., Қунелбаев М. Екі роторлық шағын гидроэлектростанция үшін сұйықтық ағынымен гидро турбиналы тордың энергетикалық өзара әрекеттесуін анықтау.....	59
Исахов А.А., Абай А., Омарова П.Т., Бекжигитова Ж.Е. Тұрғын үй аумағындағы ластаушы заттардың таралуының сандық моделдеуі.....	68
Бекболат Б., Кангужин Б., Токмагамбетов Н. Толқын теңдеуі үшін көп нүктелі аралас шекаралық есеп.....	76
Тасмамбетов Ж.Н., Раджабов Н., Убаева Ж.К. Үшінші ретті дербес туындылы екі дифференциалдық теңдеулердің бірлескен шешімі.....	83
Бекболат Б., Нурахметов Д. Б., Токмагамбетов Н., Аймал Раса Г. Х. Ойылған аймақтағы Лаплас операторының түбір функциялар жүйесінің минималдылығы	92
Марат Нуртас, Байшемиров Ж.Д. Семей полигоны аумағының температуралық режимін зерттеу және математикалық модельді және оның сандық шешімін сипаттау.....	110
Серикбаев Д., Токмагамбетов Н. Штурм-Лиувилл операторлы псевдо-параболалық теңдеу үшін кері есеп.....	122
Яковец А.Ф., Жаханишир А., Гордиенко Г.И., Жумабаев Б.Т., Литвинов Ю.Г., Абдрахманов Н. Ионосферадағы F2-қабатының вариация ауқымының максималды биіктігін айқындау статистикасы.....	129
Калимолдаев М., Ахметжанов М., Қунелбаев М. Тұрақты ток айнымалы кернеуге түрлендіру үшін түрлендіргіштікпен күн фотоэлементтерінің математикалық моделін әзірлеу және зерттеу.....	135
Жаутыков Б.О., Избасаров М., Покровский Н.С., Самойлов В.В., [Темиралиев Т.], Турсунов Р.А. 22,4 ГэВ/с және 32 ГэВ/с импульста аннигиляциямен байытылған $\bar{p}p$ - әсерлесудегі ядро-белсенді бөлшектердің корреляциясын зерттеу.....	143

СОДЕРЖАНИЕ

<i>Зазулин Д.М., Буртебаев Н., Петерсон Р.Ж., Артемов С.В., Игамов С., Керимкулов Ж.К., Алимов Д.К., Мухамеджанов Е.С., Насурлла Маулен, Сабидолда А., Насурлла Маржан, Ходжаев Р.</i> Новые результаты для радиационного захвата $p\text{-}^{12}\text{C}$ при низких энергиях.....	5
<i>Жадыранова А.А.</i> Иерархия уравнений ассоциативности WDVV для случая $n = 3$ и $N = 2$ при $V_0 = 0$ с новой системой a_i, b_i, c_i	14
<i>Шалданбаев А.Ш., Ақылбаев М.И., Шалданбаева А.А., Бейсебаева А.Ж.</i> О спектральных свойствах волнового оператора, возмущённого младшим членом.....	22
<i>Шалданбаев А.Ш., Шалданбаева А.А., Шалданбай Б.А.</i> Обратная задача оператора Штурма-Лиувилля с не разделенными краевыми условиями и симметричным потенциалом.....	30
<i>Дасибеков А., Юнусов А.А., Юнусова А.А., Корганбаев Б.Н., Куатбеков Н.А., Такибаева Г.А., Еспембетова Д.Н.</i> Уплотнение слоя грунта, обладающего упругим свойством и неоднородностью граничных условий.....	42
<i>Мамырбаев О.Ж., Шаяхметова А.С., Сейсенбекова П.Б.</i> Методология создания интеллектуальной среды повышения компетентности обучающегося по направлению it на основе байесовского подхода.....	50
<i>Калимолдаев М., Ахметжанов М., Кунелбаев М.</i> Определение силового взаимодействия гидротурбинной сетки с потоком жидкости для двухроторной микро ГЭС.....	59
<i>Исахов А.А., Абай А., Омарова П.Т., Бекжигитова Ж.Е.</i> Численное моделирование распространение загрязняющих веществ в жилых районах.....	68
<i>Бекболат Б., Кангузин Б., Токмагамбетов Н.</i> О многоточечной задаче смешанной границы для волнового уравнения.....	76
<i>Тасмамбетов Ж.Н., Раджабов Н., Убаева Ж.К.</i> Совместное решение двух дифференциальных уравнений в частных производных третьего порядка.....	83
<i>Бекболат Б., Нурахметов Д. Б., Токмагамбетов Н., Аймал Раса Г. Х.</i> О минимальности систем корневых функций оператора Лапласа в проколотой области.....	92
<i>Марат Нуртас, Байшемиров Ж.Д.</i> Исследование температурного режима территории семипалатинского полигона и описание математической модели и его численного решения.....	110
<i>Серикбаев Д., Токмагамбетов Н.</i> Обратная задача для псевдо-параболического уравнения для оператора Штурма-Лиувилля.....	122
<i>Яковец А.Ф., Жаханищир А., Гордиенко Г.И., Жумабаев Б.Т., Литвинов Ю.Г., Абдрахманов Н.</i> Статистика размаха вариаций высоты максимума F2-слоя ионосферы.....	129
<i>Калимолдаев М., Ахметжанов М., Кунелбаев М.</i> Разработка и исследование математической модели солнечного фотопреобразователя с инвертором для преобразования постоянного тока в переменное напряжение.....	135
<i>Жаутыков Б.О., Избасаров М., Покровский Н.С., Самойлов В.В., Темиралиев Т., Турсунов Р.А.</i> Исследование корреляций ядерно-активных частиц в $\bar{p}p$ -взаимодействиях, обогащенных аннигиляцией при импульсе 22,4 ГэВ/с и 32 ГэВ/с.....	143

CONTENTS

<i>Zazulin D. M., Burtabayev N., Peterson R. J., Artemov S. V., Igamov S., Kerimkulov Zh.K., Alimov D. K., Mukhamejanov E. S., Nassurlla Maulen, Sabidolda A., Nassurlla Marzhan, Khojayev R.</i> New results for the P- ¹² C radiative capture at low energies	5
<i>Zhadyranova A.A.</i> Hierarchy of WDVV associativity equations for $n = 3$ and $N = 3$ case when $V_0 = 0$ with new system a_i, b_i, c_i	14
<i>Shaldanbayev A.Sh., Akylbayev M.I., Shaldanbayeva A.A., Beisebayeva A.Zh.</i> On the spectral properties of a wave operator perturbed by a lower-order term	22
<i>Shaldanbayev A.Sh., Shaldanbayeva A.A., Shaldanbay B.A.</i> Inverse problem of the Sturm-Liouville operator with unseparated boundary value conditions and symmetric potential	30
<i>Dasibekov A., Yunusov A.A., Yunusova A.A., Korganbayev B.N., Kuatbekov N.A., Takibayeva G.A., Espembetova D.N.</i> Sealing of a layer of a sound with effect of elastic properties and inhomogeneous border conditions	42
<i>Mamyrbayev O. Zh., Shayakhmetova A.S., Seisenbekova P.B.</i> The methodology of creating an intellectual environment of increasing the competence of students based on a bayesian approach	50
<i>Kalimoldayev M., Akhmetzhanov M., Kunelbayev M.</i> Determination of the power interaction of the hydro turbine GRID with a fluid flow for a double-rotor micro hydro power plant	59
<i>Issakhov A.A., Abay A., Omarova P.T., Bekzhigitova Zh. E.</i> Numerical modeling of contaminating substances distribution in residential areas	68
<i>Bekbolat B., Kanguzhin B., Tokmagambetov N.</i> To the question of a multipoint mixed boundary value problem for a wave equation	76
<i>Tasmambetov Zh.N., Rajabov N., Ubayeva Zh.K.</i> The coordinated solution of two differential equations in private derivatives of the third order	83
<i>Bekbolat B., Nurakhmetov, D. B., Tokmagambetov N., Aimal Rasa G. H.</i> On the minimality of systems of root functions of the Laplace operator in the punctured domain	92
<i>Marat Nurtas, Baishemirov Zh.D.</i> Investigation of the temperature regime of the territory of the semipalatinsk polygon and description of the mathematical model and its numerical solution	110
<i>Serikbaev D., Tokmagambetov N.</i> An inverse problem for the pseudo-parabolic equation for a Sturm-Liouville operator	122
<i>Yakovets A.F., Jahanshir A., Gordienko G.I., Zhumabayev B.T., Litvinov Yu.G., Abdrakhmanov N.</i> Statistics of variation range of F2- layer maximum height of the ionosphere	129
<i>Kalimoldayev M., Akhmetzhanov M., Kunelbayev M.</i> Development and research of a mathematical model of a solar photo converter with an inverter for converting direct current to alternating voltage	135
<i>Izbasarov M., Pokrovsky N.S., Samoilov V.V., Temiraliyev T., Tursunov R.A., Zhautykov B.O.</i> Investigation of correlations of generated nuclear active particles in $\tilde{p}p$ - events enriched by annihilation at momenta 22.4 GeV/c AND 32 GeV/c	143

Publication Ethics and Publication Malpractice in the journals of the National Academy of Sciences of the Republic of Kazakhstan

For information on Ethics in publishing and Ethical guidelines for journal publication see <http://www.elsevier.com/publishingethics> and <http://www.elsevier.com/journal-authors/ethics>.

Submission of an article to the National Academy of Sciences of the Republic of Kazakhstan implies that the described work has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see <http://www.elsevier.com/postingpolicy>), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The National Academy of Sciences of the Republic of Kazakhstan follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/New_Code.pdf). To verify originality, your article may be checked by the Cross Check originality detection service <http://www.elsevier.com/editors/plagdetect>.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the National Academy of Sciences of the Republic of Kazakhstan.

The Editorial Board of the National Academy of Sciences of the Republic of Kazakhstan will monitor and safeguard publishing ethics.

Правила оформления статьи для публикации в журнале смотреть на сайтах:

www:nauka-nanrk.kz

<http://physics-mathematics.kz/index.php/en/archive>

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Редакторы *М. С. Ахметова, Т.А. Апендиев, Д.С. Аленов*
Верстка на компьютере *А.М. Кульгинбаевой*

Подписано в печать 10.08.2019.
Формат 60x881/8. Бумага офсетная. Печать – ризограф.
9,6 п.л. Тираж 300. Заказ 4.

NEWS

OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN

PHYSICO-MATHEMATICAL SERIES

ISSN 1991-346X

<https://doi.org/10.32014/2019.2518-1726.48>

Volume 4, Number 326 (2019), 92 – 109

UDK 517.958

B. Bekbolat¹, D. B. Nurakhmetov², N. Tokmagambetov³, G. H. Aimal Rasa⁴

^{1,3,4}Al-Farabi Kazakh National University, Almaty, Kazakhstan

²S.Seifullin Kazakh Agro Technical University, Astana, Kazakhstan

³Ghent University, Ghent, Belgium

^{1,3}Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan

bekbolat@math.kz, dauletkaznu@gmail.com,

niyaz.tokmagambetov@gmail.com, aimal.rasal4@gmail.ru

ON THE MINIMALITY OF SYSTEMS OF ROOT FUNCTIONS OF THE LAPLACE OPERATOR IN THE PUNCTURED DOMAIN

Abstract. In this paper, we consider the Laplace operator in a punctured domain, which generates a class of “new” correctly solvable boundary value problems. And for this class of problems the resolvent formula is obtained. Also described are meromorphic functions that generate the root functions of the class of problems studied. The main goal is to study the minimality of root function systems. The paper is a continuation of [8], where a description is given of correctly solvable boundary value problems for the Laplace operator in punctured domains. The Laplace operator in the punctured domain, which generates the class of “new” correctly solvable boundary value problems, is considered, and the resolvent formula is obtained for the generated problems, and meromorphic functions are described that induce systems of functions. One of these systems is a system of eigenfunctions and associated functions. The last section is devoted to the study of the minimality of the system of root functions.

Key words: Laplace operator, punctured domain, resolvent, meromorphic function, correctly solvable boundary value problems, root functions system, minimal system.

1 Introduction

Operators of the form $L + K$, where $L = \Delta$ is the Laplace operator, and K is the operator of multiplication by a generalized function, appeared in the physical works of the 30s in connection with the problem of scattering neutral particles on the nucleus, when the interaction is strong at small distances and negligibly small to medium to large [1]. The model potential of such an interaction is the Dirac δ -function. A mathematical study of the operator $-\Delta + \mu\delta(x)$ was undertaken by Berezin and Fadeev [2], Minlos and Fadeev [3], Berezin [4]. In [2, 3], the operator $-\Delta + \mu\delta(x)$ was understood as an extension of the operator $L_0 = -\Delta$ with the definition domain $D(L_0) = C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Interesting are the works [5, 6], where the following important question was studied: if the operator $-\Delta + q$ with a singular potential q is already defined, can it be approximated in some sense by operators with smooth potentials so that the corresponding operators approximate the original in the sense of resolvent convergence. The paper [7] investigated the spectral properties of the Schrödinger operator with point interactions using positive definite functions. For complete review, see marked work and links to them.

The paper is a continuation of [8], where a description is given of correctly solvable boundary value problems for the Laplace operator in punctured domains. The Laplace operator in the punctured domain, which generates the class of “new” correctly solvable boundary value problems, is considered, and the resolvent formula is obtained for the generated problems, and meromorphic functions are described that induce systems of functions. One of these systems is a system of eigenfunctions and associated functions. The last section is devoted to the study of the minimality of the system of root functions.

Consider a differential expression

$$\Delta W(x, y) := \frac{\partial^2 W(x, y)}{\partial x^2} + \frac{\partial^2 W(x, y)}{\partial y^2}$$

in the punctured area $\Omega_0 = \Omega \setminus \{(x_0, y_0)\}$, here $\Omega = \{x^2 + y^2 < 1\}$ and (x_0, y_0) is internal fixed point of area Ω . We Turn from the expression $\Delta W(x, y)$ to the operator L_{σ_1} in the space $\mathbb{L}_2(\Omega)$.

Denote by $\mathcal{D}$ the set of all functions

$$h(x, y) = h_1(x, y) + \alpha G(x, y, x_0, y_0), (x, y) \in \Omega_0,$$

where $\alpha \in \mathbb{R}$, $h_1 \in \mathbb{D} = \{h_1 \in \mathbb{W}_2^2(\Omega), h_1|_{\partial\Omega} = 0\}$. Here and after $G(x, y, x_0, y_0)$ is Green function of the Dirichlet problem for the Laplace operator in Ω [9].

For $h(x, y) \in \mathcal{D}$ we introduce functionals

$$\begin{aligned} \alpha(h) &= \frac{1}{2} \lim_{\delta \rightarrow +0} \left\{ \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial h(x_0+\delta, y)}{\partial x} - \frac{\partial h(x_0-\delta, y)}{\partial x} \right] dy + \right. \\ &\quad \left. \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial h(x, y_0+\delta)}{\partial y} - \frac{\partial h(x, y_0-\delta)}{\partial y} \right] dx \right\}, \\ \beta(h) &= \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} [h(x_0-\delta, y) - h(x_0+\delta, y)] dy, \\ \gamma(h) &= \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} [h(x, y_0-\delta) - h(x, y_0+\delta)] dx. \end{aligned}$$

Note that, the introduced functionals were first obtained in [8] to describe correctly solvable boundary value problems for the Laplace operator in the punctured domain Ω_0 .

Consider in the space $\mathbb{L}_2(\Omega)$ the operator L_{σ_1} generated by the differential equation

$$(1.1) \quad W(x, y) = f(x, y), (x, y) \in \Omega_0,$$

with external boundary condition

$$(1.2) \quad W(x, y)|_{\partial\Omega} = 0,$$

and "internal boundary conditions"

$$(1.3) \quad \begin{aligned} &\frac{1}{2} \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial W(x_0+\delta, y)}{\partial x} - \frac{\partial W(x_0-\delta, y)}{\partial x} \right] dy + \\ &+ \frac{1}{2} \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial W(x, y_0+\delta)}{\partial y} - \frac{\partial W(x, y_0-\delta)}{\partial y} \right] dx - \langle \Delta W(x, y), \sigma_1(x, y) \rangle = 0, \end{aligned}$$

$$(1.4) \quad \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} [W(x_0-\delta, y) - W(x_0+\delta, y)] dy = 0,$$

$$(1.5) \quad \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} [W(x, y_0-\delta) - W(x, y_0+\delta)] dx = 0,$$

where $f(x, y), \sigma_1(x, y) \in \mathbb{L}_2(\Omega)$, $\langle f(x, y), \sigma_1(x, y) \rangle$ mean inner product in $\mathbb{L}_2(\Omega)$.

2 Auxiliary statements

In the sequel, we will need a well-known statement from [10].

Theorem A. Let $h(x, y) \in \mathcal{D}$ and there exists functional $\alpha(h)$. Then function

$$W(x, y) = \iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta + \int_{\partial\Omega} \frac{\partial G(x, y, \xi, \eta)}{\partial \bar{n}_{\xi\eta}} h(\xi, \eta) ds_{\xi, \eta} - \\ - \alpha(h) G(x, y, x_0, y_0) - \beta(h) \frac{\partial G(x, y, x_0, y_0)}{\partial \xi} - \gamma(h) \frac{\partial G(x, y, x_0, y_0)}{\partial \eta}$$

is uniqueness solution of the problem

$$(2.1) \quad \Delta W(x, y) = f(x, y), (x, y) \in \Omega_0,$$

$$(2.2) \quad W(x, y)|_{\partial\Omega} = h(x, y)|_{\partial\Omega},$$

$$(2.3) \quad \frac{1}{2} \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial W(x_0+\delta, y)}{\partial x} - \frac{\partial W(x_0-\delta, y)}{\partial x} \right] dy + \\ + \frac{1}{2} \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial W(x, y_0+\delta)}{\partial y} - \frac{\partial W(x, y_0-\delta)}{\partial y} \right] dx = \alpha(h).$$

$$(2.4) \quad \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} [W(x_0 - \delta, y) - W(x_0 + \delta, y)] dy = \beta(h),$$

$$(2.5) \quad \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} [W(x, y_0 - \delta) - W(x, y_0 + \delta)] dx = \gamma(h).$$

Here $G(x, y, \xi, \eta)$ is Green function of the Dirichlet problem in Ω [9].

If we assume that $h(x, y) \in \mathcal{D}$ in a continuous manner in Ω_0 in the norm of $\mathbb{L}_2$ depends on the right side $f(x, y) \in \mathbb{L}_2(\Omega)$, then the outer boundary condition (2.2) takes the form

$$(2.6) \quad W(x, y)|_{\partial\Omega} = \langle \Delta W(x, y), \sigma_0(x, y) \rangle,$$

and the internal boundary conditions (2.3)-(2.5) will take the following types

$$(2.7) \quad \frac{1}{2} \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial W(x_0+\delta, y)}{\partial x} - \frac{\partial W(x_0-\delta, y)}{\partial x} \right] dy + \\ + \frac{1}{2} \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial W(\xi, y_0+\delta)}{\partial y} - \frac{\partial W(\xi, y_0-\delta)}{\partial y} \right] dx = \langle \Delta W(x, y), \sigma_1(x, y) \rangle.$$

$$(2.8) \quad \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} [W(x_0 - \delta, y) - W(x_0 + \delta, y)] dy = \langle \Delta W(x, y), \sigma_2(x, y) \rangle,$$

$$(2.8) \quad \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} [W(x, y_0 - \delta) - W(x, y_0 + \delta)] dx = \langle \Delta W(x, y), \sigma_3(x, y) \rangle,$$

where $\sigma_j(x, y) \in \mathbb{L}_2(\Omega)$, $j = \overline{0, 3}$.

For clarity of results, we assume that $\sigma_j(x, y) \equiv 0$, $j = 0, 2, 3$. For the operator L_{σ_1} the following theorem is true.

Theorem 1. Function

$$(2.10) \quad W(x, y) = \iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta + \langle f(\xi, \eta), \sigma_1(\xi, \eta) \rangle G(x, y, x_0, y_0)$$

represents the only solution for all right-hand sides $f(x, y) \in \mathbb{L}_2(\Omega)$ of problem (1.1)-(1.5).
For prove theorem 1 we will use well known lemmas from [11]:

Lemma A. For any continuously differentiable function $g(x, y)$ the following equalities are true:

$$\begin{aligned} & \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial g(x_0+\delta, y)}{\partial x} - \frac{\partial g(x_0-\delta, y)}{\partial x} \right] dy + \\ & \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial g(x, y_0+\delta)}{\partial y} - \frac{\partial g(x, y_0-\delta)}{\partial y} \right] dx = 0, \\ & \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} [g(x_0 - \delta, y) - g(x_0 + \delta, y)] dy = 0, \\ & \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} [g(x, y_0 - \delta) - g(x, y_0 + \delta)] dx = 0. \end{aligned}$$

Lemma B. For function $G(x, y, x_0, y_0)$ the following equalities are true:

$$\begin{aligned} \alpha(G) &= \frac{1}{2} \lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial G(x_0+\delta, y, x_0, y_0)}{\partial x} - \frac{\partial G(x_0-\delta, y, x_0, y_0)}{\partial x} \right] dy + \\ &+ \frac{1}{2} \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial G(x, y_0+\delta, x_0, y_0)}{\partial y} - \frac{\partial G(x, y_0-\delta, x_0, y_0)}{\partial y} \right] dx = 1, \\ \beta(G) &= \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial G(x, y_0+\delta, x_0, y_0)}{\partial y} - \frac{\partial G(x, y_0-\delta, x_0, y_0)}{\partial y} \right] dx = 0, \\ \gamma(G) &= \lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} [G(x, y_0 - \delta, x_0, y_0) - G(x, y_0 + \delta, x_0, y_0)] dx = 0. \end{aligned}$$

Now, we proof the theorem 1.

Proof. Let us show that equation (1.1) holds for $W(x, y)$. Operate the operator Laplace to the (2.10):

$$\begin{aligned} \Delta W(x, y) &= \Delta \left(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta \right) + \\ &\langle f(\xi, \eta), \sigma_1(\xi, \eta) \rangle \Delta(G(x, y, x_0, y_0)) = f(x, y) \end{aligned}$$

when $(x, y) \in \Omega_0$, as by the Green function property $\Delta(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta) = f(x, y)$ and $\Delta(G(x, y, x_0, y_0)) = 0$.

The validity of the first condition follows from the properties of the Green function.
Check the second condition

$$\alpha(W) - \langle \Delta W(x, y), \sigma_1(x, y) \rangle =$$

$$\alpha \left(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta \right) - \Delta \left(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta \right), \sigma_1(x, y) >$$

$$+ \langle f(\xi, \eta), \sigma(\xi, \eta) \rangle > \alpha(G(x, y, x_0, y_0)) - \langle f(\xi, \eta), \sigma_1(\xi, \eta) \rangle >$$

$$\langle \Delta G(x, y, x_0, y_0), \sigma_1(\xi, \eta) \rangle = 0,$$

as by the Green function property $\Delta(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta) = f(x, y)$ and $\Delta(G(x, y, x_0, y_0)) = 0$. And also by Lemma B $\alpha(G) = 1$. By Lemma B $\alpha(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta) = 0$, as a function $u(x, y) = \alpha(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta)$ is a twice differentiable function.

Check the third condition

$$\beta(W) = \beta \left(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta \right) +$$

$$\langle f(\xi, \eta), \sigma_1(\xi, \eta) \rangle > \beta(G(x, y, x_0, y_0)) = 0.$$

By Lemma A $\beta(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta) = 0$, as a function $u(x, y) = \alpha(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta)$ is a twice differentiable function. And also by Lemma B $\beta(G(x, y, x_0, y_0)) = 0$.

Check the fourth condition

$$\gamma(W) = \gamma \left(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta \right) +$$

$$\langle f(\xi, \eta), \sigma_1(\xi, \eta) \rangle > \gamma(G(x, y, x_0, y_0)) = 0.$$

By Lemma A $\gamma(G(x, y, x_0, y_0)) = 0$, as a function $u(x, y) = \alpha(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta)$ is a twice differentiable function. By Lemma B $\gamma(\iint_{\Omega} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta) = 0$.

Theorem 1 is completely proved.

3 The resolvent of correct internal boundary value problems for the Laplace operator in a punctured domain

In this section in the functional space $\mathbb{W}_2^2(\Omega_0) \cap \mathbb{C}(\Omega)$, we calculate the explicit form of the resolvent for a wider class of operators $L_{\sigma_1\sigma_2\sigma_3} := L$, generated by the differential equation (2.1) and internal boundary conditions (2.6)-(2.9) with $\sigma_0(\cdot, \cdot) \equiv 0$. The explicit form of the resolvent plays an essential role in studies of the spectral properties of the operator L . For convenience, we introduce the notation

$$T_1(x, y) := G(x, y, x_0, y_0), T_2(x, y) := \frac{\partial G(x, y, x_0, y_0)}{\partial \xi}, T_3(x, y) := \frac{\partial G(x, y, x_0, y_0)}{\partial \eta},$$

$$\kappa_j(x, y, \lambda) = L_0(L_0 - \lambda I)^{-1}T_j, j = 1, 2, 3.$$

We formulate the main result of this section.

Theorem 2. The resolvent of the operator L represents an operator-valued function of the spectral parameter λ and has the following representation:

$$(3.1) \quad (L - \lambda I)^{-1}f(x, y) = -\frac{H(f)}{d(\lambda)},$$

$$H(f) = \begin{pmatrix} \kappa_1(x, y, \lambda) & \kappa_2(x, y, \lambda) & \kappa_3(x, y, \lambda) & (L_0 - \lambda I)^{-1}f(x, y) \\ 1 - \lambda \tilde{\alpha}(\kappa_1(x, y, \lambda)) & -\lambda \tilde{\alpha}(\kappa_2(x, y, \lambda)) & -\lambda \tilde{\alpha}(\kappa_3(x, y, \lambda)) & -\tilde{\alpha}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda \tilde{\beta}(\kappa_1(x, y, \lambda)) & 1 - \lambda \tilde{\beta}(\kappa_2(x, y, \lambda)) & -\lambda \tilde{\beta}(\kappa_3(x, y, \lambda)) & -\tilde{\beta}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda \tilde{\gamma}(\kappa_1(x, y, \lambda)) & -\lambda \tilde{\gamma}(\kappa_2(x, y, \lambda)) & 1 - \lambda \tilde{\gamma}(\kappa_3(x, y, \lambda)) & -\tilde{\gamma}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \end{pmatrix},$$

$$d(\lambda) = \begin{pmatrix} 1 - \lambda \tilde{\alpha}(\kappa_1(x, y, \lambda)) & -\lambda \tilde{\alpha}(\kappa_2(x, y, \lambda)) & -\lambda \tilde{\alpha}(\kappa_3(x, y, \lambda)) \\ -\lambda \tilde{\beta}(\kappa_1(x, y, \lambda)) & 1 - \lambda \tilde{\beta}(\kappa_2(x, y, \lambda)) & -\lambda \tilde{\beta}(\kappa_3(x, y, \lambda)) \\ -\lambda \tilde{\gamma}(\kappa_1(x, y, \lambda)) & -\lambda \tilde{\gamma}(\kappa_2(x, y, \lambda)) & 1 - \lambda \tilde{\gamma}(\kappa_3(x, y, \lambda)) \end{pmatrix},$$

where

$$\begin{aligned} \kappa_j(x, y, \lambda) &:= L_0(L_0 - \lambda I)^{-1}T_j(x, y), \quad j = 1, 2, 3, \\ \tilde{\alpha}(\kappa_j(x, y, \lambda)) &= \langle \kappa_j(x, y, \lambda), \sigma_1(x, y) \rangle, \quad \tilde{\beta}(\kappa_j(x, y, \lambda)) = \\ &\quad \langle \kappa_j(x, y, \lambda), \sigma_2(x, y) \rangle, \\ \tilde{\gamma}(\kappa_j(x, y, \lambda)) &= \langle \kappa_j(x, y, \lambda), \sigma_3(x, y) \rangle, \quad \tilde{\alpha}(L_0(L_0 - \lambda I)^{-1}f(x, y)) = \\ &\quad \langle L_0(L_0 - \lambda I)^{-1}f(x, y), \sigma_1(x, y) \rangle, \\ \tilde{\beta}(L_0(L_0 - \lambda I)^{-1}f(x, y)) &= \langle L_0(L_0 - \lambda I)^{-1}f(x, y), \sigma_2(x, y) \rangle, \\ \tilde{\gamma}(L_0(L_0 - \lambda I)^{-1}f(x, y)) &= \langle L_0(L_0 - \lambda I)^{-1}f(x, y), \sigma_3(x, y) \rangle. \end{aligned}$$

Here L_0 is discreet operator, corresponding to the Dirichlet problem.

Proof. In paper [12] the following relation was proved for the resolvent of the operator L :

$$(3.2) \quad (L - \lambda I)^{-1}f(x, y) = (L_0 - \lambda I)^{-1}f(x, y) + \sum_{j=1}^3 \langle L_0(L_0 - \lambda I)^{-1}f(x, y), \sigma_j(x, y) \rangle L(L - \lambda I)^{-1}T_j(x, y).$$

Set $f(x, y) = T_1(x, y)$. Then we have

$$(L - \lambda I)^{-1}T_1(x, y) = (L_0 - \lambda I)^{-1}T_1(x, y) + \sum_{j=1}^3 \langle L_0(L_0 - \lambda I)^{-1}T_1(x, y), \sigma_j(x, y) \rangle L(L - \lambda I)^{-1}T_j(x, y).$$

We act on the obtained relation with the operator Δ . Recall that

- 1) when $u \in D(L)$ we have $\Delta u = Lu$;
- 2) when $u \in D(L_0)$ we obtain $\Delta u = L_0u$. As a result, we get

$$(3.3) \quad (1 - \lambda \langle \kappa_1(x, y, \lambda), \sigma_1(x, y) \rangle) L(L - \lambda I)^{-1}T_1(x, y) = L_0(L_0 - \lambda I)^{-1}T_1(x, y) + \sum_{j=2}^3 \lambda \langle \kappa_1(x, y), \sigma_j(x, y) \rangle L(L - \lambda I)^{-1}T_j(x, y).$$

Here it is taken into account that $(L_0 - \lambda I)^{-1}T_1(x, y) \in D(L_0)$ and $(L - \lambda I)^{-1}T_j(x, y) \in D(L)$ when $j = 1, 2, 3$.

Now, assume $f(x, y) = T_2(x, y)$. Then from similar reasoning we have

$$(L - \lambda I)^{-1}T_2(x, y) = (L_0 - \lambda I)^{-1}T_2(x, y) + \sum_{j=1}^3 \langle L_0(L_0 - \lambda I)^{-1}T_2(x, y), \sigma_j(x, y) \rangle L(L - \lambda I)^{-1}T_j(x, y).$$

We act on the obtained relation by the Laplace operator. As a result, we get that

$$(3.4) \quad (1 - \lambda \langle \kappa_2(x, y, \lambda), \sigma_2(x, y) \rangle) L(L - \lambda I)^{-1}T_2(x, y) = L_0(L_0 - \lambda I)^{-1}T_2(x, y) + \lambda \langle \kappa_2(x, y), \sigma_1(x, y) \rangle L(L - \lambda I)^{-1}T_1(x, y) + \lambda \langle \kappa_2(x, y), \sigma_3(x, y) \rangle L(L - \lambda I)^{-1}T_3(x, y).$$

When $f(x, y) = T_3(x, y)$, we have

$$(L - \lambda I)^{-1}T_3(x, y) = (L_0 - \lambda I)^{-1}T_3(x, y) + \sum_{j=1}^3 \langle L_0(L_0 - \lambda I)^{-1}T_3(x, y), \sigma_j(x, y) \rangle L(L - \lambda I)^{-1}T_j(x, y).$$

We act on the obtained relation by the Laplace operator. As a result, we get that

$$(3.5) \quad (1 - \lambda \langle \kappa_3(x, y, \lambda), \sigma_1(x, y) \rangle) L(L - \lambda I)^{-1}T_3(x, y) = L_0(L_0 - \lambda I)^{-1}T_3(x, y) + \sum_{j=1}^2 \lambda \langle \kappa_3(x, y), \sigma_j(x, y) \rangle L(L - \lambda I)^{-1}T_j(x, y).$$

Relations (3.2)-(3.5) constitute a homogeneous system of algebraic equations, which takes the matrix form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 - \lambda \tilde{\alpha}(\kappa_1(x, y, \lambda)) & -\lambda \tilde{\alpha}(\kappa_2(x, y, \lambda)) & -\lambda \tilde{\alpha}(\kappa_3(x, y, \lambda)) & -\tilde{\alpha}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda \tilde{\beta}(\kappa_1(x, y, \lambda)) & 1 - \lambda \tilde{\beta}(\kappa_2(x, y, \lambda)) & -\lambda \tilde{\beta}(\kappa_3(x, y, \lambda)) & -\tilde{\beta}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda \tilde{\gamma}(\kappa_1(x, y, \lambda)) & -\lambda \tilde{\gamma}(\kappa_2(x, y, \lambda)) & 1 - \lambda \tilde{\gamma}(\kappa_3(x, y, \lambda)) & -\tilde{\gamma}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \end{bmatrix} \begin{bmatrix} \kappa_1(x, y, \lambda) \\ \kappa_2(x, y, \lambda) \\ \kappa_3(x, y, \lambda) \\ (L_0 - \lambda I)^{-1}f(x, y) - (L - \lambda I)^{-1}f(x, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is known from the course of linear algebra that a homogeneous system of equations has a non-trivial solution for

$$\begin{vmatrix} \kappa_1(x, y, \lambda) & \kappa_2(x, y, \lambda) & \kappa_3(x, y, \lambda) & (L_0 - \lambda I)^{-1}f(x, y) - (L - \lambda I)^{-1}f(x, y) \\ 1 - \lambda\tilde{\alpha}(\kappa_1(x, y, \lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x, y, \lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x, y, \lambda)) & -\tilde{\alpha}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda\tilde{\beta}(\kappa_1(x, y, \lambda)) & 1 - \lambda\tilde{\beta}(\kappa_2(x, y, \lambda)) & -\lambda\tilde{\beta}(\kappa_3(x, y, \lambda)) & -\tilde{\beta}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda\tilde{\gamma}(\kappa_1(x, y, \lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x, y, \lambda)) & 1 - \lambda\tilde{\gamma}(\kappa_3(x, y, \lambda)) & -\tilde{\gamma}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \end{vmatrix} = 0.$$

According to the standard properties of the determinants, we can write the equality

$$\begin{vmatrix} \kappa_1(x, y, \lambda) & \kappa_2(x, y, \lambda) & \kappa_3(x, y, \lambda) & (L - \lambda I)^{-1}f(x, y) \\ 1 - \lambda\tilde{\alpha}(\kappa_1(x, y, \lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x, y, \lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x, y, \lambda)) & 0 \\ -\lambda\tilde{\beta}(\kappa_1(x, y, \lambda)) & 1 - \lambda\tilde{\beta}(\kappa_2(x, y, \lambda)) & -\lambda\tilde{\beta}(\kappa_3(x, y, \lambda)) & 0 \\ -\lambda\tilde{\gamma}(\kappa_1(x, y, \lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x, y, \lambda)) & 1 - \lambda\tilde{\gamma}(\kappa_3(x, y, \lambda)) & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} \kappa_1(x, y, \lambda) & \kappa_2(x, y, \lambda) & \kappa_3(x, y, \lambda) & (L_0 - \lambda I)^{-1}f(x, y) \\ 1 - \lambda\tilde{\alpha}(\kappa_1(x, y, \lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x, y, \lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x, y, \lambda)) & -\tilde{\alpha}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda\tilde{\beta}(\kappa_1(x, y, \lambda)) & 1 - \lambda\tilde{\beta}(\kappa_2(x, y, \lambda)) & -\lambda\tilde{\beta}(\kappa_3(x, y, \lambda)) & -\tilde{\beta}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda\tilde{\gamma}(\kappa_1(x, y, \lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x, y, \lambda)) & 1 - \lambda\tilde{\gamma}(\kappa_3(x, y, \lambda)) & -\tilde{\gamma}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \end{vmatrix}$$

This implies the assertion of Theorem 2.

From the explicit resolvent formula for the operator L , it is not difficult to see that the resolvent is a meromorphic operator-valued function, since the characteristic determinant $d(\lambda)$ may have poles in the spectrum of the operator L_0 . Since the spectrum of the Dirichlet problem for the Laplace equation is canonical restricted domains can be explicitly computed, then these poles are written out explicitly.

Corollary 1. The resolvent of the operator L can also be represented as

$$(3.6) \quad (L - \lambda I)^{-1}f(x, y) = (L_0 - \lambda I)^{-1}f(x, y) - \frac{\tilde{H}(f)}{d(\lambda)}$$

where

$$\tilde{H}(f) = \begin{bmatrix} \kappa_1(x, y, \lambda) & \kappa_2(x, y, \lambda) & \kappa_3(x, y, \lambda) & 0 \\ 1 - \lambda\tilde{\alpha}(\kappa_1(x, y, \lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x, y, \lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x, y, \lambda)) & -\tilde{\alpha}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda\tilde{\beta}(\kappa_1(x, y, \lambda)) & 1 - \lambda\tilde{\beta}(\kappa_2(x, y, \lambda)) & -\lambda\tilde{\beta}(\kappa_3(x, y, \lambda)) & -\tilde{\beta}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \\ -\lambda\tilde{\gamma}(\kappa_1(x, y, \lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x, y, \lambda)) & 1 - \lambda\tilde{\gamma}(\kappa_3(x, y, \lambda)) & -\tilde{\gamma}(L_0(L_0 - \lambda I)^{-1}f(x, y)) \end{bmatrix}$$

Corollary 2. In particular, if $\sigma_j(x, y) \equiv 0, j = 2, 3$, then the operator's resolution L_{σ_1} take the form

$$(3.7) \quad (L_{\sigma_1} - \lambda I)^{-1}f(x, y) = (L_0 - \lambda I)^{-1}f(x, y) + \frac{\tilde{\alpha}(\kappa_1(x, y, \lambda))}{d(\lambda)} \kappa_1(x, y, \lambda),$$

where

$$(3.8) \quad d(\lambda) = 1 - \lambda\tilde{\alpha}(\kappa_1(x, y, \lambda)) = 1 - \lambda < \kappa_1(x, y, \lambda), \sigma(x, y) >.$$

4 Meromorphic function generating root functions operator L_{σ_1}

In further research we will need some properties $\kappa_1(x, y, \lambda)$.

Lemma 1. Meromorphic function $\kappa_1(x, y, \lambda)$ is the main solution of the homogeneous equation

$$\Delta(\kappa_1(x, y, \lambda)) = \lambda\kappa_1(x, y, \lambda),$$

satisfying linear conditions

$$\kappa_1(x, y, \lambda)|_{\partial\Omega} = 0,$$

$$\alpha(\kappa_1(x, y, \lambda)) - \langle \Delta(\kappa_1(x, y, \lambda)), \sigma_1(x, y) \rangle = d(\lambda),$$

$$\beta(\kappa_1(x, y, \lambda)) = 0,$$

$$\gamma(\kappa_1(x, y, \lambda)) = 0,$$

where $d(\lambda)$ is define by (3.8).

Proof. Check that the function $\kappa_1(x, y, \lambda)$ is a solution of the homogeneous equation

$$\Delta(\kappa_1(x, y, \lambda)) = \Delta(T_1(x, y)) + \lambda L_0(L_0 - \lambda I)^{-1}T_1(x, y) = \lambda \kappa_1(x, y, \lambda).$$

Here it is taken into account that $\Delta(T_1(x, y)) = 0$ и $(L_0 - \lambda I)^{-1}T_1(x, y) \in D(L_0)$.

The validity of the first condition follows from the properties of the Green function.

Let us prove the validity of the second condition.

$$\alpha(\kappa_1(x, y, \lambda)) - \langle \Delta(\kappa_1(x, y, \lambda)), \sigma_1(x, y) \rangle = \alpha(T_1(x, y)) +$$

$$\lambda \alpha((L_0 - \lambda I)^{-1}T_1(x, y)) - \langle \Delta(T_1(x, y)), \sigma_1(x, y) \rangle -$$

$$\lambda \langle L_0(L_0 - \lambda I)^{-1}T_1(x, y), \sigma_1(x, y) \rangle = d(\lambda),$$

so by Lemma B $\alpha(T_1(x, y)) = 1$. From Lemma A it follows that $\alpha((L_0 - \lambda I)^{-1}T_1(x, y)) = 0$. It is also taken into account that $\Delta(T_1(x, y)) = 0$ and $(L_0 - \lambda I)^{-1}T_1(x, y) \in D(L_0)$.

Check the third condition

$$\beta(\kappa_1(x, y, \lambda)) = \beta(T_1(x, y)) + \lambda \beta((L_0 - \lambda I)^{-1}T_1(x, y)) = 0,$$

as by Lemma B $\beta(T_1(x, y)) = 0$. From Lemma A it follows that $\beta((L_0 - \lambda I)^{-1}T_1(x, y)) = 0$.

Check the fourth condition

$$\gamma(\kappa_1(x, y, \lambda)) = \gamma(T_1(x, y)) + \lambda \gamma((L_0 - \lambda I)^{-1}T_1(x, y)) = 0,$$

as by Lemma B $\gamma(T_1(x, y)) = 0$. From Lemma A it follows that $\gamma((L_0 - \lambda I)^{-1}T_1(x, y)) = 0$.

Lemma 1 is proved.

For convenience, we introduce the notation

$$(4.1) \quad \varphi_1(x, y, \lambda) = \frac{\kappa_1(x, y, \lambda)}{d(\lambda)}.$$

If relation (4.1) is taken into account, then formula (3.7) for the resolvent of the operator L_{σ_1} can be written in the following form

$$(4.2) \quad (L_{\sigma_1} - \lambda I)^{-1}f(x, y) = (L_0 - \lambda I)^{-1}f(x, y) + \tilde{\alpha}(\kappa_1(x, y, \lambda))\varphi(x, y, \lambda).$$

This implies that the poles of $\varphi_1(x, y, \lambda)$, and at the same time, the poles of the resolvent $(L_{\sigma_1} - \lambda I)^{-1}$ coincide with the zeros of the meromorphic function $d(\lambda)$. We show that the introduced solution $\kappa_1(x, y, \lambda)$ as a function of λ on the spectrum of the operator L_{σ_1} generates all its root functions.

Theorem 3. Let λ_k be an arbitrary zero of the characteristic determinant $d(\lambda)$ of multiplicity m_k . Then the functions from the following line:

$$(4.3) \quad \kappa_1(x, y, \lambda_k), \frac{1}{1!} \frac{\partial \kappa_1(x, y, \lambda_k)}{\partial \lambda}, \dots, \frac{1}{(m_k-1)!} \frac{\partial^{m_k-1} \kappa_1(x, y, \lambda_k)}{\partial \lambda^{m_k-1}},$$

have a spectral interpretation: the first nonzero zero function is a proper one, and the subsequent ones are attached, generated by the indicated eigenfunction, corresponding to the eigenvalue λ_k of the operator L_{σ_1} .

Proof. The proof of Theorem 3 is that it is necessary to check for $p = 0, \dots, m_k - 1$ relations

$$(4.4) \quad \frac{1}{p!} \frac{\partial^p \kappa_1(x, y, \lambda_k)}{\partial \lambda^p} \in D(L_{\sigma_1}),$$

$$(4.5) \quad L_{\sigma_1} \left(\frac{1}{p!} \frac{\partial^p \kappa_1(x, y, \lambda_k)}{\partial \lambda^p} \right) = \lambda_k \left(\frac{1}{p!} \frac{\partial^p \kappa_1(x, y, \lambda_k)}{\partial \lambda^p} \right) + \frac{\epsilon}{(p-1)!} \frac{\partial^{p-1} \kappa_1(x, y, \lambda_k)}{\partial \lambda^{p-1}},$$

where $\epsilon = 0$ when $p = 0$, $\epsilon = 1$ when $p > 0$. From what $d(\lambda_k) = 0$ and follows from Lemma 1 $\kappa_1(x, y, \lambda_k) \in D(L_{\sigma_1})$. Therefore, the operator relation $L_{\sigma_1}(\kappa_1(x, y, \lambda_k)) = \lambda_k \kappa_1(x, y, \lambda_k)$ coincides with a homogeneous differential equation $\Delta(\kappa_1(x, y, \lambda_k)) = \lambda_k \kappa_1(x, y, \lambda_k)$, which by definition $\kappa(x, y, \lambda)$ holds when $\lambda = \lambda_k$. Thus, if $\kappa(x, y, \lambda_k)$ is not identically zero functions, then $\kappa(x, y, \lambda_k)$ is eigenfunction of the operator L_{σ_1} .

Now let $p = 1 < m_k$. Notice, that $d(\lambda_k) = 0$, $d'(\lambda_k) = 0$ and

$$\frac{\partial \kappa_1(x, y, \lambda)}{\partial \lambda} = (L_0 - \lambda I)^{-1} T_1(x, y) + \lambda \frac{\partial}{\partial \lambda} (L_0 - \lambda I)^{-1} T_1(x, y),$$

$$d'(\lambda) = -\langle \kappa_1(x, y, \lambda), \sigma_1(x, y) \rangle - \lambda \langle \frac{\partial}{\partial \lambda} \kappa_1(x, y, \lambda), \sigma_1(x, y) \rangle.$$

Calculate

$$\begin{aligned} \Delta \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \lambda} \right) &= L_0 (L_0 - \lambda I)^{-1} T_1(x, y) + \lambda \frac{\partial}{\partial \lambda} L_0 (L_0 - \lambda I)^{-1} T_1(x, y) = \\ &= \lambda \frac{\partial}{\partial \lambda} \kappa_1(x, y, \lambda) + \kappa_1(x, y, \lambda). \end{aligned}$$

The validity of the first condition follows from the properties of the Green function.

Let us prove the validity of the second condition.

$$\alpha \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \lambda} \right) - \langle \Delta \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \lambda} \right), \sigma_1(x, y) \rangle = \alpha((L_0 - \lambda_k)^{-1} T_1(x, y)) +$$

$$\lambda_k \frac{\partial}{\partial \lambda} \alpha((L_0 - \lambda_k)^{-1} T_1(x, y)) - \langle \Delta((L_0 - \lambda_k)^{-1} T_1(x, y)), \sigma_1(x, y) \rangle$$

$$- \lambda_k \frac{\partial}{\partial \lambda} \langle \Delta((L_0 - \lambda_k)^{-1} T_1(x, y)), \sigma_1(x, y) \rangle = d'(\lambda_k) = 0,$$

so by Lemma A $\alpha((L_0 - \lambda_k I)^{-1} T_1(x, y)) = 0$. It is also taken into account that $(L_0 - \lambda_k I)^{-1} T_1(x, y) \in D(L_0)$.

Check the third condition

$$\beta \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \lambda} \right) = \beta((L_0 - \lambda_k)^{-1} T_1(x, y)) + \lambda_k \frac{\partial}{\partial \lambda} \beta((L_0 - \lambda_k)^{-1} T_1(x, y)) = 0,$$

so by Lemma A $\beta((L_0 - \lambda I)^{-1} T_1(x, y)) = 0$.

Check the fourth condition

$$\gamma(\kappa_1(x, y, \lambda)) = \gamma((L_0 - \lambda_k)^{-1} T_1(x, y)) + \lambda_k \frac{\partial}{\partial \lambda} \gamma((L_0 - \lambda_k)^{-1} T_1(x, y)) = 0,$$

so by Lemma A $\gamma((L_0 - \lambda I)^{-1} T_1(x, y)) = 0$.

Continuing the arguments for other admissible p , we obtain a complete proof of Theorem 3.

5 Projectors on rood supspace

In the monograph ([13], стр. 445) the decomposition theorem is given, from which it follows that the projector $P_k: \mathbb{L}_2(\Omega_0) \rightarrow (L_{\sigma_1} - \lambda_k I)^{m_k}$ represents the residue of the resolvent at the singular point λ_k :

$$(P_k f)(x, y) = -\frac{1}{2\pi i} \oint_{|\lambda - \lambda_k| = \delta} (L_{\sigma_1} - \lambda I)^{-1} f(x, y) d\lambda$$

with some $\delta > 0$. Since the resolvent of the Dirichlet operator has poles, for completeness of information we consider two cases.

First, we consider the case when the eigenvalue λ_k of the operator L_{σ_1} does not coincide with any eigenvalue λ_j^0 of the operator L_0 . Recalling the representation of the resolvent (3.7) from Corollary 2 and considering that the resolvent $(L_0 - \lambda I)^{-1}$ of the Dirichlet operator represents a holomorphic function of λ , projector type P_k can be specified:

$$(5.1) \quad (P_k f)(x, y) = -\frac{1}{2\pi i} \oint_{|\lambda - \lambda_k| = \delta} \frac{\kappa_1(x, y, \lambda)}{d(\lambda)} d\lambda$$

$$< f(\xi, \eta), L_0^*(L_0^* - \bar{\lambda} I)^{-1} \sigma_1(\xi, \eta) > d\lambda = \operatorname{res}_{\lambda_k} \frac{\kappa_1(x, y, \lambda)}{d(\lambda)} < f(\xi, \eta), M_1(\xi, \eta, \bar{\lambda}) >,$$

where $M_1(\xi, \eta, \bar{\lambda}) = \overline{L_0^*(L_0^* - \bar{\lambda} I)^{-1} \sigma_1(\xi, \eta)}$.

Apply the Cauchy residue theorem to the relation (5.1), we obtain

$$(5.2) \quad (P_k f)(x, y) = -\frac{1}{(m_k - 1)!} \lim_{\lambda \rightarrow \lambda_k} \frac{\partial^{m_k - 1}}{\partial \lambda^{m_k - 1}} \left(\frac{(\lambda - \lambda_k)^{m_k} < f(\xi, \eta), M_1(\xi, \eta, \bar{\lambda}) >}{d(\lambda)} \kappa_1(x, y, \lambda) \right) =$$

$$- \sum_{p=0}^{m_k - 1} \frac{1}{(m_k - 1 - p)!} \lim_{\lambda \rightarrow \lambda_k} \frac{\partial^{m_k - 1 - p}}{\partial \lambda^{m_k - 1 - p}} \left(\frac{(\lambda - \lambda_k)^{m_k} < f(\xi, \eta), M_1(\xi, \eta, \bar{\lambda}) >}{d(\lambda)} \right) \frac{1}{p!} \frac{\partial^p \kappa_1(x, y, \lambda)}{\partial \lambda^p} \Big|_{\lambda = \lambda_k} =$$

$$\sum_{p=0}^{m_k - 1} < f(\xi, \eta), -\frac{1}{(m_k - 1 - p)!} \lim_{\bar{\lambda} \rightarrow \bar{\lambda}_k} \frac{\partial^{m_k - 1 - p}}{\partial \bar{\lambda}^{m_k - 1 - p}} \left(\frac{(\bar{\lambda} - \bar{\lambda}_k)^{m_k} M_1(\xi, \eta, \bar{\lambda})}{d(\bar{\lambda})} \right) >$$

$$\frac{1}{p!} \frac{\partial^p \kappa_1(x, y, \lambda)}{\partial \lambda^p} \Big|_{\lambda = \lambda_k}$$

Analysis of formula (5.2) leads to the following notation.

$$(5.3) \quad E'_k = \{h_{k,0}(\xi, \eta), h_{k,1}(\xi, \eta), \dots, h_{k,m_k-1}(\xi, \eta)\},$$

where

$$(5.4) \quad h_{k,m_k-1-p}(\xi, \eta) = -\frac{1}{(m_k - 1 - p)!} \lim_{\bar{\lambda} \rightarrow \bar{\lambda}_k} \frac{\partial^{m_k - 1 - p}}{\partial \bar{\lambda}^{m_k - 1 - p}} \left(\frac{(\bar{\lambda} - \bar{\lambda}_k)^{m_k} M_1(\xi, \eta, \bar{\lambda})}{d(\bar{\lambda})} \right),$$

$$p = \overline{0, m_k - 1}.$$

We introduce the following family of functions

$$(5.5) \quad E' = \{E'_k: \lambda_k \text{ is arbitrary eigenvalue of the operator } L_{\sigma_1}\}.$$

Thus, we need to study the decomposition of arbitrary elements from the functional space $\mathbb{L}_2(\Omega_0)$ in the system of root functions of the operator L_{σ_1} .

$$(5.6) \quad E = \{E_k: \lambda_k \text{ is arbitrary eigenvalue of the operator } L_{\sigma_1}\}.$$

$$(5.7) \quad E_k = \{W_{k,0}(x, y), W_{k,1}(x, y), \dots, W_{k,m_k-1}(x, y)\},$$

where

$$W_{k,p}(x, y) = \left. \frac{\partial^p \kappa_1(x, y, \lambda)}{\partial \lambda^p} \right|_{\lambda=\lambda_k}$$

Now, we consider the case $\lambda_k = \lambda_k^0$.

$$\begin{aligned} (P_k f)(x, y) &= -\frac{1}{2\pi i} \oint_{|\lambda-\lambda_k|=\delta} (L_{\sigma_1} - \lambda I)^{-1} f(x, y) d\lambda = \\ &= \operatorname{res}_{\lambda_k=\lambda_k^0} ((L_0 - \lambda I)^{-1} f(x, y)) + \\ &\operatorname{res}_{\lambda_k=\lambda_k^0} \left(\tilde{\alpha}(L_0(L_0 - \lambda I)^{-1} f(x, y)) \frac{\kappa_1(x, y, \lambda)}{1 - \lambda \tilde{\alpha}(\kappa_1(x, y, \lambda))} \right). \end{aligned}$$

First, we calculate the first addend. Given the representation for the resolvent of the Dirichlet problem as an expansion in eigenfunctions

$$(5.8) \quad (L_0 - \lambda I)^{-1} f(x, y) = \sum_{m=1}^{\infty} \frac{1}{\lambda_m^0 - \lambda} f_m \omega_m^0(x, y),$$

we calculate the first addend

$$\begin{aligned} V_1 &:= \operatorname{res}_{\lambda_k=\lambda_k^0} ((L_0 - \lambda I)^{-1} f(x, y)) = \\ &\operatorname{res}_{\lambda_k=\lambda_k^0} \left(\sum_{m=1}^{\infty} \frac{1}{\lambda_m^0 - \lambda} f_m \omega_m^0(x, y) \right) = -f_k \omega_k^0(x, y). \end{aligned}$$

Calculate the second addend. Here we use the standard transform.

$$L_0(L_0 - \lambda I)^{-1} T_1(x, y) = T_1(x, y) + \lambda(L_0 - \lambda I)^{-1} T_1(x, y).$$

Then

$$\begin{aligned} V_2 &:= \operatorname{res}_{\lambda_k=\lambda_k^0} \left(\left[\tilde{\alpha}(f) + \sum_{m=1}^{\infty} \frac{\lambda}{\lambda_m^0 - \lambda} f_m \tilde{\alpha}(\omega_m^0(x, y)) \right] \right. \\ &\left. \frac{T_1(x, y) + \sum_{m=1}^{\infty} \frac{\lambda}{\lambda_m^0 - \lambda} T_{1m} \omega_m^0(x, y)}{1 - \lambda \tilde{\alpha}(T_1(x, y)) - \sum_{m=1}^{\infty} \frac{\lambda^2}{\lambda_m^0} T_{1m} \tilde{\alpha}(\omega_m^0(x, y))} \right). \end{aligned}$$

Multiply the denominator and numerator in the last relation by $\lambda_k - \lambda$. As a result, the deduction will be equal to

$$V_2 = \lambda_k^0 f_k \tilde{\alpha}(\omega_k^0(x, y)) \cdot \frac{\lambda_k^0 T_{1k} \omega_k^0(x, y)}{(\lambda_k^0)^2 T_{1k} \tilde{\alpha}(\omega_k^0(x, y))} = f_k \omega_k^0(x, y).$$

So in the case of $\lambda_k = \lambda_k^0$ the projector

$$(P_k f)(x, y) = V_1 + V_2 = -f_k \omega_k^0(x, y) + f_k \omega_k^0(x, y) = 0.$$

The main result of this section is formulated as a theorem.

Theorem 4. Let λ_k is eigenvalue of the multiplicity m_k of the operator L_{σ_1} . P_k operator on root the subspace of the operator L_{σ_1} corresponding to λ_k is determined by formula (5.2).

6 Minimality of the root function system

In this section we prove the minimality of the system of root functions of the operator L_{σ_1} in the functional space $\mathbb{L}_2(\Omega_0)$. Generally speaking, choosing one or another basis in the root subspaces $P_k \mathbb{L}_2(\Omega_0)$, it is possible to study different systems of root functions. We investigate the minimality of a concrete system (5.6). System (5.6) is generated by solutions of the differential equation (4.5), generating operator L_{σ_1} . In particular, the Green's function associated with the operator L_{σ_1} is written through them. To prove the minimality of (5.6) in the functional space $\mathbb{L}_2(\Omega_0)$, it suffices to construct a biorthogonal system of functions in this space. In the previous paragraph 5 such a system has already been built. It remains to verify that system (5.6) lies in $\mathbb{L}_2(\Omega_0)$ and satisfies the biorthogonality relation.

To begin with, we verify that the functions in (5.6) are elements of the space $\mathbb{L}_2(\Omega_0)$. Since, according to (5.4), each $h_{k,t}(\xi, \eta)$ represents a linear combination of functions $M_1(\xi, \eta, \bar{\lambda})$ and its derivatives for $\lambda = \lambda_k$ ($k = 1, 2, \dots, n$), it suffices to check that these functions $M_1(\xi, \eta, \bar{\lambda}_k)$, ($k = 1, 2, \dots, n$) belong to $\mathbb{L}_2(\Omega_0)$. The latter fact is obvious, since $M_1(\xi, \eta, \bar{\lambda}) = L_0^*(L_0^* - \bar{\lambda}I)^{-1}\sigma_1(\xi, \eta)$ and $\sigma_1(\xi, \eta)$ are an element of $\mathbb{L}_2(\Omega)$.

To establish the biorthogonality relations, we need the following lemma.

Lemma 2. For arbitrary complex numbers λ, μ the identity is correct

$$\langle \kappa_1(x, y, \lambda), M_1(x, y, \bar{\mu}) \rangle = -\frac{d(\lambda) - d(\mu)}{\lambda - \mu}.$$

Proof. For arbitrary λ, μ we calculate the following inner product

$$\begin{aligned} \lambda \langle \kappa_1(x, y, \lambda), M_1(x, y, \bar{\mu}) \rangle_{\mathbb{L}_2(\Omega_0)} &= \lambda \langle \kappa_1(x, y, \lambda), \sigma_1(x, y) \rangle + \\ &\mu (L_0^* - \bar{\mu}I)^{-1} \sigma_1(x, y) \geq \langle \Delta \kappa_1(x, y, \lambda), \sigma_1(x, y) \rangle + \\ &\mu \langle \Delta \kappa_1(x, y, \lambda), (L_0^* - \bar{\mu}I)^{-1} \sigma_1(x, y) \rangle \\ &= 1 - d(\lambda) + \mu \langle \Delta \kappa_1(x, y, \lambda), \widetilde{M}_1(x, y, \bar{\mu}) \rangle, \end{aligned}$$

where $\widetilde{M}_1(x, y, \bar{\mu}) = (L_0^* - \bar{\mu}I)^{-1} \sigma_1(x, y)$. Here, took into account the formula (3.8) for the characteristic determinant operator L_{σ_1} . We calculate separately the scalar product

$$\begin{aligned} \langle \Delta \kappa_1(x, y, \lambda), \widetilde{M}_1(x, y, \bar{\mu}) \rangle &= \langle \kappa_1(x, y, \lambda), \Delta \widetilde{M}_1(x, y, \bar{\mu}) \rangle + \\ &+ \lim_{\delta \rightarrow +0} \int_{\partial \Omega} \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \bar{n}} \widetilde{M}_1(x, y, \bar{\mu}) - \kappa_1(x, y, \lambda) \frac{\partial \widetilde{M}_1(x, y, \bar{\mu})}{\partial \bar{n}} \right) ds - \end{aligned}$$

$$\begin{aligned}
& - \lim_{\delta \rightarrow +0} \int_{\partial \Pi_\delta^0} \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \bar{n}} \widetilde{M}_1(x, y, \bar{\mu}) - \kappa_1(x, y, \lambda) \frac{\partial \widetilde{M}_1(x, y, \bar{\mu})}{\partial \bar{n}} \right) ds = \\
& = \langle \kappa_1(x, y, \lambda), \Delta \widetilde{M}_1(x, y, \bar{\mu}) \rangle - \\
& \lim_{\delta \rightarrow +0} \int_{\partial \Pi_\delta^0} \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \bar{n}} - \kappa_1(x, y, \lambda) \frac{\partial \widetilde{M}_1(x, y, \bar{\mu})}{\partial \bar{n}} \right) ds,
\end{aligned}$$

in that $\kappa_1(x, y, \lambda), \widetilde{M}_1(x, y, \bar{\mu}) \in \mathbb{W}_2^2(\Omega)$, then

$$\lim_{\delta \rightarrow +0} \int_{\partial \Omega} \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \bar{n}} \widetilde{M}_1(x, y, \bar{\mu}) - \kappa_1(x, y, \lambda) \frac{\partial \widetilde{M}_1(x, y, \bar{\mu})}{\partial \bar{n}} \right) ds = 0.$$

Here $\Pi_\delta^0 = \{(x, y) : -\delta \leq x - x_0 \leq \delta, -\delta \leq y - y_0 \leq \delta\}$. Then

$$\begin{aligned}
& (\lambda - \mu) \langle \kappa_1(x, y, \lambda), M_1(x, y, \bar{\mu}) \rangle = 1 - d(\lambda) - \\
& - \mu \lim_{\delta \rightarrow +0} \int_{\partial \Pi_\delta^0} \left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \bar{n}} \widetilde{M}_1(x, y, \bar{\mu}) - \kappa_1(x, y, \lambda) \frac{\partial \widetilde{M}_1(x, y, \bar{\mu})}{\partial \bar{n}} \right) ds.
\end{aligned}$$

Notice, that $\Delta \widetilde{M}_1(x, y, \bar{\mu}) = M_1(x, y, \bar{\mu})$, because $(L_0^* - \bar{\mu}I)^{-1} \sigma_1(x, y) \in D(L_0^*)$. Hence, when $\lambda = \mu$, we have

$$\mu \lim_{\delta \rightarrow +0} \int_{\partial \Pi_\delta^0} \left(\frac{\partial \kappa_1(x, y, \mu)}{\partial \bar{n}} \widetilde{M}_1(x, y, \bar{\mu}) - \kappa_1(x, y, \mu) \frac{\partial \widetilde{M}_1(x, y, \bar{\mu})}{\partial \bar{n}} \right) ds = 1 - d(\mu).$$

Considering the last relation, we transform the previous equality

$$\begin{aligned}
& (\lambda - \mu) \langle \kappa_1(x, y, \lambda), M_1(x, y, \bar{\mu}) \rangle = 1 - d(\lambda) - 1 + d(\mu) - \\
& \mu \lim_{\delta \rightarrow +0} \int_{\partial \Pi_\delta^0} \left[\left(\frac{\partial \kappa_1(x, y, \lambda)}{\partial \bar{n}} - \frac{\partial \kappa_1(x, y, \mu)}{\partial \bar{n}} \right) \widetilde{M}_1(x, y, \bar{\mu}) - \right. \\
& \left. (\kappa_1(x, y, \lambda) - \kappa_1(x, y, \mu)) \frac{\partial \widetilde{M}_1(x, y, \bar{\mu})}{\partial \bar{n}} \right] ds = -d(\lambda) + d(\mu) - \\
& - (\lambda - \mu) \mu \lim_{\delta \rightarrow +0} \int_{\partial \Pi_\delta^0} \frac{\partial}{\partial \bar{n}} (L_0 - \lambda I)^{-1} T_1(x, y) \cdot (L_0^* - \bar{\mu}I)^{-1} \sigma_1(x, y) ds + \\
& + (\lambda - \mu) \mu \lim_{\delta \rightarrow +0} \int_{\partial \Pi_\delta^0} (L_0 - \lambda I)^{-1} T_1(x, y) \cdot \frac{\partial}{\partial \bar{n}} (L_0^* - \bar{\mu}I)^{-1} \sigma_1(x, y) ds
\end{aligned}$$

Recall the representation for the Dirichlet problem as an expansion in eigenfunctions

$$\begin{aligned}
(L_0 - \lambda I)^{-1} T_1(x, y) &= \sum_{m=1}^{\infty} \frac{\langle T_1(x, y), \omega_m^0(x, y) \rangle}{\lambda_m^0 - \lambda} \omega_m^0(x, y) = \\
& \sum_{m=1}^{\infty} \frac{T_{1m}}{\lambda_m^0 - \lambda} \omega_m^0(x, y), \\
(L_0^* - \bar{\mu}I)^{-1} \sigma_1(x, y) &= \sum_{n=1}^{\infty} \frac{\langle \sigma_1(x, y), \omega_n^0(x, y) \rangle}{\lambda_n^0 - \bar{\mu}} \omega_n^0(x, y) = \sum_{n=1}^{\infty} \frac{\sigma_{1n}}{\lambda_n^0 - \bar{\mu}} \omega_n^0(x, y).
\end{aligned}$$

As a result, we have

$$\begin{aligned}
 & (\lambda - \mu) < \kappa_1(x, y, \lambda), M_1(x, y, \bar{\mu}) > = -d(\lambda) + d(\mu) - \\
 & -(\lambda - \mu)\mu \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{T_{1m}\sigma_{1n}}{(\lambda_m^0 - \lambda)(\lambda_n^0 - \mu)} \\
 & \lim_{\delta \rightarrow +0} \int_{\partial \Pi_{\delta}^0} \left[\frac{\partial}{\partial \bar{n}} \omega_m^0 \omega_n^0(x, y) - \omega_m^0(x, y) \frac{\partial}{\partial \bar{n}} \omega_n^0(x, y) \right] ds = -d(\lambda) + d(\mu) - \\
 & (\lambda - \mu)\mu \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{T_{1m}\sigma_{1n}}{(\lambda_m^0 - \lambda)(\lambda_n^0 - \mu)} \\
 & \lim_{\delta \rightarrow +0} \int_{\partial \Pi_{\delta}^0} [\Delta \omega_m^0 \omega_n^0(x, y) - \omega_m^0(x, y) \Delta \omega_n^0(x, y)] ds = -d(\lambda) + d(\mu) - \\
 & (\lambda - \mu)\mu \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{T_{1m}\sigma_{1n}}{(\lambda_m^0 - \lambda)(\lambda_n^0 - \mu)} (\lambda_m^0 - \lambda_n^0) \\
 & \lim_{\delta \rightarrow +0} \int_{\Pi_{\delta}^0} \omega_m^0 \omega_n^0(x, y) ds = -d(\lambda) + d(\mu)
 \end{aligned}$$

as if $m = n$, to $\lambda_m^0 - \lambda_n^0 = 0$. If $m \neq n$, then the integral is equal to zero due to the orthogonality of the eigenfunctions of the operator L_0 .

Lemma 2 is proved.

To establish the biorthogonality relations, we check the

$$< W_{n,s}(x, y), h_{k,m_k-1-p}(x, y) > = \begin{cases} 1, & \text{если } (n, s) = (k, p); \\ 0, & \text{если } (n, s) \neq (k, p). \end{cases}$$

We consider two eigenvalue λ_s and λ_k . They match pairs (s, t) and (k, p) , where $t = 0, 1, \dots, m_s - 1$ and $p = 0, 1, \dots, m_k - 1$. Note that the inner product

$$\begin{aligned}
 & < W_{n,s}(x, y), h_{k,m_k-1-p}(x, y) > = \\
 & = -\lim_{\lambda \rightarrow \lambda_s} \lim_{\mu \rightarrow \lambda_k} \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \frac{1}{(m_k-1-p)!} \frac{\partial^{m_k-1-p}}{\partial \mu^{m_k-1-p}} \left(< \kappa_1(x, y, \lambda), M_1(x, y, \bar{\mu}) > \frac{(\mu - \lambda_k)^{m_k}}{d(\mu)} \right).
 \end{aligned}$$

Recalling Lemma 2, we get equality

$$\begin{aligned}
 (6.1) \quad & < W_{n,s}(x, y), h_{k,m_k-1-p}(x, y) > = \\
 & = \lim_{\lambda \rightarrow \lambda_s} \lim_{\mu \rightarrow \lambda_k} \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \frac{1}{(m_k-1-p)!} \frac{\partial^{m_k-1-p}}{\partial \mu^{m_k-1-p}} \left(\frac{d(\lambda) - d(\mu)}{\lambda - \mu} \frac{(\mu - \lambda_k)^{m_k}}{d(\mu)} \right).
 \end{aligned}$$

We introduce the notation

$$(6.2) \quad H_{k,t}(\lambda) = \lim_{\mu \rightarrow \lambda_k} \frac{1}{t!} \frac{\partial^t}{\partial \mu^t} \left(\frac{d(\lambda) - d(\mu)}{\lambda - \mu} \frac{(\mu - \lambda_k)^{m_k}}{d(\mu)} \right)$$

We consider the function

$$F(\mu) = \frac{d(\lambda) - d(\mu)}{\lambda - \mu} \frac{(\mu - \lambda_k)^{m_k}}{d(\mu)}$$

and expand it in a neighborhood of the point $\mu = \lambda_k$ into a Taylor series. Then

$$F(\mu) = H_{k,0}(\lambda) + H_{k,1}(\lambda)(\mu - \lambda_k) + \dots + H_{k,m_k-1}(\lambda)(\mu - \lambda_k)^{m_k-1} + \dots,$$

that is, $H_{k,t}(\lambda)$ is the k -th Taylor coefficient, with a corresponding expansion, in a neighborhood of $\mu = \lambda_k$. The direct calculation of the coefficient of the Taylor series of the function $F(\mu)$ leads to the following formula for $t = 0, 1, \dots, m_k - 1$:

$$(6.3) \quad H_{k,t}(\lambda) = d(\lambda) \left(A_{k,m_k-1} \frac{1}{(\lambda - \lambda_k)^{t+1}} + A_{k,m_k-2} \frac{1}{(\lambda - \lambda_k)^t} + \dots \right. \\ \left. + A_{k,m_k-t-1} \frac{1}{\lambda - \lambda_k} \right),$$

where number $A_{k,m_k-1}, \dots, A_{k,0}$ define by identity

$$\frac{1}{d(\mu)} = \frac{A_{k,m_k-1}}{(\mu - \lambda_k)^{m_k}} + \frac{A_{k,m_k-2}}{(\mu - \lambda_k)^{m_k-1}} + \dots + \frac{A_{k,0}}{\mu - \lambda_k} + \sum_{q=0}^{\infty} B_{k,q} (\mu - \lambda_k)^q.$$

If $\lambda_s \neq \lambda_k$, that of (6.1), (6.2) and (6.3) when $n = 0, 1, \dots, m_k - 1$

$$\begin{aligned} < W_{n,s}(x, y), h_{k,m_k-1-p}(x, y) > = \lim_{\lambda \rightarrow \lambda_s} \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} H_{k,t}(\lambda) = \\ = \lim_{\lambda \rightarrow \lambda_s} \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} d(\lambda) \sum_{i=1}^{t+1} \frac{A_{k,m_k-t+i-2}}{(\lambda - \lambda_k)^i} = \frac{1}{n!} d^{(n)}(\lambda_s) \sum_{i=1}^{t+1} \frac{A_{k,m_k-t+i-2}}{(\lambda_s - \lambda_k)^i} = 0, \end{aligned}$$

in that $d^{(n)}(\lambda_s) = 0$.

Now consider the case $\lambda_s = \lambda_k$. Transform the right side of the relation (6.3)

$$(6.4) \quad H_{k,t}(\lambda) = d(\lambda) \sum_{i=1}^{t+1} \frac{A_{k,m_k-t+i-2}}{(\lambda_s - \lambda_k)^i} = \\ = d(\lambda) (\lambda - \lambda_k)^{m_k-t-1} \left(A_{k,m_k-1} \frac{1}{(\lambda - \lambda_k)^{m_k}} + A_{k,m_k-2} \frac{1}{(\lambda - \lambda_k)^{m_k-1}} + \dots \right. \\ \left. + A_{k,m_k-1-t} \frac{1}{(\lambda - \lambda_k)^{m_k-t}} \right) = d(\lambda) (\lambda - \lambda_k)^{m_k-t-1} \\ \left(\frac{1}{d(\lambda)} - A_{k,m_k-2} \frac{1}{(\lambda - \lambda_k)^{m_k-1-t}} - \dots - A_{k,0} \frac{1}{\lambda - \lambda_k} - \sum_{q=m_k}^{\infty} B_{k,q} (\lambda - \lambda_k)^q \right) = \\ (\lambda - \lambda_k)^{m_k-1-t} + \sum_{q=m_k}^{\infty} c_{kq}^t (\lambda - \lambda_k)^q, k = 0, 1, \dots, m_k - 1.$$

From relation (6.1), (6.2) and (6.3) we obtain

$$\begin{aligned} < W_{n,s}(x, y), h_{k,m_k-1-p}(x, y) > = \frac{1}{n!} \lim_{\lambda \rightarrow \lambda_k} \frac{\partial^n}{\partial \lambda^n} H_{k,m_k-1-p}(\lambda) = \\ \frac{1}{n!} \lim_{\lambda \rightarrow \lambda_k} \frac{\partial^n}{\partial \lambda^n} ((\lambda - \lambda_k)^{m_k-1-t} + \sum_{q=m_k}^{\infty} c_{kq}^t (\lambda - \lambda_k)^q). \end{aligned}$$

This implies the required assertion for $\lambda_s = \lambda_k$. Q.E.D.

Note the works of the authors [14, 15], where obtained the formulas of the first regularized traces for Laplace operator and double differentiation in the punctured areas.

ACKNOWLEDGEMENTS

The authors were supported by the Ministry of Education and Science of the Republic of Kazakhstan (MESRK) Grant AP05130994. No new data was collected or generated during the course of research.

Б. Бекболат¹, Д. Б. Нурахметов², Н. Токмагамбетов³, Г. Х. Аймак Раса⁴

^{1,3,4}Әл-Фараби атындағы ҚазҰУ, Алматы, Қазақстан;

²Сәкен Сейфуллин атындағы Қазақ ұлттық аграрлық университеті, Астана, Қазақстан;

³Гент университеті, Гент, Белгия;

^{1,3}Математика және математикалық модельдеу институты, Алматы, Қазақстан

ОЙЫЛҒАН АЙМАҚТАҒЫ ЛАПЛАС ОПЕРАТОРЫНЫҢ ТҮБІР ФУНКЦИЯЛАР ЖҮЙЕСІНІҢ МИНИМАЛДЫЛЫҒЫ

Аннотация. Біз осы мақаламызда, қисынды шешілетін шеттік есептердің «жаңа» классын туындататын, ойылған аймақтағы Лаплас операторын қарастырамыз. Осы есеп үшін резольвента формуласы алынған. Сонымен қатар, зерттелген есептің түбір функцияларын туындататын мероморфты функциялар сипатталған. Негізгі мақсатымыз түбір функциялар жүйесінің минималдылығын қарастыру. Біздің жұмыс ойылған аймақтағы Лаплас операторы үшін шеттік есептің шешімділігі сипатталған [8] жұмысының жалғасы болып табылады. Мақалада қисынды шешілетін шеттік есептердің «жаңа» классын туындататын, ойылған аймақтағы Лаплас операторы қарастырылып, туындаған есеп үшін резольвента формуласы алынған, сонымен қатар, функциялар жүйесін құратын мероморфты функциялар сипатталған. Меншікті және қосалқы функциялардың жүйесі осындай жүйенің бірі болып табылады. Соңғы бөлім түбір функциялардың минималдылығын зерттеуге арналған.

Түйін сөздер: Лаплас операторы, ойылған аймақ, резольвента, мероморфты функция, шеттік есептің қисынды шешімділігі, түбір функциялар жүйесі, минимал жүйе.

Б. Бекболат¹, Д. Б. Нурахметов², Н. Токмагамбетов³, Г. Х. Аймак Раса⁴

^{1,3,4}КазНУ им. Аль-Фараби, Алматы, Қазақстан;

²Казахский агротехнический университет имени Сакена Сейфуллина, Астана, Қазақстан;

³Гентский университет, Гент, Белгия;

^{1,3}Институт математики и математического моделирования, Алматы, Қазақстан;

О МИНИМАЛЬНОСТИ СИСТЕМ КОРНЕВЫХ ФУНКЦИЙ ОПЕРАТОРА ЛАПЛАСА В ПРОКОЛОТОЙ ОБЛАСТИ

Аннотация. В данной работе рассмотрен оператор Лапласа в проколотой области, который порождает класс "новых", корректно разрешимых краевых задач. И для этого класса задач получена формула резольвенты. Также описаны мероморфные функции, порождающие корневых функций класса исследуемых задач. Основная цель – изучение минимальности систем корневых функций. Статья является продолжением работы [8], где дано описание корректно разрешимых краевых задач для оператора Лапласа в проколотых областях. Рассмотрен оператор Лапласа в проколотой области, который порождает класс "новых", корректно разрешимых краевых задач, и для порожденных задач получена формула резольвенты, а также описаны мероморфные функции, которые индуцируют системы функций. Одна из этих систем, как раз, и является системой собственных и присоединенных функций. Последний раздел посвящен исследованию минимальности системы корневых функций.

Ключевые слова: Оператор Лапласа, проколотая область, резольвента, мероморфная функция, корректно разрешимая краевая задача, система корневых функций, минимальная система.

Information about authors:

Nurakhmetov Daulet – PhD doctor, S.Seifullin Kazakh Agro Technical University, Astana, Kazakhstan;

Tokmagambetov Niyaz – PhD doctor, Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Ghent, Belgium;

Bekbolat Bayan – Doctorate, Al-Farabi Kazakh National University, Almaty, Kazakhstan;

Aimal Rasa Ghulam Hazrat – Doctorate, Al-Farabi Kazakh National University, Almaty, Kazakhstan

REFERENCES

- [1] Landau L. D., Lifschitz E. M., *Course of theoretical physics*. M.: Nauka, 1989.
- [2] Berezin F. A., Faddeev L. D., Remark on the Scrodinger equation with singular potential, *Soviet Phys. Dokl.*, №2. 1961. P. 372–375.
- [3] Minlos R. A., Faddeev L. D., *Point interaction for a three-particle system in quantum mechanics*, *Docl. Akad. Nauk SSSR*. Vol. 141, №6. 1961. P. 1335–1338.
- [4] Berezin F. A., *About Lee 's model*, *Matematicheskii Sbornic*. Vol. 60, №4. 1963. P. 425–446.
- [5] M. I. Neiman-Zade, A. A. Shkalikov, Schrödinger operators with singular potentials from spaces of multipliers, *Mathematical notes*. Vol. 66, №5. 1999. P. 599–607.
- [6] Neiman-Zade M. I., Savchuk A. M., *Schrodinger Operators with Singular Potentials*, *Tr. Mat. Inst. Steklova*, Vol. 236. 2002. P. 262–371.
- [7] Goloschapova N. I., Zastavnyi V. P., Malamud M. M., *Positive definite functions and spectral properties of the Schrodinger operator with point interactions*, *Mathematical notes*, Vol. 90, №1. 2011. P. 149–154.
- [8] Kanguzhin B. E., Anijarov A. A., *Well-Posed Problem for the Laplace Operator in a Punctured Disk*, *Mat. Zametki*, Vol. 89, №6. 2011. P. 856–867.
- [9] Bitsadze A. V., *Equations of Mathematical Physics*, M.: Nauka, 1976.
- [10] Kanguzhin B. E., Tokmagambetov N. E., *On Regularized Trace Formulas for a Well-Posed Perturbation of the m-Laplace Operator*. *Differential Equations*. **51**:1583-1588 (2015).
- [11] Kanguzhin B. E., Tokmagambetov N. E., *Resolvents of well-posed problems for finite-rank perturbations of the polyharmonic operator in a punctured domain*. *Siberian Mathematical Journal*. **57**:265-273 (2016)
- [12] Anijarov A. A., *Resolutions of finite-dimensional perturbations of well-posed problems for the Laplace operator*, *Almaty*, 2010. C. 102.
- [13] Riesz F., Sz-Nagy B. *Lecons D'analyse fonctionnelle*. Sixiete edition, *Academiai kiado*, Budapest 1972.
- [14] Kanguzhin B. E., Tokmagambetov N. E., *Regularized trace formula for a well-perturbed Laplace operator*, *Doklady mathematics*, Vol. 460, №1. 2015. – P. 7–10.
- [15] Akhymbek M. E., Nurakhmetov D. B., *The first regularized trace for the two-fold differentiation operator in a punctured segment*, *Siberian mathematical journal*, Vol. 11. 2014. P. 626–633.
- [16] Seitmuratov A., Zharmenova B., Dauitbayeva A., Bekmuratova A. K., Tulegenova E., Ussenova G., *Numerical analysis of the solution of some oscillation problems by the decomposition method*, *News of the national academy of sciences of the republic of Kazakhstan, Series physic-mathematical*. Vol. 323, №1. 2019. P.28–37. ISSN 1991-346X <https://doi.org/10.32014/2019.2518-1726.4>