Spectrum of Volterra integral operator of the second kind

Meiramkul Amangaliyeva, Muvasharkhan Jenaliyev, Madi Ergaliev, and Murat Ramazanov

Citation: AIP Conference Proceedings **1759**, 020017 (2016); doi: 10.1063/1.4959631
View online: http://dx.doi.org/10.1063/1.4959631
View Table of Contents: http://aip.scitation.org/toc/apc/1759/1
Published by the American Institute of Physics

Spectrum of Volterra integral operator of the second kind

Meiramkul Amangaliyeva*, Muvasharkhan Jenaliyev*, Madi Ergaliev* and Murat Ramazanov†

*Institute of Mathematics and Mathematical Modeling, 050010, Almaty, Kazakhstan
†Buketov Karaganda Stated University, 100028, Karaganda, Kazakhstan

Abstract. The article addresses the singular Volterra integral equation of the second kind, which has the 'incompressible' kernel. It is shown that the corresponding homogeneous equation on $|\lambda| \geq \exp\{|\arg\lambda|\}, \quad \arg\lambda \in [-\pi,\pi]$ has a continuous spectrum, and the multiplicity of the characteristic numbers grows with increasing $|\lambda|$. We use the Carleman-Vekua regularization method. We introduce the characteristic integral equation. We prove that the initial integral equation has eigenfunctions, the multiplicity of which depends on the value of the spectral parameter λ. We prove the solvability theorem of the nonhomogeneous equation in a case when the right-hand side of the equation belongs to a certain class.

Keywords: Volterra integral equation, Spectrum, Eigenfunction
PACS: 02.30.Jr, 02.30.Rz

INTRODUCTION

In this paper we consider the singular Volterra integral equation with spectral parameter $\lambda \in \mathbf{C}$ of form

$$\varphi(t) - \lambda \int_0^t K(t,\,\tau)\,\varphi(\tau)\,d\tau = f(t), \quad t > 0, \tag{1}$$

where

$$K(t,\,\tau) = K^{(1)}(t,\,\tau) + K^{(2)}(t,\,\tau), \tag{2}$$

$$K^{(1)}(t,\,\tau) = \frac{1}{2a\sqrt{\pi}} \frac{t^\omega + \tau^\omega}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{(t^\omega + \tau^\omega)^2}{4a^2(t-\tau)}\right), \tag{3}$$

$$K^{(2)}(t,\,\tau) = \frac{1}{2a\sqrt{\pi}} \frac{t^\omega - \tau^\omega}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{(t^\omega - \tau^\omega)^2}{4a^2(t-\tau)}\right),\ \omega > 1/2. \tag{4}$$

We call such equations as the Volterra integral equations with 'incompressible' kernel [1]. It is shown that the corresponding homogeneous equation on $|\lambda| \geq \exp\{|\arg\lambda|\}, \quad \arg\lambda \in [-\pi,\pi]$ has a continuous spectrum, and the multiplicity of the characteristic numbers grows with increasing $|\lambda|$. We use the Carleman-Vekua regularization method. We introduce the characteristic integral equation. We prove that the initial integral equation has eigenfunctions, the multiplicity of which depends on the value of the spectral parameter λ. We prove the solvability theorem of the nonhomogeneous equation (1)–(4) in a case when the right-hand side of the equation belongs to a certain class.

PROPERTIES OF THE KERNEL $K(t,\,\tau)$ (2)–(4)

The kernel $K(t,\,\tau)$ (2)–(4) has the following properties:
1) $K(t,\,\tau) \geq 0$ and is continuous on $0 < \tau \leq t < \infty$;
2) $\lim\limits_{t\to t_0} \int_{t_0}^t K(t,\,\tau)\,d\tau = 0,\ t_0 \geq \varepsilon > 0$;
3) $\lim\limits_{t\to 0} \int_0^t K(t,\,\tau)\,d\tau = 1,\ \lim\limits_{t\to+\infty} \int_0^t K(t,\,\tau)\,d\tau = 1.$

The feature of equation (1) in question consists in property 3) of the kernel $K(t,\,\tau)$ and is expressed in the fact that the corresponding nonhomogeneous equation can not be solved by the successive approximations method for $|\lambda| \geq \exp\{|\arg\lambda|\}, \quad \arg\lambda \in [-\pi,\pi]$. Obviously, if $|\lambda| < \exp\{|\arg\lambda|\}, \quad \arg\lambda \in [-\pi,\pi]$ then equation (1) has a unique solution, that can be found by the successive approximations method. The case when $\lambda \in \mathbf{C}$ and $\omega = 1$ was considered in [1]. In this paper we assume that $|\lambda| \geq \exp\{|\arg\lambda|\}, \quad \arg\lambda \in [-\pi,\pi]$ and $\omega > 1/2$.

The property 3) of kernel $K(t,\,\tau)$ (2)–(4) follows from the next lemmas.

Lemma 1 *If* $\omega > \frac{1}{2}$, *then* $\lim\limits_{t\to 0}\int\limits_0^t K^{(1)}(t,\tau)d\tau = 1.$

Lemma 2 *If* $\omega > \frac{1}{2}$, *then* $\lim\limits_{t\to 0}\int\limits_0^t K^{(2)}(t,\tau)d\tau = 0.$

Lemma 3 *If* $\omega > \frac{1}{2}$, *then* $t^{3/2-\omega}\int\limits_0^t \frac{K^{(1)}(t,\tau)}{\tau^{3/2-\omega}}d\tau < C,\ 0 < t < \infty.$

Lemma 4 *If* $\omega > \frac{1}{2}$, *then* $t^{3/2-\omega}\int\limits_0^t \frac{K^{(2)}(t,\tau)}{\tau^{3/2-\omega}}d\tau < C(\omega),\ 0 < t < \infty.$

THE CHARACTERISTIC INTEGRAL EQUATION

According to the Carleman-Vekua regularization method we prove that for equation (1) the next integral equation

$$\varphi(t) - \lambda\int\limits_0^t K_0(t,\tau)\varphi(\tau)d\tau = F(t) \tag{5}$$

is characteristic, where

$$K_0(t,\tau) = \frac{1}{2a\sqrt{\pi}}\frac{(2\omega-1)^{3/2}t^{4\omega-3}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{3/2}}\exp\left[-\frac{2\omega-1}{4a^2}\frac{\left(t^{2\omega-1}+\tau^{2\omega-1}\right)^2}{t^{2\omega-1}-\tau^{2\omega-1}}\right],\ \omega > 1/2, \tag{6}$$

$$F(t) = f(t) - \lambda\int\limits_0^t \left[K_1(t,\tau)+K^{(2)}(t,\tau)\right]\varphi(\tau)d\tau, \tag{7}$$

$$K_1(t,\tau) = K_0(t,\tau) - K^{(1)}(t,\tau). \tag{8}$$

This follows from the assertions of Lemmas 6–8.

First, we note that kernel $K_0(t,\tau)$ (6) also has the property, similar properties 3) of kernel $K(t,\tau)$ (2)–(4). This property follows from Lemma 5.

Lemma 5 *If* $\omega > 1/2$, *then* $\lim\limits_{t\to 0}\int\limits_0^t K_0(t,\tau)d\tau = 1.$

Further, if we introduce a following notations

$$P^{(1)}(t,\tau) = \frac{t^\omega+\tau^\omega}{2a\sqrt{\pi}(t-\tau)^{3/2}},\ P_0(t,\tau) = \frac{2a\sqrt{\pi}(2\omega-1)^{3/2}t^{4\omega-3}}{(t^{2\omega-1}-\tau^{2\omega-1})^{3/2}},$$
$$Q^{(1)}(t,\tau) = \frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)},\ Q_0(t,\tau) = \frac{2\omega-1}{4a^2}\frac{(t^{2\omega-1}+\tau^{2\omega-1})^2}{t^{2\omega-1}-\tau^{2\omega-1}},$$

then

$$K_0(t,\tau) = P_0(t,\tau)e^{-Q_0(t,\tau)};\quad K^{(1)}(t,\tau) = P^{(1)}(t,\tau)e^{-Q^{(1)}(t,\tau)}.$$

Lemma 6 *If $\omega > 1/2$, then $\lim_{t\to 0}\int_0^t K_1(t,\tau)d\tau = 0$. In addition, we have the following estimate*

$$|K_1(t,\tau)| \le C(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}e^{-\tilde{Q}(t,\tau)} \quad (C(\omega) = const),$$

where

$$\tilde{Q}(t,\tau) = \min\{Q_0(t,\tau);\ \frac{1}{2}Q^{(1)}(t,\tau)\}.$$

Lemma 7 *We have the following estimates:*

$$\begin{aligned} |P_0(t,\tau) - P^{(1)}(t,\tau)| &\le C_2(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}, \\ P^{(1)}(t,\tau)\left|Q_0(t,\tau) - Q^{(1)}(t,\tau)\right|\exp\left\{-Q^{(1)}(t,\tau)\right\} &\le C_3(\omega)\frac{t^{2\omega-1}}{\sqrt{t-\tau}}\exp\left\{-\frac{Q^{(1)}(t,\tau)}{2}\right\}. \end{aligned}$$

Lemma 8 *If $\omega > 1/2$, then*

$$\left|K^{(2)}(t,\tau)\right| \le C_6(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}\exp\left\{-C_7(\omega)t^{2(\omega-1)}(t-\tau)\right\},$$

where functions $C_6(\omega)$ and $C_7(\omega)$ are constants continuous depending on the parameter $\omega > 1/2$.

From Lemmas 5-8 it follows directly that for equation (1) equation (5) is characteristic.

SOLVING THE CHARACTERISTIC EQUATION (5)

In equation (5) we make following changes of the independent variables (recall that $\gamma = 2\omega - 1$)

$$t = [\gamma t_1]^{-1/\gamma},\ \tau = [\gamma\tau_1]^{-1/\gamma},$$

and introduce notations ($0 < \tau_1 < t_1 < \infty$):

$$\begin{aligned} \mu(t_1) &= t_1^{\frac{\gamma/2-1}{\gamma}}\varphi([\gamma t_1]^{-1/\gamma}),\ F_1(t_1) = t_1^{\frac{\gamma/2-1}{\gamma}}F([\gamma t_1]^{-1/\gamma}), \\ k_0(t_1-\tau_1) &= \frac{1}{2a\sqrt{\pi}(\tau_1-t_1)^{3/2}}\exp\left(-\frac{1}{4a^2(\tau_1-t_1)}\right). \end{aligned} \tag{9}$$

Then equation (5) can be written as

$$\mu(t_1) - \lambda\int_{t_1}^{\infty}k_0(t_1-\tau_1)\mu(\tau_1)d\tau_1 = F_1(t_1), \quad 0 < t_1 < \tau_1 < \infty. \tag{10}$$

We have studied the equation (10) in work [2-4]. Therefore from the results [4] we have that the solution of characteristic integral equation (5) is determined as follows (where $\gamma = 2\omega - 1$):

$$\begin{aligned} \varphi(t) &= t^{\gamma/2-1}\mu([\gamma t^{\gamma}]^{-1}) = t^{\gamma/2-1}F(t) + \lambda t^{\gamma/2-1}\int_0^t \tau^{-\gamma-1}\mathbf{r}_{\lambda-}([\gamma t^{\gamma}]^{-1} - [\gamma\tau^{\gamma}]^{-1})F(\tau)d\tau \\ &\quad + t^{\gamma/2-1}\sum_{k=-N_1}^{N_2}c_k\exp(-iz_k[\gamma t^{\gamma}]^{-1}),\ t\in\mathbf{R}_+, \end{aligned} \tag{11}$$

where

$$\mathbf{r}_{\lambda-}(\theta)=2\sum_{k=-\infty}^{-(N_1+1)}\sqrt{iz_k}\exp(-iz_k\theta)+2\sum_{k=N_2+1}^{\infty}\sqrt{iz_k}\exp(-iz_k\theta)+\frac{1}{2\sqrt{\pi}(-\theta)^{3/2}}\sum_{m=1}^{\infty}\frac{m}{\lambda^m}\exp\left(\frac{m^2}{4\theta}\right),$$
$$\mathrm{Re}\,(iz_k)<0,\ |\lambda|>1,\ \theta\in\mathbf{R}_-, \tag{12}$$

the numbers $N_1,\ N_2,\ \{z_k,\ k\in\mathbf{Z}\}$ are defined by formulas

$$N_1 = \left[\frac{\ln|\lambda|+\arg\lambda}{2\pi}\right],\ N_2=\left[\frac{\ln|\lambda|-\arg\lambda}{2\pi}\right], \tag{13}$$

$$z_k = 2(\arg\lambda+2k\pi)\ln|\lambda|-i\left[\ln^2|\lambda|-(\arg\lambda+2k\pi)^2\right]. \tag{14}$$

Formula (13) follows from the boundedness of the solutions of homogeneous conditions (5), which is equivalent to the condition Re $\{iz_k\}\geq 0$ for roots z_k defined by formula (14). The number of such roots will always be the end! Detailed calculations are in [4].

Thus we have proved the following theorem.

Theorem 9 *General solution of the characteristic integral equation (5) has representation (11).*

Substituting in integral equation (11) the expression for $F(t)$ according to formulas (7)–(8), we obtain an equation:

$$\begin{aligned}\varphi(t) &= t^{\gamma/2-1}f(t)-\lambda t^{\gamma/2-1}\int_0^t\left[K_0(t,\tau)-K^{(1)}(t,\tau)+K^{(2)}(t,\tau)\right]\varphi(\tau)d\tau\\ &+\lambda t^{\gamma/2-1}\int_0^t\tau^{-\gamma-1}\mathbf{r}_{\lambda-}([\gamma t^\gamma]^{-1}-[\gamma\tau^\gamma]^{-1})\left\{f(\tau)-\lambda\int_0^\tau\left[K_0(\tau,\tau_1)-K^{(1)}(\tau,\tau_1)\right.\right.\\ &\left.\left.+K^{(2)}(\tau,\tau_1)\right]\varphi(\tau_1)d\tau_1\right\}d\tau+t^{\gamma/2-1}\sum_{k=-N_1}^{N_2}c_k\exp(-iz_k[\gamma t^\gamma]^{-1}),\ t\in\mathbf{R}_+,\end{aligned}$$

which can be rewritten as

$$\varphi(t)-\lambda t^{\gamma/2-1}\int_0^t\widehat{\mathbf{K}}(t,\tau)\varphi(\tau)d\tau=t^{\gamma/2-1}\hat{f}(t)+t^{\gamma/2-1}\sum_{k=-N_1}^{N_2}c_k\exp(-iz_k[\gamma t^\gamma]^{-1}),\ t\in\mathbf{R}_+, \tag{15}$$

where

$$\begin{aligned}\widehat{\mathbf{K}}(t,\tau) &= \widetilde{\mathbf{K}}(t,\tau)+\lambda\int_\tau^t\eta^{-\gamma-1}\mathbf{r}_{\lambda-}([\gamma t^\gamma]^{-1}-[\gamma\eta^\gamma]^{-1})\widetilde{\mathbf{K}}(\eta,\tau)d\eta,\\ \widetilde{\mathbf{K}}(t,\tau) &= -K_0(t,\tau)+K^{(1)}(t,\tau)-K^{(2)}(t,\tau),\\ \hat{f}(t) &= f(t)+\lambda\int_0^t\tau^{-\gamma-1}\mathbf{r}_{\lambda-}([\gamma t^\gamma]^{-1}-[\gamma\tau^\gamma]^{-1})f(\tau)d\tau.\end{aligned}$$

Function $\mathbf{r}_{\lambda-}(\theta)$ is determined by formula (12).

Note that Lemmas 1–8 justify the Carleman-Vekua regularization method [5] for integral equation (5), i.e., we obtain a following result.

MAIN RESULT

Theorem 10 *Integral equation (15) for any*

$$t^{3/2-\omega}f(t)\in L_\infty(0,\infty)$$

has a unique solution

$$\varphi(t)=t^{3/2-\omega}\varphi(t)\in L_\infty(0,\infty).$$

ACKNOWLEDGMENTS

This publication is supported by a grant under the grant number 0823/GF4 from the Ministry of Science and Education of the Republic of Kazakhstan.

REFERENCES

1. M. T. Jenaliyev, M. M. Amangaliyeva, M. T. Kosmakova, and M. I. Ramazanov, *Advances in Difference Equations* **2015**, 1-14 (2015), (DOI: 10.1186/s13662-015-0418-6).
2. M. T. Dzhenaliev, and M. I. Ramazanov, *Siberian Mathematical Journal* **47**, 527-547 (2006), (in Russian).
3. M. T. Dzhenaliev, and M. I. Ramazanov, *Differential Equations* **43**, 498-508 (2007), (in Russian).
4. M. T. Dzhenaliev, and M. I. Ramazanov, *Differential Equations*, **43**, 788-794 (2007), (in Russian).
5. I. N. Vekua, *Generalizing Analytical Functions*, Nauka, Moscow, 1959, (in Russian).