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Complex Analysis and Operator Theory

Horizontal Weighted Hardy–Rellich Type Inequalities on Stratified Lie Groups

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Abstract This paper is devoted to present a version of horizontal weighted Hardy–Rellich type inequality on stratified Lie groups and study some of its consequences. In particular, Sobolev type spaces are defined on stratified Lie groups and proved embedding theorems for these functional spaces.

Keywords Hardy–Rellich inequality · Caffarelli–Kohn–Nirenberg inequality · Horizontal estimate · Stratified group · Sobolev type spaces · Embedding theorem

Mathematics Subject Classification 22E30 · 43A80

1 Introduction

The main aim of this paper is to give analogues of Hardy–Rellich type inequalities on stratified groups with horizontal gradients and weights. Our results extend known Hardy and Rellich type inequalities as well as Caffarelli–Kohn–Nirenberg (CKN)

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type inequality on abelian and Heisenberg groups, for example (see e.g. [1–4]). We refer recent papers [10,11] on different dicussions for more general settings. For the convenience of the reader let us now briefly recapture the main results of this paper. Let $\mathbb{G}$ be a homogeneous stratified group of homogeneous dimension Q, and let $X_1, \ldots, X_N$ be left-invariant vector fields giving the first stratum of the Lie algebra of $\mathbb{G}$, $\nabla_H = (X_1, \ldots, X_N)$, with the sub-Laplacian

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2.$$

Denote the variables on $\mathbb{G}$ by $x = (x', x'') \in \mathbb{G}$, where x' corresponds to the first stratum. For precise definitions we refer to Sect. 2.

Thus, to summarise briefly, in this paper we establish the following results:

- **(Hardy–Rellich type inequalities)** Let $\mathbb{G}$ be a homogeneous stratified group with N being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, we have

$$\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx + (\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right)^2 \leq \int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx, \tag{1.1}$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. Moreover, for $\alpha+\beta+3 \leq N$ we have

$$\frac{|N+\alpha+\beta-1|}{2}\left\|\frac{\nabla_H f}{|x'|^{\frac{\alpha+\beta+1}{2}}}\right\|^2_{L^2(\mathbb{G})} \leq \left\|\frac{\mathcal{L}f}{|x'|^{\beta}}\right\|_{L^2(\mathbb{G})}\left\|\frac{\nabla_H f}{|x'|^{\alpha}}\right\|_{L^2(\mathbb{G})}, \tag{1.2}$$

with the sharp constant. In the special case of $\alpha = 1$ and $\beta = 0$, the inequality (1.2) gives the following stratified group version of Rellich's inequality

$$\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^2}dx \leq \left(\frac{2}{N}\right)^2\int_{\mathbb{G}}|\mathcal{L}f|^2dx, \quad 4 \leq N, \tag{1.3}$$

again with the constant $\left(\frac{2}{N}\right)^2$ being sharp. In turn, we have the following stratified group version of the weighted Hardy inequality (see [9])

$$\left\|\frac{f}{|x'|^2}\right\|_{L^2(\mathbb{G})} \leq \frac{2}{N-4}\left\|\frac{\nabla_H f}{|x'|}\right\|_{L^2(\mathbb{G})}, \quad 4 < N, \tag{1.4}$$

again with $\frac{2}{N-2}$ being the best constant. Now combining (1.4) with (1.3) we obtain

$$\left\|\frac{f}{|x'|^2}\right\|_{L^2(\mathbb{G})} \leq \frac{4}{N(N-4)}\|\mathcal{L}f\|_{L^2(\mathbb{G})}, \quad 4 < N, \tag{1.5}$$

with the sharp constant. In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \dots, \partial_{x_n})$, and $\mathcal{L} = \Delta$, so it follows the classical Rellich inequality

$$\left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{4}{n(n-4)} \|\Delta f\|_{L^2(\mathbb{R}^n)}, \quad 4 < n, \tag{1.6}$$

for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$.

- **(Another proof of L^2-CKN inequality)** The short proof of special ($p = 2$) case of horizontal L^p-Caffarelli–Kohn–Nirenberg type inequality from the paper [9] by M. Ruzhansky and the second author.
- **(Embedding results)** Let $\mathbb{G}$ be a homogeneous stratified group with N being the dimension of the first stratum, and let $\alpha,\ \beta \in \mathbb{R}$. We have the following continuous embedding
  (i)
  $$H^2_{\alpha,\beta}(\mathbb{G}) \subset D^{2,2}_{\frac{\alpha+\beta+1}{2}}(\mathbb{G}),$$
  for $\alpha + \beta - 1 \neq N$.
  (ii)
  $$D^{2,2}_{\alpha}(\mathbb{G}) \subset D^{1,2}_{\alpha+1}(\mathbb{G}),$$
  for $\alpha \leq \frac{N}{2} - 2$ with $\alpha \neq \frac{N}{2}$.
  (iii)
  $$H^1_{\alpha,\beta}(\mathbb{G}) \subset L^2_{\gamma/2}(\mathbb{G}), \tag{1.7}$$
  and
  $$H^1_{\beta,\alpha}(\mathbb{G}) \subset L^2_{\gamma/2}(\mathbb{G}), \tag{1.8}$$
  for $\gamma \neq N$.

In Sect. 2 we very briefly recall the main concepts of stratified groups and fix the notation. These discussions with Costa's ideas from [2] and [3] will play a key role in our proofs. In Sect. 3 we derive versions of Hardy–Rellich type inequalities on stratified groups and discuss their consequences including embedding results.

2 Preliminaries

A Lie group $\mathbb{G} = (\mathbb{R}^n, \circ)$ is called a stratified (Lie) group if it satisfies the following conditions:

(a) For some natural numbers $N + N_2 + \cdots + N_r = n$, that is $N = N_1$, the decomposition $\mathbb{R}^n = \mathbb{R}^N \times \cdots \times \mathbb{R}^{N_r}$ is valid, and for every $\lambda > 0$ the dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\delta_\lambda(x) \equiv \delta_\lambda(x', x^{(2)}, \dots, x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)})$$

is an automorphism of the group $\mathbb{G}$. Here $x' \equiv x^{(1)} \in \mathbb{R}^N$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \ldots, r$.

(b) Let N be as in (a) and let $X_1, \ldots, X_N$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, \ldots, N$. Then

$$\operatorname{rank}(\operatorname{Lie}\{X_1, \ldots, X_N\}) = n,$$

for every $x \in \mathbb{R}^n$, i.e. the iterated commutators of $X_1, \ldots, X_N$ span the Lie algebra of $\mathbb{G}$.

That is, we say that the triple $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ is a stratified group. See also e.g. [6] for discussions from the Lie algebra point of view. Here r is called a step of $\mathbb{G}$ and the left invariant vector fields $X_1, \ldots, X_N$ are called the (Jacobian) generators of $\mathbb{G}$. The number

$$Q = \sum_{k=1}^{r} k N_k,$$

is called the homogeneous dimension of $\mathbb{G}$. The second order differential operator

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2, \quad N_1 = N, \tag{2.1}$$

is called the (canonical) sub-Laplacian on $\mathbb{G}$. The sub-Laplacian $\mathcal{L}$ is a left invariant homogeneous hypoelliptic differential operator and it is known that $\mathcal{L}$ is elliptic if and only if the step of $\mathbb{G}$ is equal to 1. We also recall that the standard Lebesque measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [6, Proposition 1.6.6]). The left invariant vector field X_j has an explicit form and satisfies the divergence theorem, see e.g. [6] for the derivation of the exact formula: more precisely, we can write

$$X_k = \frac{\partial}{\partial x_k'} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}\left(x', \ldots, x^{(l-1)}\right) \frac{\partial}{\partial x_m^{(l)}}, \tag{2.2}$$

see also [6, Section 3.1.5] for a general presentation. We will also use the following notations

$$\nabla_H := (X_1, \ldots, X_N)$$

for the horizontal gradient,

$$\operatorname{div}_H v := \nabla_H \cdot v$$

for the horizontal divergence,

$$\mathcal{L}_p f := \operatorname{div}_H(|\nabla_H f|^{p-2} \nabla_H f), \quad 1 < p < \infty, \tag{2.3}$$

for the horizontal p-Laplacian (or p-sub-Laplacian), and

$$|x'| = \sqrt{x_1'^2 + \cdots + x_N'^2}$$

for the Euclidean norm on $\mathbb{R}^N$.

The explicit representation (2.2) allows us to have the identity

$$\operatorname{div}_H\left(\frac{x'}{|x'|^\gamma}\right) = \frac{\sum_{j=1}^N |x'|^\gamma X_j x_j' - \sum_{j=1}^N x_j' \gamma |x'|^{\gamma-1} X_j |x'|}{|x'|^{2\gamma}} = \frac{N-\gamma}{|x'|^\gamma} \tag{2.4}$$

for all $\gamma \in \mathbb{R},\ |x'| \neq 0$.

3 Horizontal Hardy–Rellich Type Inequalities and Embedding Results

Below we adopt all the notation introduced in Sect. 2 concerning stratified groups and the horizontal operators.

Theorem 3.1 *Let $\mathbb{G}$ be a homogeneous stratified group with N being the dimension of the first stratum, and let $\alpha,\ \beta \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ we have*

$$\begin{aligned}&\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx + (\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right)^2\\ &\quad\leq \int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx,\end{aligned} \tag{3.1}$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. Moreover, for $\alpha+\beta+3 \leq N$ we have

$$\begin{aligned}&\frac{|N+\alpha+\beta-1|}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx\\ &\quad\leq \left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{\frac{1}{2}},\end{aligned} \tag{3.2}$$

with the sharp constant.

Let us define the following Sovolev type spaces on the stratified Lie group $\mathbb{G}$:

- Let $D_\gamma^{1,2}(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{D_\gamma^{1,2}(\mathbb{G})} = \left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\gamma}}dx\right)^{\frac{1}{2}}.$$

- Let $D^{2,2}_{\gamma}(\mathbb{G})$ be the completion of $C_0^{\infty}(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{D^{2,2}_{\gamma}(\mathbb{G})} = \left(\int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\gamma}}dx\right)^{\frac{1}{2}}.$$

- Let $H^2_{\alpha,\beta}(\mathbb{G})$ be the completion of $C_0^{\infty}(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{H^2_{\alpha,\beta}(\mathbb{G})} = \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}} + \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\right)^{\frac{1}{2}}.$$

Theorem 3.2 *Let $\mathbb{G}$ be a homogeneous stratified group with N being the dimension of the first stratum, and let $\alpha,\ \beta \in \mathbb{R}$. We have the following continuous embedding*

(i)

$$H^2_{\alpha,\beta}(\mathbb{G}) \subset D^{2,2}_{\frac{\alpha+\beta+1}{2}}(\mathbb{G}),$$

for $\alpha+\beta-1 \neq N$.

(ii)

$$D^{2,2}_{\alpha}(\mathbb{G}) \subset D^{1,2}_{\alpha+1}(\mathbb{G}),$$

for $\alpha \leq \frac{N}{2}-2$ with $\alpha \neq \frac{N}{2}$.

In the abelian case $\mathbb{G}=(\mathbb{R}^n,+)$, we have $N=n$, $\nabla_H=\nabla=(\partial_{x_1},\ldots,\partial_{x_n})$, so (3.1) implies the Hardy–Rellich type inequality (see e.g. [2,5]) for $\mathbb{G}\equiv\mathbb{R}^n$:

$$\left(\frac{n-(\alpha+\beta+3)}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{\|x\|^{\alpha+\beta+1}}dx + (\alpha+\beta+1)\int_{\mathbb{R}^n} \frac{(x\cdot\nabla f)^2}{\|x\|^{\alpha+\beta+3}}dx\right)^2 \leq \int_{\mathbb{R}^n} \frac{|\Delta f|^2}{\|x\|^{2\beta}}dx \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{\|x\|^{2\alpha}}dx, \tag{3.3}$$

for all $f \in C_0^{\infty}(\mathbb{R}^n\backslash\{0\})$, and $\|x\|=\sqrt{x_1^2+\cdots+x_n^2}$.

When $\alpha=1$ and $\beta=0$, the inequality (3.2) gives the following stratified group version of Rellich's inequality

$$\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^2}dx \leq \left(\frac{2}{N}\right)^2 \int_{\mathbb{G}} |\mathcal{L}f|^2 dx, \quad 4 \leq N, \tag{3.4}$$

with $\left(\frac{2}{N}\right)^2$ being the best constant.

Directly from the inequality (3.2), choosing α and β, we can obtain a number of Heisenberg–Pauli–Weyl type uncertainly inequalities which have various consequences and applications. For instance,

- $$\frac{|N+2\alpha|}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2(\alpha+1)}}dx \le \left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2(\alpha+1)}}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{\frac{1}{2}}$$

  for $\alpha \le \frac{N}{2}-2$ and any $f \in H^2_{\alpha,\alpha+1}(\mathbb{G})$.
- $$\frac{|N-2|}{2}\int_{\mathbb{G}}|\nabla_H f|^2 dx \le \left(\int_{\mathbb{G}}|x'|^{2(\alpha+1)}|\nabla_H f|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\alpha}}dx\right)^{\frac{1}{2}}$$

  for $3 \le N$ and any $f \in D^{1,2}_0(\mathbb{G})$.
- $$\frac{N}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^2}dx \le \left(\int_{\mathbb{G}}|\nabla_H f|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^2}dx\right)^{\frac{1}{2}}$$

  for any $f \in D^{1,2}_1(\mathbb{G})$.
- $$\frac{N-1}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|}dx \le \left(\int_{\mathbb{G}}|x'|^2|\nabla_H f|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^2}dx\right)^{\frac{1}{2}}$$

  for $2 \le N$ and any $f \in D^{1,2}_{1/2}(\mathbb{G})$.
- $$\frac{N-1}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|}dx \le \left(\int_{\mathbb{G}}|\nabla_H f|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}|\mathcal{L}f|^2 dx\right)^{\frac{1}{2}}$$

  for $2 \le N$ and any $f \in D^{1,2}_{1/2}(\mathbb{G})$.

Proof of Theorem 3.1 For all $s \in \mathbb{R}^n$ we have

$$\int_{\mathbb{G}}\left|\frac{\nabla_H f}{|x'|^{\alpha}}+s\frac{x'}{|x'|^{\beta+1}}\mathcal{L}f\right|^2 dx \ge 0,$$

that is,

$$\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx + 2s\int_{\mathbb{G}}\frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\mathcal{L}f dx + s^2\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx \ge 0. \tag{3.5}$$

Since

$$\int_{\mathbb{G}}\frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\mathcal{L}f dx = \int_{\mathbb{G}}\mathrm{div}_H(\nabla_H f)\left(\frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\right)dx$$

by using the divergence theorem (see, e.g. [7] for Heisenberg group discussions and [8] for general discussions) and (2.4) we obtain

$$\int_{\mathbb{G}} \operatorname{div}_H(\nabla_H f)\left(\frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\right)dx = -\frac{1}{2}\int_{\mathbb{G}} \frac{x'}{|x'|^{\alpha+\beta+1}}\cdot\nabla_H(|\nabla_H f|^2)dx$$
$$-\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx + (\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx.$$

Again by the divergence theorem and (2.4) we get

$$-\frac{1}{2}\int_{\mathbb{G}} \frac{x'}{|x'|^{\alpha+\beta+1}}\cdot\nabla_H(|\nabla_H f|^2)dx = \frac{N-(\alpha+\beta+1)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx.$$

Thus

$$\int_{\mathbb{G}} \frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\mathcal{L}f\,dx$$
$$= \frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx + (\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx. \tag{3.6}$$

Therefore, the Eq. (3.5) can be restated as

$$s^2\int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx + 2s\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx\right.$$
$$\left.+(\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right) + \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx \geq 0. \tag{3.7}$$

Denoting

$$a := \int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx,$$
$$b := \frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx + (\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx,$$

and

$$c := \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx$$

we arrive at

$$as^2 + 2bs + c \geq 0, \tag{3.8}$$

which is equivalent to $b^2 - ac \leq 0$. Thus, we have

$$\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx+(\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right)^2 \leq \int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx. \tag{3.9}$$

This shows the inequality (3.1). Now it remains to prove (3.2). It can be proved similarly. We refer [9] for a different proof of (3.2). □

Proof of Theorem 3.2 Since $N \neq \alpha+\beta-1$, from (3.2) we obtain

$$\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\frac{(\alpha+\beta+1)}{2}}}dx \leq \frac{2}{|N+\alpha+\beta-1|}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{\frac{1}{2}} \leq \frac{2}{|N+\alpha+\beta-1|}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx+\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right),$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$. This proves Part (i). Part (ii) also implies from the inequality (3.2), namely assuming $\alpha+\beta+3 \leq N$ and letting $\beta = \alpha+1,\ \alpha \neq \frac{N}{2}$. □

The following theorem was proved by M. Ruzhansky and the second author in [9].

Theorem 3.3 [9] (Caffarelli–Kohn–Nirenberg type inequalities) *Let $\mathbb{G}$ be a homogeneous stratified group N being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, and all $1 < p < \infty$, we have*

$$\frac{|N-\gamma|}{p}\left\|\frac{f}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\mathbb{G})}^p \leq \left\|\frac{\nabla_H f}{|x'|^{\alpha}}\right\|_{L^p(\mathbb{G})}\left\|\frac{f}{|x'|^{\frac{\beta}{p-1}}}\right\|_{L^p(\mathbb{G})}^{p-1} \tag{3.10}$$

where $\gamma = \alpha+\beta+1$ and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. If $\gamma \neq N$ then the constant $\frac{|N-\gamma|}{2}$ is sharp.

Let us restated the special $p = 2$ case of the above theorem and show its another proof using our techniques.

Corollary 3.4 *Let $\mathbb{G}$ be a homogeneous stratified group N being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, we have*

$$\frac{|N-\gamma|}{2}\left\|\frac{f}{|x'|^{\frac{\gamma}{2}}}\right\|_{L^2(\mathbb{G})}^2 \leq \left\|\frac{\nabla_H f}{|x'|^{\alpha}}\right\|_{L^2(\mathbb{G})}\left\|\frac{f}{|x'|^{\beta}}\right\|_{L^2(\mathbb{G})} \tag{3.11}$$

where $\gamma = \alpha+\beta+1$ and then the constant $\frac{|N-\gamma|}{2}$ is sharp.

Proof Given $f \in C_0^\infty(\mathbb{G}\backslash x' = 0)$ arbitrary and $\alpha, \beta \in \mathbb{R}$, we have

$$\int_{\mathbb{G}} \left| \frac{\nabla_H f}{|x'|^\beta} + s\frac{x'}{|x'|^{\alpha+1}} f \right|^2 dx \geqslant 0 \tag{3.12}$$

for every $s \in \mathbb{R}$.

$$\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\beta}} dx + s^2 \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx + 2s \int_{\mathbb{G}} f \frac{x' \cdot \nabla_H f}{|x'|^\gamma} dx \geqslant 0 \tag{3.13}$$

by using divergence theorem

$$\begin{aligned}
&\int_{\mathbb{G}} f \frac{x' \cdot \nabla_H f}{|x'|^\gamma} dx = -\frac{N-\gamma}{2} \int_{\mathbb{G}} \frac{|f|^2}{|x'|^\gamma} dx \\
&a = \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx, \quad b = [N-\gamma] \int_{\mathbb{G}} \frac{|f|^2}{|x'|^\gamma}, \ c = \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\beta}} dx \\
&as^2 - bs + c \geqslant 0
\end{aligned}$$

This is equivalent to $b^2 - 4ac \leqslant 0$

$$[N-\gamma]^2 \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^\gamma} \right)^2 \leqslant 4 \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx \right) \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\beta}} dx \right). \tag{3.14}$$

□

Remark 3.5 Let us denote by $H^1_{\alpha,\beta}(\mathbb{G})$ the completion of $C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ with respect to the weighted Sobolev type norm

$$\|f\|_{H^1_{\alpha,\beta}} := \left(\int_{\mathbb{G}} \left[\frac{|f|^2}{|x'|^{2\alpha}} + \frac{|\nabla_H f|^2}{|x'|^{2\beta}} \right] dx \right)^{1/2} \tag{3.15}$$

and by $L^2_\alpha(\mathbb{G})$ the completion of $C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ with respect to the weighted Lebesgue norm

$$\|f\|_{L^2_\alpha} := \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx \right)^{1/2} \tag{3.16}$$

Then the inequality 3.11 implies that, for $\gamma \neq N$, we have the continuous embedding

$$H^1_{\alpha,\beta}(\mathbb{G}) \subset L^2_{\gamma/2}(\mathbb{G}). \tag{3.17}$$

Moreover, since the right-hand side above is symmetric with respect to the parameters α, β we also have the continuous embedding

$$H^1_{\beta,\alpha}(\mathbb{G}) \subset L^2_{\gamma/2}(\mathbb{G}). \tag{3.18}$$

Next, we present some consequences of 3.3.

Corollary 3.6 *The inequalities below hold true with sharp constants*:

- *For any* $f \in D^{1,2}(\mathbb{G})$ *and* $\alpha = 1$, $\beta = 0$ *it follows that*

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|f|^2}{|x'|^2} dx \leq \int_{\mathbb{G}} |\nabla f|^2 dx. \tag{3.19}$$

- *For any* $f \in H^1_{\beta+1,\beta}(\mathbb{G})$ *and* $\alpha = \beta + 1$ *it follows that*

$$\left(\frac{N-2(\beta+1)}{2}\right)^2 \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2(\beta+1)}} dx \leq \int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^{2\beta}} dx. \tag{3.20}$$

- *For any* $f \in H^1_{\alpha,\alpha+1}(\mathbb{G})$ *and* $\beta = \alpha + 1$ *it follows that*

$$\begin{aligned}\left(\frac{N-2(\alpha+1)}{2}\right)^2 &\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2(\alpha+1)}} dx \\ &\leq \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^{2(\alpha+1)}} dx\right)^{1/2}.\end{aligned} \tag{3.21}$$

- *For any* $f \in H^1_{-(\beta+1),\beta}(\mathbb{G})$ *and* $\alpha = -(\beta+1)$, *then* $f \in L^2(\mathbb{G})$ *and*

$$\left(\frac{N}{2}\right) \int_{\mathbb{G}} |u|^2 dx \leq \left(\int_{\mathbb{G}} |x'|^{2(\beta+1)} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^{2\beta}} dx\right)^{1/2}. \tag{3.22}$$

- *For any* $f \in H^1_{0,1}(\mathbb{G})$ *and* $\alpha = 0$, $\beta = 1$, *then* $f \in L^2_1(\mathbb{G})$ *and*

$$\left|\frac{N-2}{2}\right| \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^2} dx\right)^{1/2}. \tag{3.23}$$

- *For any* $f \in H^1_{-1,1}(\mathbb{G})$, $N > 1$ *and* $\alpha = -1$, $\beta = 1$, *then* $f \in L^2_{1/2}(\mathbb{G})$ *and*

$$\left(\frac{N-1}{2}\right) \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |x'|^2 |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^2} dx\right)^{1/2}. \tag{3.24}$$

- *For any* $f \in H^1(\mathbb{G}) = H^1_{0,0}(\mathbb{G})$, $N > 1$ *and* $\alpha = 0$, $\beta = 0$, *then* $f \in L^2_{1/2}(\mathbb{G})$ *and*

$$\left(\frac{N-1}{2}\right) \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} |\nabla f|^2 dx\right)^{1/2}. \tag{3.25}$$

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