Lateral boundary conditions for the Klein-Gordon-Fock equation

Kanat S. Tulenov and Dostilek Dauitbek

Citation: AIP Conference Proceedings **1759**, 020144 (2016); doi: 10.1063/1.4959758
View online: http://dx.doi.org/10.1063/1.4959758
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1759?ver=pdfcov
Published by the AIP Publishing

Lateral boundary conditions for the Klein-Gordon-Fock equation

Kanat S. Tulenov*,† and Dostilek Dauitbek*,†

*Institute of Mathematics and Mathematical Modeling, 050010, Almaty, Kazakhstan
†Al-Farabi Kazakh National University, 050040, Almaty, Kazakhstan

Abstract. In this paper we consider an initial-boundary value problem for the Klein-Gordon-Fock equation. We prove the uniqueness of the solution and find lateral boundary conditions for the Klein-Gordon-Fock equation.

Keywords: Klein-Gordon-Fock equation, Fundamental solution, Boundary conditions
PACS: 02.30.Em, 02.30.Jr

INTRODUCTION

In [1], authors considered one dimensional potential

$$u(x,t) = \int_\Omega (-\frac{1}{2}|x-y|) f(y) dy \tag{1}$$

in $\Omega = (0,1)$ and the equation

$$-u^{''}(x) = f(x),\ x \in \Omega, \tag{2}$$

and they showed that if equation (2) is solved with the boundary conditions

$$u^{'}(0) + u^{'}(1) = 0,\ -u^{'}(1) + u(0) + u(1) = 0, \tag{3}$$

then a unique solution of this boundary value problem is found in the form (1).

This simple method allows us to find equivalent boundary conditions for one-dimensional potential integrals. However this task becomes difficult for PDE, and in works [1, 2] boundary conditions of the volume potentials for elliptic and hyperbolic equations are obtained and some their applications are showed. In particular, in [2], by using a new nonlocal boundary value problem, which is equivalent to the Newton potential, authors found explicitly all eigenvalues and eigenfunctions of the Newton potential in the 2-disk and the 3-ball (see [3] and the references therein). The aim of this work is to find lateral boundary conditions for the Klein-Gordon-Fock equation as shown in [1]. Unlike elliptic and parabolic cases, where they have obtained nonlocal boundary conditions for the corresponding volume potentials, and some other nonclassic nonlocal boundary conditions for initial boundary value problems of hyperbolic equations (for example, see [4]), we get a local initial boundary value problem for the Klein-Gordon-Fock equation.

MAIN RESULT

In the bounded domain $\Omega \equiv \{(x,t) : (0,l) \times (0,T)\}$ we consider the potential

$$u(x,t) = \int_\Omega \varepsilon(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau, \tag{4}$$

where $\varepsilon(x-\xi,t-\tau) = \frac{1}{2}\theta(t-\tau-|x-\xi|)J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})$ is a fundamental solution of Cauchy problem for Klein-Gordon-Fock equation and J_0 is Bessel function (see [5]); i.e.,

$$\frac{\partial^2 \varepsilon(x-\xi,t-\tau)}{\partial t^2} - \frac{\partial^2 \varepsilon(x-\xi,t-\tau)}{\partial x^2} + m_0^2 \varepsilon = \delta(x-\xi,t-\tau),$$

International Conference on Analysis and Applied Mathematics (ICAAM 2016)
AIP Conf. Proc. 1759, 020144-1–020144-5; doi: 10.1063/1.4959758
Published by AIP Publishing. 978-0-7354-1417-4/$30.00

$$\frac{\partial^2 \varepsilon(x-\xi,t-\tau)}{\partial\tau^2}-\frac{\partial^2 \varepsilon(x-\xi,t-\tau)}{\partial\xi^2}+m_0^2\varepsilon=\delta(x-\xi,t-\tau),$$

$$\varepsilon(x-\xi,t-\tau)|_{\tau=t}=\left.\frac{\partial\varepsilon(x-\xi,t-\tau)}{\partial t}\right|_{\tau=t}=\left.\frac{\partial\varepsilon(x-\xi,t-\tau)}{\partial\tau}\right|_{\tau=t}=0$$

if $f(x,t)\in L_2(\Omega)$ then $u(x,t)\in W_2^1(\Omega)\cap W_2^1(\partial\Omega)$ and this potential (4) satisfies equation (see [5])

$$\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial^2 u(x,t)}{\partial x^2}+m_0^2u(x,t)=f(x,t),\quad (x,t)\in\Omega \tag{5}$$

with the initial conditions

$$u(x,0)=u_t(x,0)=0,\quad 0<x<l. \tag{6}$$

The potential (4) is widely used to solve various initial-boundary problems for the Klein-Gordon-Fock equation. Here we find the lateral boundary conditions of the above potential (4). Main result of this paper is as follows.

Theorem 1. *If $f(x,t)\in L_2(\Omega)$, then potential (4) satisfies the following lateral boundary conditions for $0<t<T$:*

$$\begin{aligned}
u(0,t) \;=\;& \frac{1}{2}\int_0^l \frac{\partial u(\xi,t-\xi)}{\partial\tau}d\xi \\
&-\frac{1}{2}\int_0^l d\xi\int_0^{t-\xi}\frac{m_0(t-\tau)}{\sqrt{(t-\tau)^2-\xi^2}}J_1\left(m_0\sqrt{(t-\tau)^2-\xi^2}\right)\frac{\partial u(\xi,\tau)}{\partial\tau}d\tau \\
&-\frac{1}{2}\int_0^t\left[J_0\left(m_0\sqrt{(t-\tau)^2-l^2}\right)\frac{\partial u(l,\tau)}{\partial\xi}-J_0(m_0(t-\tau))\frac{\partial u(0,\tau)}{\partial\xi}\right]d\tau \\
&+\frac{1}{2}\int_0^t d\tau\int_0^l\frac{m_0\xi}{\sqrt{(t-\tau)^2-\xi^2}}J_1\left(m_0\sqrt{(t-\tau)^2-\xi^2}\right)\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi \\
&-\frac{1}{2}\int_t^{t-l}\left[J_0\left(m_0\sqrt{(t-\tau)^2-l^2}\right)\frac{\partial u(l,\tau)}{\partial\xi}-\frac{\partial u(\tau-t,\tau)}{\partial\xi}\right]d\tau \\
&+\frac{1}{2}\int_t^{t-l}d\tau\int_{\tau-t}^l\frac{m_0\xi}{\sqrt{(t-\tau)^2-\xi^2}}J_1\left(m_0\sqrt{(t-\tau)^2-\xi^2}\right)\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi \\
&-\frac{1}{2}\int_{t-l}^t\left[\frac{\partial u(t-\tau,\tau)}{\partial\xi}-\frac{\partial u(\tau-t,\tau)}{\partial\xi}\right]d\tau \\
&+\frac{1}{2}\int_{t-l}^t d\tau\int_{\tau-t}^{t-\tau}\frac{m_0\xi}{\sqrt{(t-\tau)^2-\xi^2}}J_1\left(m_0\sqrt{(t-\tau)^2-\xi^2}\right)\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi \\
&+\frac{m_0^2}{2}\int_0^l d\xi\int_0^{t-\xi}J_0(m_0\sqrt{(t-\tau)^2-\xi^2})u(\xi,\tau)d\tau
\end{aligned} \tag{7}$$

$$\begin{aligned}
u(l,t) \;=\;& \frac{1}{2}\int_0^l\frac{\partial u(\xi,t+\xi-l)}{\partial\tau}d\xi \\
&-\frac{1}{2}\int_0^l d\xi\int_0^{t+\xi-l}\frac{m_0(t-\tau)}{\sqrt{(t-\tau)^2-(l-\xi)^2}}J_1\left(m_0\sqrt{(t-\tau)^2-(l-\xi)^2}\right)\frac{\partial u(\xi,\tau)}{\partial\tau}d\tau \\
&-\frac{1}{2}\int_0^{t-l}\left[J_0(m_0(t-\tau))\frac{\partial u(l,\tau)}{\partial\xi}-J_0(m_0\sqrt{(t-\tau)^2-l^2})\frac{\partial u(0,\tau)}{\partial\xi}\right]d\tau \\
&+\frac{1}{2}\int_0^{t-l}d\tau\int_0^l\frac{m_0(\xi-l)}{\sqrt{(t-\tau)^2-(l-\xi)^2}}J_1\left(m_0\sqrt{(t-\tau)^2-(l-\xi)^2}\right)\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi \\
&-\frac{1}{2}\int_{t-l}^t\left[J_0(m_0(t-\tau))\frac{\partial u(l,\tau)}{\partial\xi}-\frac{\partial u(\tau-t+l,\tau)}{\partial\xi}\right]d\tau \\
&+\frac{1}{2}\int_{t-l}^t d\tau\int_{\tau-t+l}^l\frac{m_0(\xi-l)}{\sqrt{(t-\tau)^2-(l-\xi)^2}}J_1\left(m_0\sqrt{(t-\tau)^2-(l-\xi)^2}\right)\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi \\
&+\frac{m_0^2}{2}\int_0^l d\xi\int_0^{t+\xi-l}J_0\left(m_0\sqrt{(t-\tau)^2-(l-\xi)^2}\right)u(\xi,\tau)d\tau,
\end{aligned} \tag{8}$$

Conversely, if a function $u(x,t) \in W_2^1(\Omega) \cap W_2^1(\partial\Omega)$ satisfies equation (5)*, the initial conditions* (6)*, and the lateral boundary conditions (4)-(5), then the function $u(x,t)$ uniquely defines the potential* (4).

Proof. We use techniques from [1]. Consider the one-dimensional potential (4) in the bounded domain $\Omega \equiv \{(x,t) : (0,l) \times (0,T)\}$ with the boundary S,

$$u(x,t) = \int_\Omega \varepsilon(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau.$$

Since $\tau = t - x + \xi$ and $\tau = t + x - \xi$ are characteristics, the integral vanishes outside of characteristic domain. Therefore, we integrate by inside of characteristic domain. Assuming that $u(x,t) \in W_2^1(\Omega)$, taking into account properties of the fundamental solution, and integrating by part, we calculate

$$\begin{aligned}
u(x,t) &= \int_\Omega \varepsilon(x-\xi,t-\tau) f(\xi,\tau) d\xi d\tau \\
&= \frac{1}{2}\int_0^x d\xi \int_0^{t+\xi-x} J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial^2 u(\xi,\tau)}{\partial\tau^2} d\tau \\
&+\frac{1}{2}\int_x^l d\xi \int_0^{t-\xi+x} J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial^2 u(\xi,\tau)}{\partial\tau^2} d\tau \\
&-\frac{1}{2}\int_0^{t-x} d\tau \int_0^{l} J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2} d\xi \\
&-\frac{1}{2}\int_{t-x}^{t-l+x} d\tau \int_{\tau-t+x}^{l} J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2} d\xi \\
&-\frac{1}{2}\int_{t-l+x}^{t} d\tau \int_{\tau-t+x}^{t-\tau+x} J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2} d\xi \\
&+\frac{m_0^2}{2}\int_0^x d\xi \int_0^{t+\xi-x} J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2}) u(\xi,\tau) d\tau \\
&+\frac{m_0^2}{2}\int_x^l d\xi \int_0^{t-\xi+x} J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2}) u(\xi,\tau) d\tau \\
&= \frac{1}{2}\int_0^x \frac{\partial u(\xi,t+\xi-x)}{\partial\tau} d\xi - \frac{1}{2}\int_0^x d\xi \int_0^{t+\xi-x} \frac{m_0(t-\tau)}{\sqrt{(t-\tau)^2-(x-\xi)^2}} \\
&\times J_1(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial u(\xi,\tau)}{\partial\tau} d\tau \\
&+\frac{1}{2}\int_x^l \frac{\partial u(\xi,t-\xi+x)}{\partial\tau} d\xi - \frac{1}{2}\int_x^l d\xi \int_0^{t-\xi+x} \frac{m_0(t-\tau)}{\sqrt{(t-\tau)^2-(x-\xi)^2}} \\
&\times J_1(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial u(\xi,\tau)}{\partial\tau} d\tau \\
&-\frac{1}{2}\int_0^{t-x} [J_0(m_0\sqrt{(t-\tau)^2-(x-l)^2})\frac{\partial u(l,\tau)}{\partial\xi} - J_0(m_0\sqrt{(t-\tau)^2-x^2})\frac{\partial u(0,\tau)}{\partial\xi}] d\tau \\
&+\frac{1}{2}\int_0^{t-x} d\tau \int_0^l \frac{m_0(\xi-x)}{\sqrt{(t-\tau)^2-(x-\xi)^2}} J_1(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial u(\xi,\tau)}{\partial\xi} d\xi \\
&-\frac{1}{2}\int_{t-x}^{t-l+x} [J_0(m_0\sqrt{(t-\tau)^2-(x-l)^2})\frac{\partial u(l,\tau)}{\partial\xi} - \frac{\partial u(\tau-t+x,\tau)}{\partial\xi}] d\tau \\
&+\frac{1}{2}\int_{t-x}^{t-l+x} d\tau \int_{\tau-t+x}^{l} \frac{m_0(\xi-x)}{\sqrt{(t-\tau)^2-(x-\xi)^2}} J_1(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial u(\xi,\tau)}{\partial\xi} d\xi \\
&-\frac{1}{2}\int_{t-l+x}^{t} [\frac{\partial u(t-\tau+x,\tau)}{\partial\xi} - \frac{\partial u(\tau-t+x,\tau)}{\partial\xi}] d\tau \\
&+\frac{1}{2}\int_{t-l+x}^{t} d\tau \int_{\tau-t+x}^{t-\tau+x} \frac{m_0(\xi-x)}{\sqrt{(t-\tau)^2-(x-\xi)^2}} J_1(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})\frac{\partial u(\xi,\tau)}{\partial\xi} d\xi
\end{aligned}$$

$$+\frac{m_0^2}{2}\int_0^x d\xi\int_0^{t+\xi-x}J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})u(\xi,\tau)d\tau$$
$$+\frac{m_0^2}{2}\int_x^l d\xi\int_0^{t-\xi+x}J_0(m_0\sqrt{(t-\tau)^2-(x-\xi)^2})u(\xi,\tau)d\tau$$

for all $(x,t)\in\Omega$.

Now we consider this identity when $(x,t)\to S$. Taking the limit as $x\to 0$, we have

$$\begin{aligned}
u(0,t) \quad = \quad & \frac{1}{2}\int_0^l \frac{\partial u(\xi,t-\xi)}{\partial\tau}d\xi-\frac{1}{2}\int_0^l d\xi\int_0^{t-\xi}\frac{m_0(t-\tau)}{\sqrt{(t-\tau)^2-\xi^2}}J_1(m_0\sqrt{(t-\tau)^2-\xi^2})\frac{\partial u(\xi,\tau)}{\partial\tau}d\tau\\
&-\frac{1}{2}\int_0^t[J_0(m_0\sqrt{(t-\tau)^2-l^2})\cdot\frac{\partial u(l,\tau)}{\partial\xi}-J_0(m_0(t-\tau))\cdot\frac{\partial u(0,\tau)}{\partial\xi}]d\tau\\
&+\frac{1}{2}\int_0^t d\tau\int_0^l\frac{m_0\xi}{\sqrt{(t-\tau)^2-\xi^2}}\cdot J_1(m_0\sqrt{(t-\tau)^2-\xi^2})\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi\\
&-\frac{1}{2}\int_t^{t-l}[J_0(m_0\sqrt{(t-\tau)^2-l^2})\cdot\frac{\partial u(l,\tau)}{\partial\xi}-\frac{\partial u(\tau-t,\tau)}{\partial\xi}]d\tau\\
&+\frac{1}{2}\int_t^{t-l}d\tau\int_{\tau-t}^l\frac{m_0\xi}{\sqrt{(t-\tau)^2-\xi^2}}J_1(m_0\sqrt{(t-\tau)^2-\xi^2})\cdot\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi\\
&-\frac{1}{2}\int_{t-l}^t[\frac{\partial u(t-\tau,\tau)}{\partial\xi}-\frac{\partial u(\tau-t,\tau)}{\partial\xi}]d\tau\\
&+\frac{1}{2}\int_{t-l}^t d\tau\int_{\tau-t}^{t-\tau}\frac{m_0\xi}{\sqrt{(t-\tau)^2-\xi^2}}J_1(m_0\sqrt{(t-\tau)^2-\xi^2})\cdot\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi\\
&+\frac{m_0^2}{2}\int_0^l d\xi\int_0^{t-\xi}J_0(m_0\sqrt{(t-\tau)^2-\xi^2})\cdot u(\xi,\tau)d\tau
\end{aligned}$$

for $0<t<T$. Similarly,

$$\begin{aligned}
u(l,t) \quad = \quad & \frac{1}{2}\int_0^l\frac{\partial u(\xi,t+\xi-l)}{\partial\tau}d\xi\\
&-\frac{1}{2}\int_0^l d\xi\int_0^{t+\xi-l}\frac{m_0(t-\tau)}{\sqrt{(t-\tau)^2-(l-\xi)^2}}J_1(m_0\sqrt{(t-\tau)^2-(l-\xi)^2})\frac{\partial u(\xi,\tau)}{\partial\tau}d\tau\\
&-\frac{1}{2}\int_0^{t-l}[J_0(m_0(t-\tau))\frac{\partial u(l,\tau)}{\partial\xi}-J_0(m_0\sqrt{(t-\tau)^2-l^2})\frac{\partial u(0,\tau)}{\partial\xi}]d\tau\\
&+\frac{1}{2}\int_0^{t-l}d\tau\int_0^l\frac{m_0(\xi-l)}{\sqrt{(t-\tau)^2-(l-\xi)^2}}J_1(m_0\sqrt{(t-\tau)^2-(l-\xi)^2})\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi\\
&-\frac{1}{2}\int_{t-l}^t[J_0(m_0(t-\tau))\cdot\frac{\partial u(l,\tau)}{\partial\xi}-\frac{\partial u(\tau-t+l,\tau)}{\partial\xi}]d\tau\\
&+\frac{1}{2}\int_{t-l}^t d\tau\int_{\tau-t+l}^l\frac{m_0(\xi-l)}{\sqrt{(t-\tau)^2-(l-\xi)^2}}J_1(m_0\sqrt{(t-\tau)^2-(l-\xi)^2})\frac{\partial u(\xi,\tau)}{\partial\xi}d\xi\\
&+\frac{m_0^2}{2}\int_0^l d\xi\int_0^{t+\xi-l}J_0(m_0\sqrt{(t-\tau)^2-(l-\xi)^2})\cdot u(\xi,\tau)d\tau
\end{aligned}$$

for $0<t<T$.

Hence, the one-dimensional potential (4) satisfies the lateral boundary conditions (7)-(8).

Conversely, if a function $u(x,t)\in W_2^1(\Omega)\cap W_2^1(\partial\Omega)$ satisfies equation (5), the initial conditions (6), and the lateral boundary conditions (7-(8), then the function $u(x,t)$ uniquely defines the potential (4).

To summarize, the initial-boundary value problem (5)-(6) has a unique solution and the solution coincides with the potential (4). This completes the proof. □

Remark 1. *When $m_0=0$, by taking derivative from $u(x,t)$ at x and t, we easily obtain the main result of [1].*

ACKNOWLEDGMENTS

This publication is supported by the target program 0085/PTSF-14 and 4075/GF4 from the Ministry of Science and Education of the Republic of Kazakhstan.

REFERENCES

1. T. Sh. Kal'menov, and D. Suragan, *Electron. J. Diff. Equ.* **48**, 1-6 (2014).
2. T. Sh. Kal'menov, and D. Suragan, *Doklady Mathematics* **80**, 646-649 (2009).
3. M. A. Sadybekov, and N. S. Imanbaev, *Differential Equations* **48**, 896-900 (2012).
4. L. A. Pulkina, and M. V. Strigun, *Vestnik SamGU. Estestvenno-Nauchnaya Ser.* **83**, 46-56 (2011).
5. V. S. Vladimirov, *Equations of Mathematical Physics*, Nauka, 1991.