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# Duality property of the noncommutative $\ell_\infty$ and $\ell_1$ valued symmetric Hardy spaces

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**Abstract.** In this paper, we consider the noncommutative $H_E(\mathscr{A};\ell_\infty)$ and $H_E(\mathscr{A};\ell_1)$ spaces and obtain some result on duality for these spaces.

**Keywords:** von Neumann algebra, Subdiagonal algebras, Noncommutative vector valued symmetric Hardy spaces, Duality
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## INTRODUCTION

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{M}$ be a finite von Neumann algebra on the Hilbert space equipped with a normal faithful tracial state $\tau$. The set of all $\tau$-measurable operators will be denoted by $L_0(\mathscr{M})$. The set $L_0(\mathscr{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product [1]. For each $x$ on $\mathscr{H}$ affiliated with $\mathscr{M}$, all spectral projection $e_s^\perp(|x|) = \chi_{(s;\infty)}(|x|)$ corresponding to the interval $(s;\infty)$ belong to $\mathscr{M}$, and $x \in L_0(\mathscr{M})$ if and only if $\chi_{(s;\infty)}(|x|) < \infty$ for some $s \in \mathbf{R}$. Recall the definition of the decreasing rearrangement (or generalized singular numbers) of an operator $x \in L_0(\mathscr{M})$: For $t > 0$

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}, t > 0,$$

where

$$\lambda_s(x) = \tau(e_s^\perp(|x|)), s > 0.$$

The function $s \mapsto \lambda_s(x)$ is called the distribution function of $x$. For more details on generalized singular value function of measurable operators we refer to [2, 3]. We now recall the definition of a symmetric operator space $L_E(\mathscr{M})$ buildup with respect to a noncommutative measure space $(\mathscr{M},\tau)$ and a symmetric Banach function space.

By a symmetric quasi Banach space on $[0;1]$ we mean a quasi Banach lattice $E$ of measurable functions on $[0;1]$ satisfying the following properties:

***(i)*** $E$ contains all simple functions;
***(ii)*** if $x \in E$ and $y$ is measurable function such that $|y|$ is equi-distributed with $|x|$, then $y \in E$ and $\|x\|_E = \|y\|_E$.

For convenience we shall always assume $E$ additionally satisfies

$$0 \leq x_n \uparrow x, x_n, x \in E \Rightarrow \|x_n\|_E \uparrow \|x\|_E.$$

Here $x \prec\prec y$ as usual denotes the submajorization in the sense of Hardy-Littlewood-Polya: for all $t > 0$

$$\int_0^t \mu_s(x)ds \leq \int_0^t \mu_s(y)ds.$$

To see examples, $L_p$, Orlich, Lorentz and Marcinkiewicz spaces are rearrangement invariant Banach function spaces.
The Köthe dual of a symmetric Banach function space $E$ on $[0,1]$ is the Banach space $E^\times$ given by

$$E^\times = \{x \in L_0[0,1] : \sup\{\int_0^1 |x(t)y(t)|dt : \|x\|_E \leq 1\} < \infty\}$$

with the norm

$$\|y\| = \sup\{\int_0^1 |x(t)y(t)|dt : \|x\|_E \leq 1\},\ y \in E^\times.$$

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The space $E^{\times}$ is fully symmetric and has the Fatou property. It is isometrically isomorphic to a closed subspace of $E^*$ via the map

$$y \to L_y,\;\; L_y(x) = \int_0^1 x(t)y(t)dt \;\; (x \in E).$$

A symmetric Banach space $E$ on $[0,1]$ has the Fatou property if and only if $E = E^{\times\times}$ isometrically. It has order continuous norm if and only if it is separable, which is also equivalent to the statement $E^* = E^{\times}$.

Let $E$ be a symmetric quasi Banach space on $[0;1]$. We define

$$L_E(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \mu.(x) \in E\}$$

together with the norm

$$\|x\|_{L_E(\mathscr{M})} = \|\mu.(x)\|_E.$$

Then $(L_E(\mathscr{M}); \|\cdot\|_{L_E(\mathscr{M})})$ is a quasi-Banach space (cf. [4–6]). We will use the following duality theorem proved in [7, Theorem 5.3 and Remark 5.4].

**Theorem 1.** *Let $\mathscr{M}$ be a semi-finite von Neumann algebra and let E be a separable symmetric Banach function space on $\mathbf{R}_+$. If $y = (y_k) \in L_E(\mathscr{M};\ell_\infty)$ satisfies $y_k \geq 0$ for all k. Then*

$$L_E(\mathscr{M};\ell_1)^* = L_{E^{\times}}(\mathscr{M};\ell_\infty)$$

*isometrically with respect to the duality bracket*

$$\langle x,y\rangle = \sum_{k\geq 1} \tau(x_k y_k),$$

*where $x \in L_E(\mathscr{M};\ell_1)$ and $y \in L_{E^{\times}}(\mathscr{M};\ell_\infty)$.*

Now, let $E$ be a quasi-Banach lattice. and let $0 < r < \infty$. Then $E$ is said to be $r$-convex (resp. $r$-concave) if there exists a constant $C > 0$ such that for all finite sequence $(x_n)$ in $E$

$$\left\| \left( \sum_{k=1}^n |x_k|^r \right)^{1/r} \right\|_E \leq C \left( \sum_{k=1}^n \|x_k\|_E^r \right)^{1/r},$$

and

$$\left( \sum_{k=1}^n \|x_k\|_E^r \right)^{1/r} \leq C \left\| \left( \sum_{k=1}^n |x_k|^r \right)^{1/r} \right\|_E,$$

respectively; as usual the best constant $C > 0$ is denoted by $M^{(r)}(E)$ resp. $M_{(r)}(E)$. We recall that for $r_1 \leq r_2$

$$M^{r_1}(E) \leq M^{r_2}(E),$$

and

$$M_{r_2}(E) \leq M_{r_1}(E).$$

To see example: each $L_p(\mu)$ is p-convex and p-concave with constant 1, and as a sequence $M^{(2)}(L_p(\mu)) = 1$ for $2 \leq p$ and $M_{(2)}(L_p(\mu)) = 1$ for $p \leq 2$. For all needed information on convexity and concavity we once again refer to [8]. If $M^{max(1,r)}(E) = 1$, then the $r$-th power

$$E^r := \{x \in L_0(\Omega) : |x|^{1/r} \in E\}$$

endowed with the norm

$$\|x\|_{E^r} = \||x|^{1/r}\|_E^r$$

is again a Banach function space which is $1/min(1,r)$-convex. Since for each operator $x \in L_0(\mathscr{M})$

$$\mu(|x|^r) = \mu(x)^r,$$

we conclude for every symmetric Banach function space $E$ on the interval $[0,1]$ which satisfies $M^{max(1,r)}(E)=1$ that

$$L_{E^r}(\mathscr{M}) := \{x \in L_0(\mathscr{M}) : |x|^{1/r} \in L_E(\mathscr{M})\},$$

and

$$\|x\|_{L_{E^r}(\mathscr{M})} = \|\mu(|x|)\|_{E^r} = \|\mu(|x|^{1/r})\|_E^r = \||x|^{1/r}\|_{L_E(\mathscr{M})}^r.$$

For details see [4]. Let $\mathscr{D}$ be a von Neumann subalgebra of $\mathscr{M}$, and let $\Phi : \mathscr{M} \to \mathscr{D}$ be the unique normal faithful conditional expectation such that $\tau \circ \Phi = \tau$. A finite subdiagonal algebra of $\mathscr{M}$ with respect to $\Phi$ is a $w^*$-closed subalgebra $\mathscr{A}$ of $\mathscr{M}$ satisfying the following conditions:

***(i)*** $\mathscr{A} + \mathscr{A}^*$ is $w^*$-dense in $\mathscr{M}$;

***(ii)*** $\Phi$ is multiplicative on $\mathscr{A}$, i.e., $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a,b \in \mathscr{A}$;

***(iii)*** $\mathscr{A} \cap \mathscr{A}^* = \mathscr{D}$, where $\mathscr{A}^*$ is the family of all adjoint elements of the element of $\mathscr{A}$, i.e., $\mathscr{A}^* = \{a^* : a \in \mathscr{A}\}$.

The algebra $\mathscr{D}$ is called the diagonal of $\mathscr{A}$. It's proved by Exel [9] that a finite subdiagonal algebra $\mathscr{A}$ is automatically maximal. Given $0 < p \leq \infty$ we denote by $L_p(\mathscr{M})$ the usual noncommutative $L_p$-spaces associated with $(\mathscr{M},\tau)$. Recall that $L_\infty(\mathscr{M}) = \mathscr{M}$, equipped with the operator norm. The norm of $L_p(\mathscr{M})$ will be denoted by $\|\cdot\|_p$. For $p < \infty$ we define $H_p(\mathscr{A})$ to be closure of $\mathscr{A}$ in $L_p(\mathscr{M})$, and for $p = \infty$ we simply set $H_\infty(\mathscr{A}) = \mathscr{A}$ for convenience. These are so called Hardy spaces associated with $\mathscr{A}$. They are noncommutative extensions of the classical Hardy space on the torus $\mathbf{T}$. We refer to [10] and [11] for more examples. These noncommutative Hardy spaces have received a lot of attention since Arveson's pioneer work. For references see [10, 12–14] whereas more references on previous works can be found in the survey paper [1].

**Definition 1.** *[15] Let E be a symmetric quasi Banach space on* $[0;1]$ *and* $\mathscr{A}$ *be a finite subdiagonal subalgebra of* $\mathscr{M}$. *Then* $H_E(\mathscr{A}) = \overline{\mathscr{A}}^{\|\cdot\|_{L_E(\mathscr{M})}}$ *called symmetric Hardy space associated with* $\mathscr{A}$. *We denote* $\overline{\mathscr{A}_0}^{\|\cdot\|_{L_E(\mathscr{M})}}$ *by* $H_E^0(\mathscr{A})$.

The theory of vector-valued noncommutative $L_p$-spaces are introduced by Pisier in [16] for the case $\mathscr{M}$ is hyperfinite and later by Junge [17](see also [18]) for the general case. The noncommutative symmetric $L_E(\mathscr{M};\ell_\infty)$ and $L_E(\mathscr{M};\ell_1)$ spaces are introduced by Defant in [19] and Dirksen in [7]. Now we give the definition of the noncommutative symmetric $\ell_\infty$ and $\ell_1$ valued Hardy spaces which have been defined in [20–22]

**Definition 2.** *(i) We define* $H_E(\mathscr{A},\ell_\infty)$ *as the space of all sequences* $x = (x_n)_{n\geq 1}$ *in* $H_E(\mathscr{A})$ *which admit a factorization of the following form: there are* $a,b \in H_{E^{1/2}}(\mathscr{A})$, *and a bounded sequence* $y = (y_n) \subset \mathscr{A}$ *such that* $x_n = ay_nb, \forall n \geq 1$. *Given* $x \in H_E(\mathscr{A},\ell_\infty)$ *define*

$$\|x\|_{H_E(\mathscr{A},\ell_\infty)} := \inf\{\|a\|_{H_{E^{1/2}}(\mathscr{A})} \sup_n \|y_n\|_\infty \|b\|_{H_{E^{1/2}}(\mathscr{A})}\},$$

*where the infimum runs over all factorizations of* $(x_n)$ *as above. Moreover, let us define* $H_E(\mathscr{A};\ell_\infty^c)$ *(here c should remind on the word "column") as the space of all* $(x_n)_{n\geq 1}$ *in* $H_E(\mathscr{A})$ *for which there are* $b \in H_E(\mathscr{A})$ *and bounded sequence* $(y_n)_{n\geq 1}$ *in* $\mathscr{M}$ *such that* $x_n = y_nb$ *and*

$$\|x\|_{H_E(\mathscr{A},\ell_\infty)} := \inf\{\sup_n \|y_n\|_\infty \|b\|_{H_E(\mathscr{A})}\}.$$

*Similarly, we define the row version* $H_E(\mathscr{A};\ell_\infty^r)$ *all sequences which allow a uniform factorization* $x_n = ay_n$, *again with* $a \in H_E(\mathscr{A})$ *and* $(y_n)_{n\geq 1}$ *uniformly bounded in* $\mathscr{M}$.

*(ii) We define* $H_E(\mathscr{A};\ell_1)$ *as the space of all sequences* $x = (x_n)_{n\geq 1}$ *in* $H_E(\mathscr{A})$ *which can be decomposed as* $x_n = \sum_{k=1}^\infty u_{kn}v_{nk}, \forall n \geq 1$ *for two families* $(u_{kn})_{k,n\geq 1}$ *and* $(v_{nk})_{n,k\geq 1}$ *in* $H_{E^{1/2}}(\mathscr{A})$ *such that*

$$\sum_{k,n=1}^\infty u_{kn}u_{kn}^* \in L_E(\mathscr{M}) \text{ and } \sum_{n,k=1}^\infty v_{nk}^*v_{nk} \in L_E(\mathscr{M}).$$

*In this space we define the following form:*

$$\|x\|_{H_E(\mathscr{A};\ell_1)} := \inf\{\|\sum_{k,n=1}^\infty u_{kn}u_{kn}^*\|_{H_E(\mathscr{A})}^{1/2} \|\sum_{n,k=1}^\infty v_{nk}^*v_{nk}\|_{H_E(\mathscr{A})}^{1/2}\},$$

*where the infimum runs over all decompositions of x as above.*

## MAIN RESULTS

**Proposition 2.** *Let E be a separable symmetric quasi Banach function space on* $[0;1]$*. Then we have the following:*

$$H_E(\mathscr{A};\ell_\infty)=\{(x_n)\in L_E(\mathscr{M};\ell_\infty):\ \Sigma_{n=1}^{\infty}\tau(x_ny_n)=0,\quad \forall\ (y_n)\in H^0_{E^\times}(\mathscr{A};\ell_1)\}$$

*and*

$$H^0_E(\mathscr{A};\ell_\infty)=\{(x_n)\in L_E(\mathscr{M};\ell_\infty):\ \Sigma_{n=1}^{\infty}\tau(x_ny_n)=0,\quad \forall\ (y_n)\in H_{E^\times}(\mathscr{A};\ell_1)\}.$$

*Proof.* The inclusion $H_E(\mathscr{A};\ell_\infty)\subset\{(x_n)\in L_E(\mathscr{M};\ell_\infty):\ \Sigma_{n=1}^{\infty}\tau(x_ny_n)=0,\quad \forall\ (y_n)\in H^0_{E^\times}(\mathscr{A};\ell_1)\}$ is clearly. Let

$$(z_n)\in\{(x_n)\in L_E(\mathscr{M};\ell_\infty):\ \Sigma_{n=1}^{\infty}\tau(x_ny_n)=0\quad \forall\ (y_n)\in H^0_{E^\times}(\mathscr{A};\ell_1)\}$$

and $c\in\mathscr{A}_0$. For $n\in\mathscr{N}$, set $y_k=0,(k\neq n)$ and $y_n=c$, then $(y_k)\in H^0_{E^\times}(\mathscr{A};\ell_1)$. Hence for all $n\in\mathscr{N}$,

$$\tau(z_nc)=0\quad \forall c\in\mathscr{A}_0.$$

By (1.2) in [13], we get $(z_n)\subset H_E(\mathscr{A})$. Using Lemma 1 in [20], we obtain that $(z_n)\in H_E(\mathscr{A};\ell_\infty)$. The latter equality follows from the continuity of $\Phi$ on $H_E(\mathscr{A};\ell_\infty)$. □

By Proposition 1 in [20], arguments similar to proof of Proposition 2, we get the following result.

**Proposition 3.** *Let E be an r-convex symmetric quasi Banach function space on* $[0;1]$ *for some* $0<r<\infty$ *and E do not contain* $c_0$ *or separable. Then*

$$H_E(\mathscr{A};\ell_1)=\{x\in L_E(\mathscr{M};\ell_1):\sum_{n=1}^{\infty}\tau(x_ny_n^*)=0,\ for\ all\ (y_n^*)\in H^0_{E^\times}(\mathscr{A};\ell_\infty)\}.$$

*Moreover,*

$$H^0_E(\mathscr{A};\ell_1)=\{x\in L_E(\mathscr{M};\ell_1):\sum_{n=1}^{\infty}\tau(x_ny_n^*)=0,\ for\ all\ (y_n^*)\in H_{E^\times}(\mathscr{A};\ell_\infty)\}.$$

**Theorem 4.** *Let E be an r-convex symmetric Banach function space on* $[0;1]$ *for some* $0<r<\infty$ *and E do not contain* $c_0$ *or separable. Then*

$$(i)\ \ (H_E(\mathscr{A};\ell_1))^*=L_{E^\times}(\mathscr{M};\ell_\infty)/J(H^0_{E^\times}(\mathscr{A};\ell_\infty))$$

*isometrically via the following duality bracket*

$$((x_n),(y_n))=\sum_{n=1}^{\infty}\tau(y_n^*x_n)$$

*for* $x\in H_E(\mathscr{A};\ell_1)$ *and* $y\in H_{E^\times}(\mathscr{A};\ell_\infty)$*, where* $J(H^0_{E^\times}(\mathscr{A};\ell_\infty))=\{x^*:\ x\in H^0_{E^\times}(\mathscr{A};\ell_\infty)\}$.

$$(ii)\ \ (L_E(\mathscr{M};\ell_1)/J(H^0_p(\mathscr{A};\ell_1)))^*=H_{E^\times}(\mathscr{A};\ell_\infty)$$

*isometrically via the following duality bracket*

$$((x_n),(y_n))=\sum_{n=1}^{\infty}\tau(y_n^*x_n)$$

*for* $x\in H_E(\mathscr{A};\ell_1)$ *and* $y\in H_{E^\times}(\mathscr{A};\ell_\infty)$*, where* $J(H^0_{E^\times}(\mathscr{A};\ell_1))=\{x^*:\ x\in H^0_{E^\times}(\mathscr{A};\ell_1)\}$.

*Proof.* By Theorem 1 it is clear that

$$(H_E(\mathscr{A};\ell_1))^*=L_{E^\times}(\mathscr{M};\ell_\infty)/(H_E(\mathscr{A};\ell_1))^\perp\quad\text{and}\quad(L_E(\mathscr{M};\ell_1)/^\perp(H_{E^\times}(\mathscr{A};\ell_\infty)))^*=H_{E^\times}(\mathscr{A};\ell_\infty),$$

where

$$(H_E(\mathscr{A};\ell_1))^\perp=\{(x_n)\in L_{E^\times}(\mathscr{M};\ell_\infty):\ \sum_{n=1}^{\infty}\tau(y_n^*x_n)=0\quad\forall\ (y_n)\in H_E(\mathscr{A};\ell_1)\},$$

and

$$^{\perp}(H_{E^{\times}}(\mathscr{A};\ell_{\infty}))=\{(x_n)\in L_E(\mathscr{M};\ell_1):\ \sum_{n=1}^{\infty}\tau(y_n^*x_n)=0\quad\forall\,(y_n)\in H_{E^{\times}}(\mathscr{A};\ell_{\infty})\}.$$

On the other hand, by Proposition 2 and Proposition 3, we have that

$$^{\perp}(H_{E^{\times}}(\mathscr{A};\ell_{\infty}))=J(H_E^0(\mathscr{A};\ell_1)),\quad (H_E(\mathscr{A};\ell_1))^{\perp}=J(H_{E^{\times}}^0(\mathscr{A};\ell_{\infty})).$$

From that the desired results follow. □

**Remark 1.** *Let $\mathscr{M}=L^{\infty}(\mathbf{T})$, $\mathscr{A}=H^{\infty}(\mathbf{T})$ and let*

$$\Phi(a)=\left(\int adt\right)1,\ \ \tau(a)=\left(\int adt\right)\ \forall a\in\mathscr{M}.$$

*Then $\mathscr{A}$ is a finite subdiagonal algebra in $\mathscr{M}$ and $\mathscr{A}$ is maximal. Let $1<p<\infty$, $1/p+1/p'=1$. Then*

$$L_p(\mathscr{M};\ell_{\infty})=\{(y_n)_{n\geq 1}\subset L_p(\mathbf{T})\mid \sup_n|y_n|\in L_p(\mathbf{T})\},$$

$$H_p(\mathscr{A};\ell_{\infty})=\{(y_n)_{n\geq 1}\subset H_p(\mathbf{T})\mid \sup_n|y_n|\in L_p(\mathbf{T})\},$$

*and*

$$\|(x_n)\|_{L_p(\mathscr{M};\ell_{\infty})}=\|\sup_n|x_n|\|_{L_p(\mathbf{T})},\quad \|(y_n)\|_{H_p(\mathscr{A};\ell_{\infty})}=\|\sup_n|y_n|\|_{L_p(\mathbf{T})}.$$

*If $H_p(\mathscr{A};\ell_1)^*=H_{p'}(\mathscr{A};\ell_{\infty})$, then $L_{p'}(\mathscr{M};\ell_{\infty})/J(H_{p'}^0(\mathscr{A};\ell_{\infty})$ is equivalent to $H_{p'}(\mathscr{A};\ell_{\infty})$. Hence the Hilbert transform $\mathscr{H}$ is bounded projection from $L_{p'}(\mathscr{M};\ell_{\infty})$ to $H_{p'}(\mathscr{A};\ell_{\infty})$, i. e.,*

$$\|\sup_n|\mathscr{H}x_n|\|_{H_{p'}(\mathbf{T})}\leq C_{p'}\|\sup_n|x_n|\|_{L_{p'}(\mathbf{T})}\quad\forall(x_n)\in L_{p'}(\mathbf{T}).$$

*This means that $\mathscr{H}\otimes id$ is bounded on $L_{p'}(\mathbf{T},\ell_{\infty})$. By Lemma 2 in [23],we get $\ell_{\infty}\in UMD$. This is a contradiction. In general, $H_E(\mathscr{A};\ell_1)^*\neq H_{E^{\times}}(\mathscr{A};\ell_{\infty})$ (see [22]).*

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