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Abstract. The problem of optimal control for the three-sector economic model of a cluster is considered. Task statement is to determine the optimal distribution of investment and manpower in moving the system from a given initial state to desired final state. To solve the optimal control problem with finite-horizon planning, in case of fixed ends of trajectories, with box constraints, the method of Lagrange multipliers of a special type is used. This approach allows to represent the desired control in the form of synthesis control, depending on state of the system and current time. The results of numerical calculations for an instance of three-sector model of the economy show the effectiveness of the proposed method.

Keywords: Economic model, Cluster, Investments, Manpower, Optimal control problem, Method of Lagrange multipliers
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INTRODUCTION

Problems of optimal control for dynamic systems with fixed ends of trajectories are often used in practical applications. The essence of these problems is to move the system from a given initial state to the desired final state in a finite time interval, while minimizing control costs. Various mathematical formulations of optimal control problems and their classification are given in [1, 2]. Optimal control problems can be solved by using Pontryagin's principle of maximum [3], Bellman's dynamic programming method [4] or Krotov's sufficient optimality conditions [5].

It should be noted that the main feature of the optimal control problem considered in this paper, is that trajectories of the system must pass through given points at the initial and final moments of time (i. e. left and right ends of trajectories are fixed). The task is to construct a synthesized control on a finite time interval taking into account the control value constraints.

This approach is used to solve the problem of optimal distribution of investment and manpower in a three-sector economic model of a cluster [6, 7]. In contrast to [8], where optimal control problems on an infinite-horizon are set out, we consider here a fixed time interval. Using mathematical modeling and optimal control theory allows to create a competitive cluster structure [9–11].

OPTIMAL CONTROL PROBLEM

We consider the control system described by the differential equation of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad (t_0 \le t \le T) \tag{1}$$

with the given initial and final states

$$x(t_0) = x_0, \tag{2}$$

$$x(T) = 0, \tag{3}$$

and the following box constraints on control

$$\alpha(t) \le u(t) \le \beta(t), \quad (t_0 \le t \le T), \tag{4}$$

where $x(t)$ is n-vector of the object state, $u(t)$ is a m-vector of piecewise continuous controls, $A(t)$, $B(t)$ are matrices of dimensions $(n\times n)$, $(n\times m)$, respectively (elements of these matrices are continuous functions), $\alpha(t)$, $\beta(t)$ are piecewise continuous m-vector functions. The dynamics of this system is considered in the time interval $[t_0, T]$, where

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t_0 and T are known initial and final moments of time. It is assumed that system (1) is completely controllable at time t_0.

Suppose given a quadratic objective functional of type

$$J(u) = \frac{1}{2}\int_{t_0}^{T} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]\,dt, \tag{5}$$

where $Q(t)$ is a positive semidefinite $(n \times n)$-matrix; $R(t)$ is a positive definite $(m \times m)$-matrix.

Problem. It is required to find the synthesizing control $u(t) = u(x(t),t)$ that satisfies the box constraints (4) and brings the system (1) from a given initial state (2) to final state (3) (the origin) within the fixed interval of time $[t_0, T]$, minimizing the functional (5). A method of Lagrange multipliers of special type is used to solve problem (1)-(5), it allows to represent the optimal control as a sum of feedback and programmed controls [12].

ALGORITHM FOR SOLVING THE PROBLEM

To find the optimal trajectory of the system movement and optimal control for problem (1)-(5) we use the following algorithm.

1. Integrate the following system of differential equations in the interval $[t_0, T]$:

$$\dot{K}(t) = -A'(t)K(t) - K(t)A(t) + K(t)S(t)K(t) - Q(t), \quad K(T) = K_T, \tag{6}$$

$$\dot{W}(t,T) = [A(t) - S(t)K(t)]W(t,T) + W(t,T)[A(t) - S(t)K(t)]' - S(t), \quad W(T,T) = 0, \tag{7}$$

where K_T is an arbitrary positive semidefinite matrix; $S(t) = B(t)R^{-1}(t)B'(t)$. As a result of system (6), (7) integration the matrices $K_0 = K(t_0)$ and $W_0 = W(t_0)$ are determined, and as well the vector

$$q_0 = W_0^{-1}x_0. \tag{8}$$

2. Integrate the following system of differential equations in the interval $[t_0, T]$:

$$\begin{aligned}
&\dot{K}(t) = -A'(t)K(t) - K(t)A(t) + K(t)S(t)K(t) - Q(t), \quad K(t_0) = K_0,\\
&\dot{W}(t,T) = [A(t) - S(t)K(t)]W(t,T) + W(t,T)[A(t) - S(t)K(t)]' - S(t), \quad W(t_0,T) = W_0,\\
&\dot{x}(t) = [A(t) - S(t)K(t)]x(t) + B(t)\varphi(x(t),t) - S(t)q(t), \quad x(t_0) = x_0,\\
&\dot{q}(t) = -[A(t) - S(t)K(t)]'^{-1}(t,T)B(t)\varphi(x(t),t), \quad q(t_0) = q_0.
\end{aligned} \tag{9}$$

The choice of a vector q_0 in the initial condition $q(t_0) = q_0$ in form (8) ensures that the terminal condition $x(T) = 0$ (3) is satisfied. Solution $x(t)$ obtained from (9) corresponds to the desired optimal trajectory of the system.

Function $\varphi(t)$ in the process of integration is calculated by the formula

$$\varphi(x(t),t) = R^{-1}(t)[\lambda_1(x(t),t) - \lambda_2(x(t),t)],$$

where

$$\lambda_1(x(t),t) = R(t)\max\{0; \alpha(t) - \omega(x(t),t)\}, \quad \lambda_2(x(t),t) = R(t)\max\{0; \omega(x(t),t) - \beta(t)\}$$
$$\omega(x(t),t) = -R^{-1}B'(t)[K(t)x(t) + q(t)].$$

Finally, the optimal control $u(t)$ is defined as $u(x(t),t) = \omega(x(t),t) + \varphi(x(t),t)$.

THREE-SECTOR ECONOMIC MODEL OF A CLUSTER

As an example, consider the optimal control problem for the three-sector economy model consisting of material sector ($i = 0$), capital generating sector ($i = 1$), and consumer sector ($i = 2$). It is assumed that in each sector produced its aggregate product: material sector–objects of labor (fuel, electricity, raw and other materials), capital generating sector–means of labor (machines, equipment and industrial buildings, etc.); consumer sector–consumer goods.

Let the amount of production output in each i-th sector is described by the Cobb–Douglas functions [6]

$$X_i = F_i(K_i, L_i) = A_i K_i^{\alpha_i} L_i^{1-\alpha_i}, \quad (i = 0,\ 1,\ 2), \tag{10}$$

where X_i is a production output, K_i is an investment capital, L_i is a number of employees, α_i is an elasticity coefficient of funds, $(1-\alpha_i)$ is an elasticity coefficient of labour.

The dynamics of fixed assets can be described by the following differential equations

$$\dot{K}_i = -\mu_i K_i + I_i, \quad K_i(0) = K_i^0, \quad (i = 0,\ 1,\ 2), \tag{11}$$

where μ_i is a proportion of withdrawn fixed assets, I_i is an investment.

We have the following balance equations:

$$\begin{aligned} X_1 &= I_0 + I_1 + I_2, \\ L &= L_0 + L_1 + L_2, \\ X_0 &= \beta_0 X_0 + \beta_1 X_1 + \beta_2 X_2, \end{aligned} \tag{12}$$

where $L = L^0 e^{\nu t}$ is a labour resources, ν is an annual growth rate of employees; β_i is a direct material costs per output unit of i-th sector ($i = 0,\ 1,\ 2$).

We will use the following notation: $s_i = I_i/X_1$ is share of sectors in investment resource distribution, $\theta_i = L_i/L$ is share of sectors in workforce distribution, $k_i = K_i/L_i$ is sectors' capital-labour ratio, $f_i(k_i) = X_i/L_i = A_i k_i^{\alpha_i}$ is labour productivity of i-th sector, $x_i = X_i/L = \theta_i f_i(k_i)$, $\lambda_i = \mu_i + \nu$, ($i = 0,\ 1,\ 2$). Using this notation, the three-sector economic model of a cluster (10-(12) can be described as the system of six differential and algebraic equations [13]:

$$\begin{aligned} \dot{k}_i &= -\lambda_i k_i + (s_i/\theta_i) x_1, \quad k_i(0) = k_i(0) = k_i^0, \quad \lambda_i > 0, \quad (i = 0,\ 1,\ 2), \\ x_i &= \theta_i A_i k_i^{\alpha_i}, \quad A_i > 0, \quad 0 < \alpha_i < 1, \quad (i = 0,\ 1,\ 2), \end{aligned} \tag{13}$$

and plus three balance equations

$$\begin{aligned} & s_0 + s_1 + s_2 = 1, \quad s_0 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0, \\ & \theta_0 + \theta_1 + \theta_2 = 1, \quad \theta_0 \geq 0, \quad \theta_1 \geq 0, \quad \theta_2 \geq 0, \\ & (1-\beta_0) x_0 = \beta_1 x_1 + \beta_2 x_2, \quad \beta_0 \geq 0, \quad \beta_1 \geq 0. \quad \beta_2 \geq 0. \end{aligned} \tag{14}$$

The system's state is described by the vector (k_0, k_1, k_2), and $(s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$ is a control vector. The initial state of the system is (k_0^0, k_1^0, k_2^0), where k_i^0 is a capital-labour ratio of i-th sector at $t = 0$. We'll consider the problem of moving the system into the state (k_0^*, k_1^*, k_2^*) within the time interval $[0, T]$. As a desired final state we choose the steady state of the system, which can be determined by equating to zero the right sides of differential equations (13):

$$k_1^* = \left(\frac{s_1 A_1}{\lambda_1}\right)^{\frac{1}{1-\alpha_1}}, \quad k_0^* = \frac{s_0 \theta_1 A_1 (k_1^*)^{\alpha_1}}{\lambda_0 \theta_0}, \quad k_2^* = \frac{s_2 \theta_1 A_1 (k_1^*)^{\alpha_1}}{\lambda_2 \theta_2}. \tag{15}$$

The values of capital-labor ratios k_i^*, ($i = 0,\ 1,\ 2$) in the steady state (15) depend on controls $(s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$, which optimal values $(s_0^*, s_1^*, s_2^*, \theta_0^*, \theta_1^*, \theta_2^*)$ can be determined by solving a nonlinear programming problem to maximize a specific consumption $(x_2 \to \max)$ [12]. It should be noted that balanced growth of all three sectors is provided in the equilibrium state of system (15), where the production output will increase with the equal annual growth rate ν.

Using three balance ratios (14), we reduce the task to a problem with three controls denoted as (s_1, ν_2, θ_1). Let's write the system of differential equations (13) using deviations from the steady state of the system

$$\dot{y}_i = f_i(y, u), \quad y_i(0) = y_i^0, \quad (i = 1,\ 2,\ 3), \tag{16}$$

where $y_1 = k_0 - k_0^*, y_2 = k_1 - k_1^*,\ y_3 = k_2 - k_2^*,\ u_1 = s_1 - s_1^*, u_2 = \nu_2 - \nu_2^*,\ u_3 = \theta_1 - \theta_1^*,$ and

$$\begin{aligned}
f_1(y,u) &= -\lambda_0(y_1+k_1^*) + \frac{(u_2+\nu_2^*)(1-u_1-s_1^*)A_1(y_2+k_1^*)^{\alpha_1}\left[(1-\beta_0)A_0(y_1+k_0^*)^{\alpha_0}+\beta_2A_2(y_3+k_2^*)^{\alpha_2}\right]}{\beta_1A_1(y_2+k_1^*)^{\alpha_1}+\beta_2\left[(u_3+\theta_1^*)^{-1}-1\right]A_2(y_3+k_2^*)^{\alpha_2}},\\
f_2(y,u) &= -\lambda_1(y_2+k_1^*)+(u_1+s_1^*)A_1(y_2+k_1^*)^{\alpha_1},\\
f_3(y,u) &= -\lambda_2(y_3+k_2^*) + \frac{(1-u_2-\nu_2^*)(1-u_1-s_1^*)A_1(y_2+k_1^*)^{\alpha_1}\left[(1-\beta_0)A_0(y_1+k_0^*)^{\alpha_0}+\beta_2A_2(y_3+k_2^*)^{\alpha_2}\right]}{(1-\beta_0)\left[(u_3+\theta_1^*)^{-1}-1\right]A_0(y_1+k_0^*)^{\alpha_0}-\beta_1A_1(y_2+k_1^*)^{\alpha_1}}.
\end{aligned}$$

Here $y = (y_1,y_2,y_3)'$ means the object's state vector, $u = (u_1,u_2,u_3)'$ is a control vector. Linearizing the system (16), we obtain the vector differential equation of the form

$$\dot{y}(t) = Ay(t)+Bu(t),\quad y(0) = y^0,\quad (0 \le t \le T), \tag{17}$$

where the elements of matrices $A = \| a_{ij} \|_{3\times3}$ and $B = \| b_{ij} \|_{3\times3}$ are defined by the formulas

$$a_{ij} = \frac{\partial f_i(y,u)}{\partial y_j},\quad a_{ij} = \frac{\partial f_i(y,u)}{\partial u_j},\quad (i, j = 1,\ 2,\ 3)$$

at $y = (0,0,0)'$ and $u = (0,0,0)'$.

The initial and final states of the system are given as

$$y(0) = y^0,\quad y(T) = 0. \tag{18}$$

Note that the desired final state of the system is a steady state $y(T) = 0$, where the specific consumption is maximal. The control vector components satisfy to box constraints of the following form

$$-s_1^* \le u_1 \le 1-s_1^*,\quad -\nu_2^* \le u_2 \le 1-\nu_2^*,\quad -\theta_1^* \le u_3 \le 1-\theta_1^* \tag{19}$$

which are obtained from the original constraints: $0 \le s_1 \le 1,\ 0 \le \nu_2 \le 1,\ 0 \le \theta_1 \le 1$.

Let's consider the following optimal control problem: find a control that brings system (17) from a given initial state $y(0) = y^0$ to the steady state $y(T) = 0$ within the finite time interval $[0,T]$, minimizing the objective functional

$$J(u) = \frac{1}{2}\int_0^T [y'(t)Qy(t)+u'(t)Ru(t)]dt, \tag{20}$$

where Q and R are positive semidefinite and positive definite (3×3)-matrices, respectively.

We consider the linear-quadratic problem (LQ-problem) for the system with fixed ends of trajectories: $y(0) = y^0$, $y(T) = 0$, i.e. the optimal trajectory must pass through these two points. In addition, there are box constraints (19) on the control values. Note that the proposed method of solving the LQ-problem (17)-(20) allows to find the optimal control in the form of synthesizing control $u = u(y,t)$, that depends on the state of system y and the current time t.

As a result of solving the nonlinear programming problem in order to maximize the spesific consumption, the following values of controls were obtained:

$$s_0^* = 0.2763,\quad s_1^* = 0.4476,\quad s_2^* = 0.2761,\quad \theta_0^* = 0.3944,\quad \theta_1^* = 0.2562,\quad \theta_2^* = 0.3494,$$

and the capital-labor ratios were calculated using the formulas (15)

$$k_0^* = 966.44,\quad k_1^* = 2410.15,\quad k_2^* = 1090.12.$$

Further calculations were carried to find a numerical solution of optimal control problem (17)-(20) with initial conditions $y_1(0) = -80,\ y_2(0) = -560,\ y_3(0) = -70$ and planning horizon $T = 10$. The results of numerical calculations are shown in Figures. 1, 2.

As it is seen in Figure 1, the found control brings the system to the steady state $y(T) = 0$ at the final time moment T. The control belongs to the boundary of domain U defined by the inequalities (19) during time interval $[0,\ t_1]$ and locates inside of U in time interval $(t_1,T]$ (see Figure 2), where $t_1 \approx 0.68$.

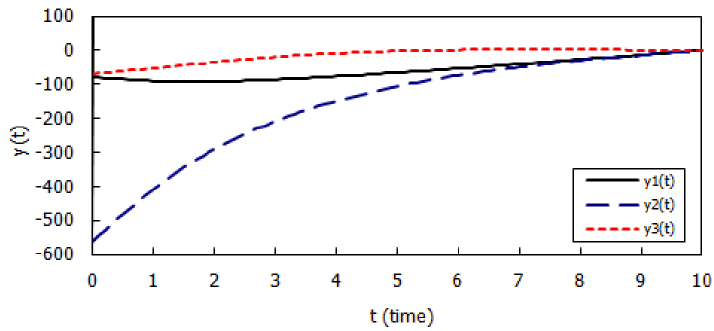

FIGURE 1. Graph of the optimal trajectories

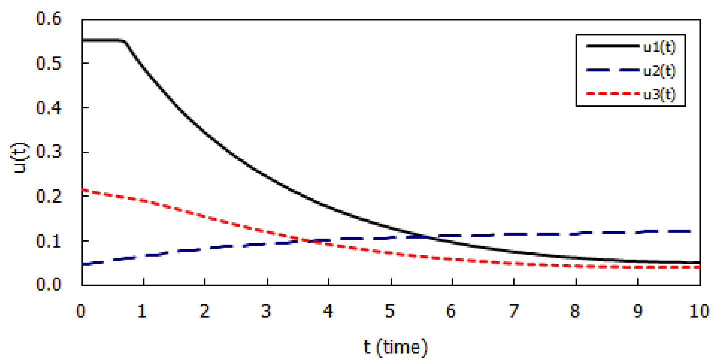

FIGURE 2. Graph of the optimal control

CONCLUSIONS

This work considers the three-sector economic model of a cluster, which can be described by a system of differential and algebraic equations, and balance ratios. The problem of finding the optimal control, which moves the system from a given initial state to desired final state within the fixed time interval. As the final state of the system is selected the steady state, where the labor productivity in the second sector (commodities sector) is maximal.

The peculiarities of the problem are: it is solved for the finite time interval; the left and right ends of trajectories are fixed; there are constraints on the control values; the synthesizing control depending on the state of the object and the current time is searched. The problem is solved using the Lagrange multipliers of special type. That allows to represent the required optimal control as sum of control with feedback and programming control . A numerical example shows that the obtained control brings the system to the steady state with good enough accuracy; the values obtained in the numerical calculations are: $y_1(T) \approx 0.475 \cdot 10^{-3}$, $y_2(T) \approx 0.112 \cdot 10^{-3}$, $y_3(T) \approx -0.159 \cdot 10^{-3}$.

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