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Regularization of the continuation problem for elliptic equations

S. I. Kabanikhin, Y. S. Gasimov, D. B. Nurseitov, M. A. Shishlenin, B. B. Sholpanbaev and S. Kasenov

Abstract. We investigate the continuation problem for the elliptic equation. The continuation problem is formulated in operator form $Aq = f$. The singular values of the operator A are presented and analyzed for the continuation problem for the Helmholtz equation. Results of numerical experiments are presented.

Keywords. Helmholtz equation, inverse problem, singular values, degree of ill-posedness.

2010 Mathematics Subject Classification. 65N20, 65N21, 35J05, 35Q93.

1 Introduction

The Cauchy problem for the elliptic equation is a well-known example of an ill-posed problem. The solution is unique, but does not depend continuously on the Cauchy data in standard norms [1, 6, 7].

The Cauchy problem for the Helmholtz equation was investigated theoretically by F. John in [6]. He showed that the conditional stability estimate for k is the best logarithmic estimate. It was demonstrated in [4, 5] that ill-posedness of the Cauchy problem for the Helmholtz equation depends on the wave number k and increases as k increases. There is a subspace of the data space on which the Cauchy problem is well-posed, and this subspace grows with larger k (a subspace of stability). For more general geometries, authors studied the ill-posedness by computing the singular values of some operators associated with corresponding well-posed (direct) boundary value problems.

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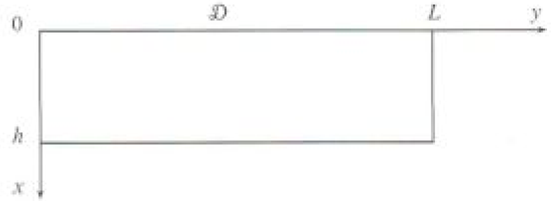

Figure 1. Domain $\Omega = (0, h) \times \mathcal{D}$.

Numerical calculations using various regularization methods are presented, e.g., in the following papers: quasi-reversibility method [2, 11], frequency space cut-off [18], iterative methods [9, 12, 15], regularization methods [3, 10, 16, 17, 19].

It was shown [19] that the Cauchy problem for the Helmholtz equation depends on different smoothness situations, the best possible accuracy may be of Hölder type, of logarithmic type, or of some other type.

First we consider the continuation problem for the elliptic equation in the cylinder (see Figure 1):

$$u_{xx} + L(y)u = 0, \quad (x, y) \in \Omega, \tag{1.1}$$

$$u(0, y) = f(y), \quad y \in \mathcal{D}, \tag{1.2}$$

$$u_x(0, y) = 0, \quad y \in \mathcal{D}, \tag{1.3}$$

$$u|_{\partial\mathcal{D}} = 0, \quad x \in (0, h), \tag{1.4}$$

with the condition

$$f|_{\partial\mathcal{D}} = 0, \quad x \in (0, h). \tag{1.5}$$

Here $\Omega = (0, h) \times \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^n$ is a connected bounded domain with Lipschitz boundary $\partial\mathcal{D}$. The operator $L(y)$ has the form

$$L(y)u = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u}{\partial y_j}\right) - c(y)u,$$

with the coefficients $a_{ij}(y)$ and $c(y)$ satisfying the relations

$$C_1 \sum_{j=1}^{n} v_j^2 \le \sum_{i,j=1}^{n} a_{ij}(y) v_i v_j, \quad \text{for all } v_j \in \mathbb{R},$$

$$a_{ij} = a_{ji}, \quad i, j = 1, \dots, n, \quad a_{ij} \in C^1(\overline{\mathcal{D}}),$$

$$0 \le c(y) \le C_2, \quad c \in C(\overline{\mathcal{D}}).$$

Here C_1 and C_2 are positive constants.

In Section 2 we describe and justify a gradient type algorithm for the continuation problem (1.1)–(1.5).

In Section 3 we investigate the continuation problem for the Helmholtz equation for simple geometry and homogeneous media.

In Section 4 we introduce and analyze a complex-valued cost functional such that the adjoint problem has exactly the same form as the direct problem.

In Section 5 the numerical results are presented and analyzed.

2 Optimization approach

Let us consider the ill-posed continuation problem (1.1)–(1.5) as an inverse problem to the following direct well-posed problem:

$$u_{xx} + L(y)u = 0, \quad (x, y) \in \Omega, \tag{2.1}$$

$$u_x(0, y) = 0, \quad y \in \mathcal{D}, \tag{2.2}$$

$$u(h, y) = q(y), \quad y \in \mathcal{D}, \tag{2.3}$$

$$u|_{\partial\mathcal{D}} = 0, \quad x \in (0, h), \tag{2.4}$$

with the condition

$$q|_{\partial\mathcal{D}} = 0, \quad x \in (0, h). \tag{2.5}$$

In the direct problem (2.1)–(2.5) it is required to find $u(x, y)$ in Ω from the function $q(y)$ given on a part of the boundary $x = h$ of the domain Ω.

The inverse problem consists of finding $q(y)$ from (2.1)–(2.5) and the additional information

$$u(0, y) = f(y). \tag{2.6}$$

The inverse problem (2.1)–(2.6) and the continuation problem (1.1)–(1.5) are equivalent to each other in the following sense: if we solve the inverse problem, we find the solution of the continuation problem $u(x, y)$ and *vice versa*.

Let us remind several results from the direct [14] and inverse [7] problems theory.

Definition 2.1. A function $u \in L_2(\Omega)$ is called a generalized solution to the direct problem (2.1)–(2.5) if for any $w \in H^2(\Omega)$ such that

$$w_x(0, y) = 0, \quad y \in \mathcal{D},$$

$$w(h, y) = 0, \quad y \in \mathcal{D},$$

$$w|_{\partial\mathcal{D}} = 0, \quad x \in (0, h),$$

u satisfies the equality

$$\int_{\Omega} u(w_{xx} + L(y)w)\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathcal{D}} q(y)w_x(h, y)\,\mathrm{d}y = 0.$$

Theorem 2.2 (on the well-posedness of the direct problem and existence of the trace $u(0, y)$). *If $q \in L_2(\mathcal{D})$, then the direct problem (2.1)–(2.5) has a unique generalized solution $u \in L_2(\Omega)$ which has the trace $u(0, y) \in L_2(\mathcal{D})$ and the following estimates hold true:*

$$\|u\|_{L_2(\Omega)} \le M_1\|q\|_{L_2(\mathcal{D})},$$
$$\|u(0, y)\|_{L_2(\mathcal{D})} \le \|q\|_{L_2(\mathcal{D})}.$$

Here M_1 is a positive constant.

Theorem 2.3 (Conditional stability estimate). *Let $q, f \in L_2(\mathcal{D})$. If the continuation problem (1.1)–(1.5) has a solution $u \in C^2(\overline{\Omega})$, then it satisfies the inequality* [7, 13]

$$\int_{\mathcal{D}} u^2(x, y)\,\mathrm{d}y \le \|q\|^{2x/h}_{L_2(\mathcal{D})}\|f\|^{2(h-x)/h}_{L_2(\mathcal{D})}, \quad x \in (0, h).$$

Together with the direct problem (2.1)–(2.5) we consider the adjoint problem

$$\psi_{xx} + L(y)\psi = 0, \qquad (x, y) \in \Omega, \tag{2.7}$$
$$\psi_x(0, y) = \mu(y), \qquad y \in \mathcal{D}, \tag{2.8}$$
$$\psi(h, y) = 0, \qquad y \in \mathcal{D}, \tag{2.9}$$
$$\psi|_{\partial\mathcal{D}} = 0, \quad x \in (0, h). \tag{2.10}$$

The problem is to find $\psi(x, y)$ by given $\mu(y)$.

Theorem 2.4 (on the well-posedness of the adjoint problem and existence of the trace $\psi_x(h, y)$). *If $\mu \in L_2(\mathcal{D})$, then problem (2.7)–(2.10) has a unique generalized solution $\psi \in L_2(\Omega)$ which has the trace $\psi_x(h, y) \in L_2(\mathcal{D})$ and the following estimates hold true:*

$$\|\psi\|_{L_2(\Omega)} \le M_2\|\mu\|_{L_2(\mathcal{D})},$$
$$\|\psi_x(h, y)\|_{L_2(\mathcal{D})} \le \|\mu\|_{L_2(\mathcal{D})}.$$

Here M_2 is a positive constant.

We introduce the operator

$$A : q(y) \to u(0, y),$$

where $u(x, y)$ is a solution to the direct problem (2.1)–(2.5).

Then the adjoint operator A^* has the form

$$A^* : \mu(y) \to \psi_x(h, y),$$

where $\psi(x, y)$ is a solution to the adjoint problem (2.7)–(2.10).

It follows from Theorems 2.2 and 2.4 that the operators A and A^* map $L_2(\mathcal{D})$ into $L_2(\mathcal{D})$. Therefore, the inverse problem (2.1)–(2.6) can be written in the operator form

$$Aq = f. \tag{2.11}$$

We will find the solution of (2.11) by minimizing the cost functional

$$J(q) = \|Aq - f\|^2_{L_2(\mathcal{D})}.$$

Let us consider the method of Landweber iteration

$$q_{n+1} = q_n - \alpha J'q_n, \quad n = 0, 1, 2, \dots,$$

where α is the descent parameter

$$\alpha \in \left(0, \frac{1}{2\|A\|^2}\right)$$

and $J'q_n$ is the gradient of the functional $J(q_n)$,

$$J'q = 2A^*(Aq - f).$$

Note that $J'q$ is calculated by

$$(J'q)(y) = \psi_x(h, y),$$

where $\psi(x, y)$ is a solution to the adjoint problem (2.7)–(2.10) with

$$\mu(y) = 2[u(0, y) - f(y)].$$

Theorem 2.5 (The rate of convergence with respect to the functional). *Let the problem $Aq = f$ have the exact solution $q_e \in L_2(\mathcal{D})$. Then there exists a constant $M_3 > 0$ such that the following estimate holds true:*

$$J(q_n) \le \frac{M_3}{n}, \quad n = 1, 2, \dots.$$

Theorem 2.6. *Let the problem $Aq = f$ have the exact solution $q_e \in L_2(\mathcal{D})$. Then there exists a constant $M_4 > 0$ such that the sequence $\{u_n\}$ of solutions to the direct problem (2.1)–(2.5) for the corresponding iterations q_n converges to the* *exact solution $u_e \in L_2(\Omega)$ to problem* (1.1)–(1.5) *and the following estimate holds* [8]:

$$\int_{\mathcal{D}} (u_n(x,y) - u_e(x,y))^2 \, dy \le M_4 n^{\frac{x-h}{h}}, \quad x \in (0,h).$$

Theorem 2.7. *Let the problem $Aq = f$ have the exact solution $q_e \in L_2(\mathcal{D})$. Let $\|f - f^\delta\| \le \delta$. Then there exist constants $M_5 > 0$ and $M_6 > 0$ such that the sequence $\{u_n\}$ of solutions to the direct problem* (2.1)–(2.5) *for the corresponding iterations ${u^\delta}_n$ converges to the exact solution $u_e \in L_2(\Omega)$ to problem* (1.1)–(1.5) *and the following estimate holds* [8]:

$$\int_{\mathcal{D}} (u_n^\delta(x,y) - u_e(x,y))^2 \, dy \le M_5 \beta(n)\delta + M_6 n^{\frac{x-h}{h}}, \quad x \in (0,h). \tag{2.12}$$

Here

$$\beta(n) = \frac{(1 + 2\alpha\|A\|^2)^{n-1} - 1}{\|A\|}.$$

The same results were obtained for the steepest descent method and conjugate-gradient method [7, 8].

The estimate (2.12) shows that the sequence u_n^δ is a regularizing sequence with n as the regularization parameter. Since the first term monotonically tends to infinity and the second term monotonically tends to zero as $n \to \infty$, the stopping number n_* can be chosen as follows. Differentiating the right-hand side of (2.12) with respect to n, we then can find the root n_r of the equation

$$M_5\delta \frac{\ln(1 + 2\alpha\|A\|^2)}{\|A\|}(1 + 2\alpha\|A\|^2)^{n-1} + M_6 \frac{x-h}{h} n^{\frac{x-2h}{h}} = 0 \tag{2.13}$$

and choose the stopping number n_s as the natural number which is the nearest to the root n_r of equation (2.13).

3 SVD analysis of continuation problem

Let us consider the continuation problem for the Helmholtz equation for simple geometry and homogeneous media:

$$\Delta u + ku = 0, \qquad x \in (0,h),\ y \in (0,\pi), \tag{3.1}$$

$$u(0,y) = f(y), \qquad y \in (0,\pi), \tag{3.2}$$

$$u_x(0,y) = 0, \qquad y \in (0,\pi), \tag{3.3}$$

$$u(x,0) = u(x,\pi) = 0, \quad x \in (0,h). \tag{3.4}$$

Here

$$k = \varepsilon\omega^2 - i\sigma\omega,$$

ω is a frequency, ε and σ are positive constants.

The continuation problem (3.1)–(3.4) consists of finding a function $u(x, y)$ in the domain $x \in (0, h)$, $y \in (0, \pi)$ by the given boundary conditions (3.2)–(3.4).

Let us formulate the continuation problem in the form of inverse problem. We introduce the direct problem:

$$\Delta u + ku = 0, \qquad x \in (0, h),\ y \in (0, \pi), \tag{3.5}$$

$$u_x(0, y) = 0, \quad u(h, y) = q(y), \quad y \in (0, \pi), \tag{3.6}$$

$$u(x, 0) = u(x, \pi) = 0, \qquad x \in (0, h). \tag{3.7}$$

Inverse problem. Find a function $q(y)$ using the additional information

$$u(0, y) = f(y), \quad y \in (0, \pi). \tag{3.8}$$

The operator statement of the inverse problem (3.5)–(3.8) can be written in the form $Aq = f$, where $A : H^{\frac{1}{2}}(0, \pi) \to H^{\frac{1}{2}}(0, \pi)$, see [7, 12].

Let us find the solution to the direct problem (3.5)–(3.7). We suppose that $q(y)$ has the form

$$q(y) = \sum_{m=1}^{\infty} q^{(m)} \sin(my)$$

and find the direct problem solution as a Fourier series

$$u(x, y) = \sum_{m=1}^{\infty} u^{(m)}(x) \sin(my)$$

solving the sequence of direct problems:

$$u_{xx}^{(m)} + k_m u^{(m)} = 0, \quad x \in (0, h), \tag{3.9}$$

$$u_x^{(m)}(0) = 0, \quad u^{(m)}(h) = q^{(m)}. \tag{3.10}$$

Here

$$k_m = \varepsilon\omega^2 - m^2 - i\sigma\omega.$$

The general solution of equation (3.9) has the following form:

$$u^{(m)}(x) = C_1 e^{\lambda_m x} + C_2 e^{-\lambda_m x}.$$

Here

$$\sqrt{-k_m} = \pm\lambda_m$$

and

$$\lambda_m = \alpha_m + i\beta_m$$

with

$$\alpha_m = \sqrt{\frac{\sqrt{(m^2 - \varepsilon\omega^2)^2 + \sigma^2\omega^2} + m^2 - \varepsilon\omega^2}{2}},$$

$$\beta_m = \sqrt{\frac{\sqrt{(m^2 - \varepsilon\omega^2)^2 + \sigma^2\omega^2} - m^2 + \varepsilon\omega^2}{2}}.$$

Therefore the solution of problem (3.9)–(3.10) is given by the formula

$$u^{(m)}(x) = \frac{\cosh(\lambda_m x)}{\cosh(\lambda_m h)} q^{(m)}.$$

Then the solution of the direct problem (3.5)–(3.7) is given by the following Fourier series:

$$u(x, y) = \sum_{m=1}^{\infty} \frac{\cosh(\lambda_m x)}{\cosh(\lambda_m h)} q^{(m)} \sin(my).$$

Therefore the solution of inverse problem (3.5)–(3.8) is given by the Fourier series expansion

$$q(y) = \sum_{m=1}^{\infty} f^{(m)} \cosh(\lambda_m h) \sin(my). \tag{3.11}$$

Thus the singular values of the operator A have the form

$$\sigma_m(A) = \frac{1}{|\cosh(\lambda_m h)|} = \frac{\sqrt{2}}{\sqrt{\cosh(2\alpha_m h) + \cos(2\beta_m h)}}. \tag{3.12}$$

Remark. Formulas (3.11) and (3.12) show how one can choose the number m to use the cut-SVD regularization. Namely, if $\| f^m - f^m_\delta \| < \delta$, then α_m should not be greater than $|\ln \delta| / h$.

Let us consider several particular cases of singular values of the operator A.

Example 3.1. Laplace equation: $\varepsilon = 0$, $\sigma = 0$,

$$\sigma_m(A) = \frac{1}{\cosh(mh)}.$$

Example 3.2. Parabolic equation: $\varepsilon = 0, \sigma \neq 0$,

$$\sigma_m(A) = \frac{\sqrt{2}}{\sqrt{\cosh(2\alpha_m h) + \cos(2\beta_m h)}},$$

$$\alpha_m = \sqrt{\frac{\sqrt{m^4 + \sigma^2\omega^2} + m^2}{2}},$$

$$\beta_m = \sqrt{\frac{\sqrt{m^4 + \sigma^2\omega^2} - m^2}{2}}.$$

Example 3.3. Helmholtz equation [5]: $\varepsilon \neq 0, \sigma = 0$.

$$\sigma_m(A) = \begin{cases} \frac{1}{|\cos(\sqrt{k_m}h)|}, & m^2 \leq \varepsilon\omega^2, \\ \frac{1}{\cosh(\sqrt{k_m}h)}, & \varepsilon\omega^2 < m^2. \end{cases}$$

Example 3.4. In the general case of singular values (3.12) when $\sigma \to 0$ we arrive at Example 3.3.

The singular values depend on the number k, see [5]. In the low frequency domain $m^2 \leq \varepsilon\omega^2$ the singular values of A are bounded from below by 1, while for the high frequency domain the singular values decay exponentially. The most important fact is that in the low frequency domain the operator A is continuously invertible and this domain increases with k.

4 Complex-valued cost functional

Let us describe the numerical method for the continuation problem (3.1)–(3.4) in the case when the direct problem and adjoint problem have the same form.

We introduce a new cost functional

$$J_1(q) = (Aq - f)^2 = \int_0^\pi [u(0, y) - f(y)]^2 \, dy.$$

Note that the functions $u(x, y)$ and $f(y)$ are complex-valued functions. The functional $J_1(q)$ maps $H^{\frac{1}{2}}(0, \pi)$ into the set of complex numbers $\mathbb{C}$. We construct a sequence $\{q_n\}_{n \in \mathbb{N}}$ such that $|J_1(q_n)| \to 0$ when $n \to \infty$.

Note that the traditional functional

$$J(q) = |Aq - f|^2, \quad J : H^{\frac{1}{2}}(0, \pi) \to \mathbb{R}_+,$$

is connected to $J(q) = |J_1(q)|$.

If the exact solution of $Aq = f$ exists, then $J(q)$ vanishes $J(q_e) = 0$. If the solution of $Aq = f$ is unique, then $J(q)$ does not have more than one zero. It follows from the definition that $J_1(q) \to 0$ when $q \to q_e$ is equivalent to $J(q) \to 0$ when $q \to q_e$.

Therefore we can consider the complex-valued cost functional $J_1(q)$ and minimize it by the following procedure:

(1) Let $q_0(y)$ be the initial guess.

(2) Suppose that $q_n(y)$ is known.

(3) Solve the direct problem

$$\begin{aligned} &\Delta u_n + k u_n = 0, \\ &u_{nx}(0,y) = 0, \quad u_n(h,y) = q_n(y), \\ &u_n(x,0) = u_n(x,\pi) = 0. \end{aligned}$$

(4) Solve the adjoint problem

$$\begin{aligned} &\Delta \psi_n + k\psi_n = 0, \\ &\psi_{nx}(0,y) = 2[u_n(0,y) - f(y)], \quad \psi_n(h,y) = 0, \\ &\psi_n(x,0) = \psi_n(x,\pi) = 0. \end{aligned}$$

(5) Find an approximate solution on the next step $n+1$:

$$q_{n+1}(y) = q_n(y) - \alpha_n \overline{J'(q_n)}(y). \tag{4.1}$$

Here $\overline{J'(q_n)}$ is the complex conjugate of the gradient $J'(q_n)$, $\alpha_n \in \mathbb{C}$ is a complex descent parameter and $|\alpha_n|$ is small enough,

$$\alpha_n = \alpha_{n1} \cdot \operatorname{sgn} \operatorname{Re} J(q_n) + \mathrm{i} \cdot \alpha_{n2} \cdot \operatorname{sgn} \operatorname{Im} J(q_n), \tag{4.2}$$

where α_{n1} and α_{n2} are positive real numbers and

$$\operatorname{sgn} a = \begin{cases} 1, & a > 0, \\ 0, & a = 0, \\ -1, & a < 0. \end{cases}$$

Remark. Using this minimizing procedure, we obtain that $|J(q_n)| \to 0$ as $n \to \infty$.

It follows from (4.1) that

$$q_{n+1} - q_n = -\alpha_n \overline{J'(q_n)}. \tag{4.3}$$

From the definition of the Fréchet derivative we have

$$J(q_{n+1}) - J(q_n) = J'(q_n)(q_{n+1} - q_n) + o(|q_{n+1} - q_n|) \tag{4.4}$$

and substituting (4.3) into equation (4.4), we obtain

$$J(q_{n+1}) - J(q_n) = -\alpha_n J'(q_n)\overline{J'(q_n)} + o(|J'(q_n)|)$$
$$= -\alpha_n |J'(q_n)|^2 + o(|J'(q_n)|).$$

Therefore, using minimizing procedure (4.1) with choosing the appropriate descent parameter (4.2), we obtain that $J(q_n) \to 0$ when $n \to \infty$.

5 Numerical experiments

In this section we demonstrate that continuation of the solution up to the depth $x = h$ helps detect two inclusions located beneath the line $x = h$.

Indeed, let us consider the homogeneous domain $(0, h_1) \times (0, L)$ (see Figure 2) with fixed $\varepsilon = 1$ and $\sigma = 0$ in the medium. First we solve the direct problem (see the trace $u(0, y)$) in Figure 2) with the source function

$$u_x(0, y) = g(y) = 25 \cdot 10^5 \theta\left(0.002 - \left|y - \frac{L}{2}\right|\right)$$

in the domain $(0, h_1) \times (0, L)$.

Second we solve the continuation problem in $(0, h) \times (0, L)$ with Cauchy data

$$u(0, y) = f(y), \quad u_x(0, y) = g(y).$$

The result of the continuation is demonstrated in Figure 2.

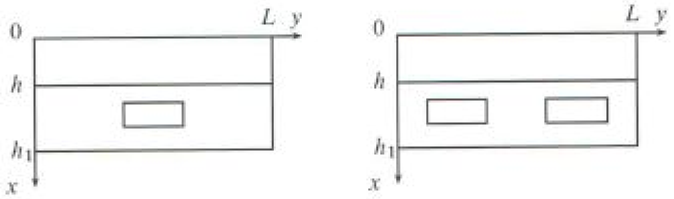

Figure 2. Domain with one inclusion in the center (left figure) and two inclusions (right figure) beneath the boundary $x = h$.

We apply the method of steepest descent for minimizing the cost functional and finite element method for the direct and the adjoint problem solution. There are 427,336 triangles and 214,767 vertexes in the domain. We fixed $h = 1$, $h_1 = 3$, $L = 6$. We consider two cases with one inclusion (see Figure 2 (left)) and with two inclusions (see Figure 2 (right)) beneath the boundary with $\varepsilon = 40$. Initial guess in all numerical experiments is zero.

First case: one inclusion in the center which is located in $x \in [1, 2]$ and $y \in [2, 4]$ (Figure 2 (left)). Second case: two inclusions which are located in $x \in [1.5, 2.5]$, $y \in [4, 5.5]$ and $x \in [1.5, 2.5]$, $y \in [4, 5.5]$ (Figure 2 (right)).

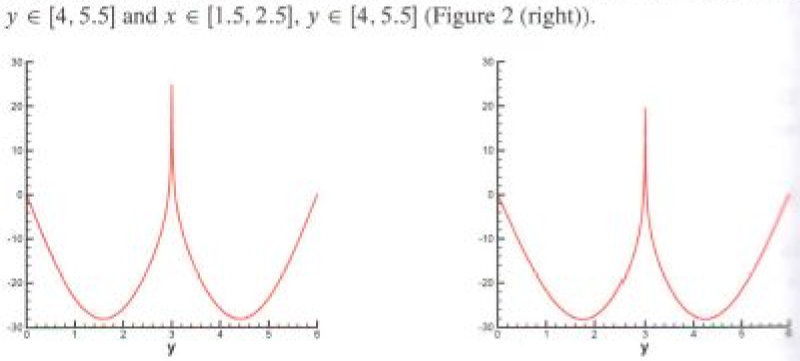

Figure 3. On the left-hand side figure: $u(0, y)$ in a medium with one inclusion. On the right-hand side figure: $u(0, y)$ in a medium with two inclusions.

On the left-hand side of Figure 3 there is measured the function $u(0, y)$ on the surface $x = 0$ in a medium with one inclusion in the center beneath the boundary $x = h$. On the right-hand side of Figure 3 there is measured the function $u(0, y)$ on the surface $x = 0$ in a medium with two inclusions beneath the boundary $x = h$.

On the left-hand side of Figure 4 there are the function $q(y) = u(h, y)$ (solid curve) and the reconstructed $q^{(250)}(y)$ (dashed curve) in a medium with one inclusion in the center beneath the boundary $x = h$.

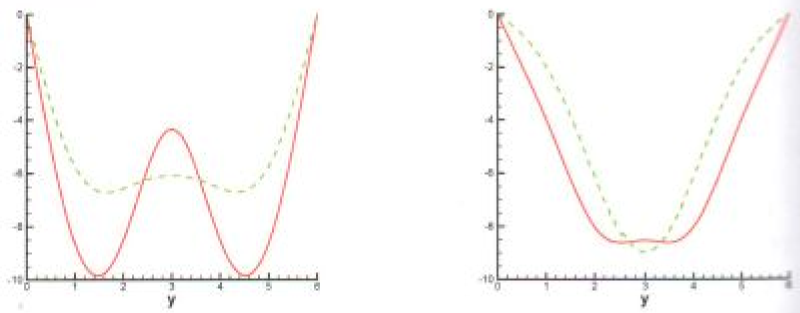

Figure 4. On the left-hand side: $q(y) = u(h, y)$ (solid curve) and the reconstructed $q^{(250)}(y)$ (dashed curve) in a medium with one inclusion in the center beneath the boundary. On the right-hand side: $q(y) = u(h, y)$ (solid curve) and the reconstructed $q^{(250)}(y)$ (dashed curve) in a medium with two inclusions beneath the boundary.

On the right-hand side of Figure 4 there are the function $q(y) = u(h, y)$ (solid curve) and the reconstructed $q^{(250)}(y)$ (dashed curve) in a medium with two inclusions beneath the boundary $x = h$.

We see that the boundary condition $q(y)$ depends on the inclusion outside the domain and the continuation procedure makes more clear if there are inclusions beneath the boundary.

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