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On estimates of solutions of the linear stationary problem of magnetohydrodynamics problem in Sobolev spaces

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Abstract. In this paper, we get estimates in Sobolev spaces for solutions of stationary linear problem arising in magnetohydrodynamics. The problem is studied in the multiply connected domains.

Keywords: Magnetohydrodynamics, L_p-estimate, Sobolev spaces, Multi-connected domains.
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STATEMENT OF THE PROBLEM

Let Ω_1, bounded domain in $\mathbb{R}^3$ with a smooth boundary S_1, be strictly interior subdomain Ω from abroad S and let $\Omega_2 = \Omega \backslash \Omega_1$. In this paper, we consider the linear problem that is the system of Maxwell's equations with excluded bias current

$$
\begin{aligned}
&rot\vec{H}(x) - \sigma\vec{E}(x) = \vec{j}(x),\\
&div\vec{H}(x) = 0, \qquad (1)\\
&rot\vec{E}(x) = 0
\end{aligned}
$$

at a given $\vec{j}(x)$, $x \in \Omega_1$. Thus, $rot\vec{H} = 0$, $div\vec{H} = 0$, $x \in \Omega_2$ and fair

$$
\begin{aligned}
&[H_n] = 0,\ [\vec{H}_\tau] = 0,\ x \in S_1 \qquad (2)\\
&H_n = 0,\ x \in S.
\end{aligned}
$$

Under $[u]$ jump of the function $u(x)$, $x \in \Omega_1 \bigcup \Omega_2$ on the surface $S_1:\ [u] = u^1(x) - u^2(x)$, $u^{(i)} = u(x)|_{x\in\Omega_i}$, $H_n = \vec{H}\cdot\vec{n}$ and $\vec{H}_\tau = \vec{H} - \vec{n}H_n$ are normal and tangential components of the vector $\vec{H}(x)$ on S and S_1, μ is a piecewise constant function, equal μ_i in Ω_i, $i = 1,2$, $\mu_i > 0$.

Problem (1)-(2) arises in the study of problems of magneto hydrodynamics, [1–3] in which Ω_1 is an area, filled with a viscous incompressible electrically conducting fluid, Ω_2 is a vacuum surrounding, S is a perfectly conducting surface, $\vec{H}(x)$ is the magnetic field strength. Relations (1) represent a linearized stationary equations of Maxwell (with exceptional bias currents) and (2) represent the standard conditions at the boundary of the magnetic field. We assume the field Ω_1 and Ω_2 simply connected. Then equations $rot\vec{H} = 0$, $div\vec{H} = 0$ in Ω_2 entails $\vec{H}^2(x) = \nabla\varphi(x)$, where $\varphi(x)$ is a solution of the following Neumann problem

$$
\begin{aligned}
&\nabla^2\varphi(x) = 0,\ x \in \Omega_2,\ \left.\frac{\partial\varphi}{\partial n}\right|_{x\in S} = 0,\\
&\mu_2 \left.\frac{\partial\varphi}{\partial n}\right|_{x\in S_1} = \mu_1 \left.\vec{H}^{(1)}\vec{n}\right|_{x\in S_1}, \qquad (3)
\end{aligned}
$$

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and problem (1)-(2) can be written as

$$
\begin{gathered}
\frac{1}{\sigma} rotrot\vec{H}^{(1)}(x)=\vec{g}(x),\;\; div\vec{H}^{(1)}(x)=0,\\
\vec{H}^{(2)}(x)=\nabla\varphi(x),\\
\nabla^2\varphi(x)=0,\;\; x\in\Omega_2,\;\; \frac{\partial\varphi}{\partial n}\Big|_{x\in S}=0,\\
\mu_2\frac{\partial\varphi}{\partial n}-\mu_1\vec{H}^{(1)}\vec{n}\Big|_{x\in S_1}=0,\\
\vec{H}_\tau^{(1)}=\nabla_\tau\varphi(x),\;\; x\in S_1,
\end{gathered}
\tag{4}
$$

where $\vec{g}(x)=\frac{1}{\sigma}rot\vec{j}(x)$.

Hence, $\vec{H}^2(x)$ is completely determined by $\vec{H}^1\cdot\vec{n}\Big|_{x\in S_1}$. Conditions on the surface S_1 for the vector $\vec{H}$ can be briefly written as $\vec{H}_\tau(x)=B(\vec{H}\cdot\vec{n})$, where $B-$nonlocal linear operator. We use annotation of functional spaces and norms accepting in [4, 5].

Theorem 1. *Suppose that $\vec{g}(x)\in L_p(\Omega_1)$ and the conditions*

$$
\nabla\cdot\vec{g}=0,\;\nabla\cdot\vec{H}(x)=0,\;x\in\Omega_1, \tag{5}
$$

$$
\vec{H}_\tau^{(1)}=B(\vec{H}^{(1)}\cdot\vec{n})
$$

hold. Then, problem (1)-(2) *has a unique solution $\vec{H}^{(1)}\in W_p^2(\Omega_1)$ and it satisfies*

$$
||\vec{H}^{(1)}||_{w_p^2(\Omega_1)}\le c||\vec{g}||_{L_{p(\Omega_1)}}. \tag{6}
$$

Recall that $W_p^r(\Omega_1),\; r=[r]+\lambda,\; 0<\lambda<1$ is the space with the norm

$$
||\upsilon||_{w_p^r(\Omega_1)}\le\left(\sum_{0\le j\le[r]}||D^j\upsilon||^p_{L_p(\Omega_1)}+\sum_{|j|=[r]}\int_{\Omega_1}\int_{\Omega_1}|D^j\upsilon(x)-D^j\upsilon(y)|^p\frac{dxdy}{|x-y|^3+p\lambda}\right)^{1/p}.
$$

It is easy to check that (6) implies the same estimate for $\vec{H}^2(x)$. Indeed, the solution of problem (3) satisfies

$$
||\nabla\varphi||_{w_p^2(\Omega_2)}\le c||\vec{H}^{(1)}\cdot\vec{n}||_{w_p^{1-1/p}(S_1)}\le c||\vec{H}^{(1)}||_{w_p^2(\Omega_1)}. \tag{7}
$$

Furthermore, since

$$
\mu_2\int_{\Omega_2}\nabla\varphi\nabla\eta dx=-\int_S\mu_1\vec{H}^{(1)}\vec{n}\eta ds=-\mu_1\int_{\Omega_1}\vec{H}^{(1)}\vec{n}\eta ds=-\mu_1\int_{\Omega_1}\vec{H}^{(1)}\nabla\eta dx
$$

for any $\eta\in W_p^1(\Omega)$, we obtain

$$
||\nabla\varphi||_{L_p(\Omega_2)}\le c||\vec{H}^{(1)}||_{L_p(\Omega_1)}. \tag{8}
$$

From (8)

$$
||\vec{H}^{(2)}||_{w_p^2(\Omega_2)}\le c||\vec{H}^{(1)}||_{w_p^2(\Omega_1)}. \tag{9}
$$

We also have $\vec{H}^2=\nabla\varphi$, where $\varphi(x)$ is the weak solution of the Neumann problem

$$
\begin{gathered}
\nabla^2\varphi=0,\;x\in\Omega_2,\\
\frac{\partial\varphi}{\partial n}\Big|_S=0,\;\; \mu_2\frac{\partial\varphi}{\partial n}-\mu_1\vec{H}^{(1)}\vec{n}\Big|_{S_1}=0,
\end{gathered}
\tag{10}
$$

i.e., the function $\varphi(x)$ satisfies the following integral identity, for all test function $\eta \in J_2^1(\Omega_1)\bigcap J_2^1(\Omega_2)$, satisfying boundary conditions (10)

$$\mu_2 \int\limits_{\Omega_2} \nabla\varphi\nabla\eta dx + \int\limits_{\Omega_1} \mu_1 H^{(1)} \cdot \nabla\eta dx = 0. \tag{11}$$

Solenoidal condition (for example $\nabla\vec{g} = 0$) understood in the usual meaning as $\int\limits_{\Omega_1} \vec{g} \cdot \nabla\eta dx = 0$ for any smooth η vanishing on S_1.

Condition (5) means for $p > 3/2$ as equality trace function $\vec{H}(x)$ and on $S : \vec{H}_\tau^1 = \nabla_\tau\varphi = \vec{H}_\tau^2 \in W_p^{2-3/p}(S_1)$. At $p < 3/2$ it makes no sense, and if $p = 3/2$ understood as an integral limitations

$$\int\limits_{\Omega_2} \left(\vec{k} - \vec{H}^{(2)} - \vec{n} \cdot \vec{n}^* \left(\vec{k} - \vec{H}^{(2)}\right) \rho^{-1}(x)\right) dx,$$

where $\rho(x)$ is a smooth function, equal $dist(x, S_1)$ around S_1, $\vec{n}^*$ is a smooth extension of the normal $\overrightarrow{n}$ inside Ω_2, $\vec{k} \in W_{3/2}^{2/3}(\Omega_2)$ is continuation of the vector field $\vec{H}^1 \in W_{3/2}^{2/3}(\Omega_1)$ with preservation of class.

Remark 1. *For applications to the magneto hydrodynamics most interesting case $p > 3/2$.*

PROBLEM (1)-(2) IN MULTIPLY CONNECTED DOMAINS Ω_1 AND Ω

We turn to a discussion of problem (1)-(2). In the case of many areas of connectedness convenient consider it in the form

$$\begin{gathered} rot\vec{E} = 0,\ div\vec{H}(x) = 0,\ x \in \Omega_1 \bigcup \Omega_2, \\ rot\vec{H} = \sigma\vec{E} + \vec{j}(x),\ \ x \in \Omega_1, \\ rot\vec{H}(x) = 0,\ div\vec{E} = 0,\ x \in \Omega_2, \\ [\mu\vec{H} \cdot \vec{n}] = 0,\ [\vec{H}_\tau] = 0,\ \ [\vec{E}_\tau] = 0,\ x \in S_1, \\ \vec{H} \cdot \vec{n} = 0,\ \ \vec{E}_\tau = 0,\ \ x \in S, \end{gathered} \tag{12}$$

where $\vec{j}(x)$ is given and $\vec{E}$ is additional unknown vector field.

It is clear that, $\vec{E}$ easily eliminated from (12) by (1)-(2) with $\vec{g}(x) = \sigma^{-1} rot\,\vec{j}$. Thus, $\vec{H}^1(x)$ satisfies

$$\sigma^{-1} rot rot \vec{H}^{(1)} = \sigma^{-1} rot\,\vec{j}(x),\ div\vec{H}^{(1)} = 0,\ \ x \in \Omega_1, \tag{13}$$

$$\mu_1 \vec{H}^{(1)}\vec{n} = \mu_2 \frac{\partial\varphi}{\partial n},\ \ \vec{H}_\tau^{(1)} = \nabla_\tau\varphi + \vec{u}_\tau(x),\ \ x \in S_1,\ \ \vec{H}^{(1)}(x) = 0, \tag{14}$$

where function φ, as above, a solution of (3). In addition, it is easy to check that $\vec{H}(x)$ satisfies the integral identity

$$\int\limits_{\Omega_1} rot\vec{H} \cdot rot\,\psi dx = \int\limits_{\Omega_1} \vec{j}(x) rot\,\vec{\psi}(x) dx, \tag{15}$$

where $\vec{\psi}$ is any vector field of the $rot\,\vec{\psi} \in W_2^1(\Omega_1)\bigcap W_2^1(\Omega_2)$, $rot\,\vec{\psi} = 0$ in Ω_2 and continuous tangential component on S_1. Let $\vec{u}_m^*$ be solenoidal smooth extension $\vec{u}_m$ in the area Ω_1. In (15) putting $\vec{\psi} = \vec{u}_m^*$, we get

$$-\int\limits_{\Omega_1} rot rot \vec{H}^{(1)} \cdot \vec{u}_m^* dx + \int\limits_{\Omega_1} rot\,\vec{j}(x) \cdot \vec{u}_m^* dx = \int\limits_{S_1} (rot\vec{H}^{(1)} - \vec{j})(\vec{n} \times \vec{u}_m) dS,$$

that by (13) and $\vec{H}^2 = \nabla\varphi + \vec{u}(x),\ \vec{u}(x) = \sum_{j=1}^{h+h_1} K_j \cdot \vec{u}_j(x)$ is reduced to

$$\mu_2 \sum_{j=1}^{h+h_1} C_{mj} k_j^{'} = -\int_{S_1} \left(\sigma^{-1} rot\vec{H}^{(1)} - \sigma^{-1}\vec{j}\right)(\vec{n} \times \vec{u}_m) dS, \tag{16}$$

where h and h_1 are the first Betti numbers of Ω and Ω_1.

We show that $\vec{H}$ is reduced to the evaluation $\vec{H}^1(x)$, satisfying (16) and

$$\sum_{j=1}^{h+h_1} k_j C_{mi} = \int_{\Omega_2} \vec{H}^2(x) \cdot \vec{u}_m(x) dx,$$

where $C_{mj} = \int_{\Omega_2} u_m(x)\vec{u}_j(x)dx$ are elements of a positive definite matrix.

Problem (13), (14) differs from (4) only in the presence of heterogeneity in the boundary condition. In the same way as above, we can prove

$$\begin{aligned} ||\vec{H}^{(1)}||_{W_p^2(\Omega)} &\leq c\left[||rot\,\vec{j}(x)||_{L_p(\Omega_1)} + ||\vec{u}||_{W_p^{2-1/p}(S_1)} + ||\vec{H}^{(1)}||_{L_p(\Omega_1)}\right] \\ &\leq c\left(||rot\,\vec{j}||_{L_p(\Omega_1)} + ||\vec{H}^{(1)}||_{L_p(\Omega_1)}\right). \end{aligned}$$

Furthermore, we obtain (9) for $\varphi(x)$ the following inequality

$$||\nabla\varphi||_{W_p^2(\Omega_1)} \leq c||\vec{H}^{(1)}||_{W_p^2(\Omega_1)},$$

and hence

$$||\vec{H}^{(2)}||_{W_p^2(\Omega_1)} \leq c||\vec{H}^{(1)}||_{W_p^2(\Omega_1)}.$$

Next, we use the interpolation inequality [4]

$$||rot\vec{H}^{(1)}||_{L_p(S_1)} \leq \varepsilon||D^2\vec{H}^{(1)}||_{L_p(\Omega_1)} + c(\varepsilon)||\vec{H}^{(1)}||_{L_p}.$$

Combining these inequalities, we obtain the estimate

$$\sum_{i=1}^{2} ||\vec{H}^{(i)}||_{W_p^2(\Omega_i)} \leq c(\Omega_i)(||rot\,\vec{j}||_{L_p(\Omega_1)} + ||\vec{j}(x)||_{L_p(\Omega_1)}). \tag{17}$$

Using (17) from system (1), we get the estimate

$$\sum_{i=1}^{2} ||\vec{E}^{(i)}(x)||_{W_p(\Omega_i)} \leq c[||rot\vec{H}||_{W_p^1(\Omega_1)} + ||\vec{j}(x)||_{W_p^1(\Omega_1)}] \leq c\left(\sum_{i=1}^{2} ||\vec{H}^{(i)}||_{W_p^1(\Omega_1)}\right). \tag{18}$$

for the vector field $\vec{E}(x)$. Thus, we have proved the following theorem.

Theorem 2. *If in* (12) *vectors* $\vec{j}(x), rot\,\vec{j}(x) \in L_p(\Omega_1)$, *then the electric and magnetic fields* $\vec{E}(x) \in W_p^1(\Omega_i)$ *and* $\vec{H}(x) \in W_p^2(\Omega_i)$, $i = 1, 2$, *and the estimates* (17) *and* (18) *hold.*

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