The Problem of Tree-Body-Points with Masses, Changing Isotropically in Different Specific Rates

Mukhtar Minglibayev [1,2], Gulnara Mayemerova [3]

[1] al-Farabi Kazakh National University
al-Farabi av., 71, Almaty, 050040, R. Kazakhstan
minglibayev@mail.ru

[2] Fessenkov Astrophysical Institute
Observatory, 23, Almaty, 050020, R. Kazakhstan
minglibayev@mail.ru

[3] al-Farabi Kazakh National University
al-Farabi av., 71, Almaty, 050040, R. Kazakhstan
mayemerova@gmail.com

Abstract. *Non-autonomous canonical equations of secular perturbations are obtained in different systems of variables. Perturbing functions expressed in terms of osculating canonical elements with the use of system MATHEMATICA. Full first-order secular perturbations are obtained by the method of Picard.*

1 Introduction

Real space bodies are non-stationary ones. Their mass, size, shape and structure of the mass distribution within the body change in time [1]–[3]. These processes take place intensively in the binary and multiple systems [4]. In this connection we analytically investigate the three-body problem with masses varying isotropic at the same specific rate. Bodies are assumed to be material points. Secular perturbations in the corresponding three-body problems with variable masses are investigated on the basis of perturbation theory based on aperiodic motion on quasiconic section (see [5]).

2 Problem statement

Let's consider a system of three reciprocally gravitating bodies with masses, comparable to each other and changing isotropically in different:

$$m_0 = m_0(t), \quad m_1 = m_1(t), \quad m_2 = m_2(t), \quad \frac{\dot{m}_i}{m_i} \neq \frac{\dot{m}_j}{m_j}, \quad i \neq j. \qquad (2.1)$$

Using Jacobi coordinates [5]–[6], one can write equations of motion of the system in the form:

$$\mu_1 \ddot{\vec{r}}_1 = \mathrm{grad}_{\vec{r}_1} U, \quad \mu_2 \ddot{\vec{r}}_2 = \mathrm{grad}_{\vec{r}_2} U - \mu_2 (2\dot{\nu}_1 \dot{\vec{r}}_1 + \ddot{\nu}_1 \vec{r}_1), \qquad (2.2)$$

where

$$\mu_1 = \mu_1(t) = \frac{m_1 m_0}{m_0 + m_1} \neq const, \quad \mu_2 = \mu_2(t) = \frac{m_2(m_1 + m_0)}{m_0 + m_1 + m_2} \neq const, \tag{2.3}$$

$$U = f\left(\frac{m_0 m_1}{r_{01}} + \frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}}\right), \tag{2.4}$$

$$r_{01}^2 = x_1^2 + y_1^2 + z_1^2 = r_1^2, \quad r_2^2 = x_2^2 + y_2^2 + z_2^2, \tag{2.5}$$

$$r_{01}^2 = (x_2 + \nu_1 x_1)^2 + (y_2 + \nu_1 y_1)^2 + (z_2 + \nu_1 z_1)^2, \tag{2.6}$$

$$r_{02}^2 = (x_2 - \nu_0 x_1)^2 + (y_2 - \nu_0 y_1)^2 + (z_2 - \nu_0 z_1)^2, \tag{2.7}$$

$$\nu_1 = \nu_1(t) = \frac{m_1}{m_0 + m_1} \neq const, \quad \nu_0 = \nu_0(t) = \frac{m_0}{m_0 + m_1} \neq const, \tag{2.8}$$

and f is the gravitational constant.

Equations of motion (2.2) can be analyzed in the framework of the perturbation theory based on aperiodic motion on quasielliptic section $(0 \leq e < 1)$ [5].

3 The equations of motion in analogues of the Jacobi osculating elements

Equations (2.2) can be written as:

$$\mu_1 \ddot{\vec{r}}_1 = \mathrm{grad}_{\vec{r}_1}\left(f\frac{m_1 m_0}{r_1}\right) + b_1 \vec{r}_1 + \mathrm{grad}_{\vec{r}_1} R_1, \tag{3.1}$$

$$\mu_2 \ddot{\vec{r}}_2 = \mathrm{grad}_{\vec{r}_2}\left(f\frac{m_2(m_1 + m_0)}{r_2}\right) + b_2 \vec{r}_2 + \mathrm{grad}_{\vec{r}_2} R_2, \tag{3.2}$$

where

$$b_1 = b_1(t) = \mu_1 \frac{\ddot{\gamma}_1}{\gamma_1}, \quad \gamma_1 = \gamma_1(t) = \frac{m_0(t_0)}{m_0(t)}, \tag{3.3}$$

$$b_2 = b_2(t) = \mu_2 \frac{\ddot{\gamma}_2}{\gamma_2}, \quad \gamma_2 = \gamma_2(t) = \frac{m_0(t_0) + m_1(t_0)}{m_0(t) + m_1(t)}, \tag{3.4}$$

$$R_1 = -\frac{1}{2} b_1 r_1^2 + W, \quad R_2 = -\frac{1}{2} b_2 r_2^2 + W - V, \tag{3.5}$$

$$W = f\left(\frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}} - \frac{m_2(m_0 + m_1)}{r_2}\right), \tag{3.6}$$

$$V = \mu_2 \left[(2\dot{\nu}_1 \dot{x}_1 + \ddot{\nu}_1 x_1) x_2 + (2\dot{\nu}_1 \dot{y}_1 + \ddot{\nu}_1 y_1) y_2 + (2\dot{\nu}_1 \dot{z}_1 + \ddot{\nu}_1 z_1) z_2\right]. \tag{3.7}$$

Using the following notations:

$$\begin{aligned} K_1 &= \frac{1}{2}\mu_1 \left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2\right), \quad K_2 = \frac{1}{2}\mu_2 \left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right), \\ U_1 &= f\frac{m_1 m_0}{\sqrt{x_1^2 + y_1^2 + z_1^2}} + \frac{1}{2} b_1 \left(x_1^2 + y_1^2 + z_1^2\right) + R_1, \\ U_2 &= f\frac{m_2(m_0 + m_1)}{\sqrt{x_2^2 + y_2^2 + z_2^2}} + \frac{1}{2} b_2 \left(x_2^2 + y_2^2 + z_2^2\right) + R_2, \end{aligned} \tag{3.8}$$

and transformation to new variables:

$$x_i = \gamma_i \rho_i \cos\varphi_i \cos\theta_i, \quad y_i = \gamma_i \rho_i \cos\varphi_i \sin\theta_i, \quad z_i = \gamma_i \rho_i \sin\varphi_i, \tag{3.9}$$

then the equations of motion in the variables:

$$\begin{aligned}
\rho_i, \quad P_{\rho_i} &= \frac{\partial K_i}{\partial \dot{\rho}_i} = \mu_i \gamma_i^2 \dot{\rho}_i + \mu_i \gamma_i \dot{\gamma}_i \rho_i, \\
\varphi_i, \quad P_{\varphi_i} &= \frac{\partial K_i}{\partial \dot{\varphi}_i} = \mu_i \gamma_i^2 \rho_i^2 \dot{\varphi}_i, \\
\theta_i, \quad P_{\theta_i} &= \frac{\partial K_i}{\partial \dot{\theta}_i} = \mu_i \gamma_i^2 \rho_i^2 \cos^2\varphi_i \dot{\theta}_i,
\end{aligned} \tag{3.10}$$

can be written in the form:

$$\left.\begin{aligned}
\dot{\rho}_i &= \frac{\partial H_i}{\partial P_{\rho_i}}, \quad \dot{P}_{\rho_i} = -\frac{\partial H_i}{\partial \rho_i} + \frac{\dot{\mu}_i}{\mu_i} P_{\rho_i}, \\
\dot{\varphi}_i &= \frac{\partial H_i}{\partial P_{\varphi_i}}, \quad \dot{P}_{\varphi_i} = -\frac{\partial H_i}{\partial \varphi_i} + \frac{\dot{\mu}_i}{\mu_i} P_{\varphi_i}, \\
\dot{\theta}_i &= \frac{\partial H_i}{\partial P_{\theta_i}}, \quad \dot{P}_{\theta_i} = -\frac{\partial H_i}{\partial \theta_i} + \frac{\dot{\mu}_i}{\mu_i} P_{\theta_i},
\end{aligned}\right\} \tag{3.11}$$

respectively

$$\begin{aligned}
U_1^* &= f\frac{m_1 m_0}{\gamma_1 \rho_1} + \frac{1}{2}\left(b_1 \gamma_1^2 + \mu_1 \dot{\gamma}_1^2\right)\rho_1^2 + R_1, \\
U_2^* &= f\frac{m_2(m_0 + m_1)}{\gamma_2 \rho_2} + \frac{1}{2}\left(b_2 \gamma_2^2 + \mu_2 \dot{\gamma}_2^2\right)\rho_2^2 + R_2, \\
H_i &= \frac{1}{2\mu_i \gamma_i^2}\left[(P_{\rho_i} - \mu_i \gamma_i \dot{\gamma}_i \rho_i)^2 + \frac{P_{\varphi_i}^2}{\rho_i^2} + \frac{P_{\theta_i}^2}{\rho_i^2 \cos^2\varphi_i}\right] - U_i^*, \quad i = 1, 2.
\end{aligned} \tag{3.12}$$

Introducing the new impulses:

$$\begin{aligned}
P_{\rho_i} = \psi_i \tilde{P}_{\rho_i}, \quad P_{\varphi_i} &= \psi_i \tilde{P}_{\varphi_i}, \quad P_{\theta_i} = \psi_i \tilde{P}_{\theta_i}, \\
\psi_i = \psi_i(t) &= \frac{\mu_i}{\mu_{i0}} = \frac{\mu_i(t)}{\mu_i(t_0)}, \quad i = 1, 2,
\end{aligned} \tag{3.13}$$

we rewrite the equations of the motion in the form:

$$\left.\begin{aligned}
&\dot{\rho}_i = \frac{\partial \tilde{H}_i}{\partial \tilde{P}_{\rho_i}}, \quad \dot{\varphi}_i = \frac{\partial \tilde{H}_i}{\partial \tilde{P}_{\varphi_i}}, \quad \dot{\theta}_i = \frac{\partial \tilde{H}_i}{\partial \tilde{P}_{\theta_i}}, \\
&\dot{\tilde{P}}_{\rho_i} = -\frac{\partial \tilde{H}_i}{\partial \rho_i}, \quad \dot{\tilde{P}}_{\varphi_i} = -\frac{\partial \tilde{H}_i}{\partial \varphi_i}, \quad \dot{\tilde{P}}_{\theta_i} = -\frac{\partial \tilde{H}_i}{\partial \theta_i}, \quad i = 1, 2,
\end{aligned}\right\} \tag{3.14}$$

where

$$\tilde{H}_i = \tilde{H}_{i0} + \tilde{H}_{i1}, \quad i = 1, 2, \tag{3.15}$$

$$\begin{aligned}\tilde{H}_{10} &= \frac{\psi_1}{2\mu_1\gamma_1^2}\left[\left(\tilde{P}_{\rho_1} - \frac{\mu_1\gamma_1\dot{\gamma}_1}{\psi_1}\rho_1\right)^2 + \frac{\tilde{P}_{\psi_1}^2}{\rho_1^2} + \frac{\tilde{P}_{\theta_1}^2}{\rho_1^2\cos^2\varphi_1}\right] - \\ &- f\frac{m_1 m_0}{\psi_1\gamma_1\rho_1} - \frac{1}{\psi_1}\left(b_1\gamma_1^2 + \mu_1\dot{\gamma}_1^2\right)\rho_1^2, \\ \tilde{H}_{20} &= \frac{\psi_2}{2\mu_2\gamma_2^2}\left[\left(\tilde{P}_{\rho_2} - \frac{\mu_2\gamma_2\dot{\gamma}_2}{\psi_2}\rho_2\right)^2 + \frac{\tilde{P}_{\psi_2}^2}{\rho_2^2} + \frac{\tilde{P}_{\theta_2}^2}{\rho_2^2\cos^2\varphi_2}\right] - \\ &- f\frac{m_2(m_0+m_1)}{\psi_2\gamma_2\rho_2} - \frac{1}{\psi_2}\left(b_2\gamma_2^2 + \mu_2\dot{\gamma}_2^2\right)\rho_2^2.\end{aligned} \tag{3.16}$$

$$\tilde{H}_{11} = -\frac{1}{\psi_1}R_1, \quad \tilde{H}_{21} = -\frac{1}{\psi_2}R_2. \tag{3.17}$$

When $R_1 = R_2 = 0$ equations (3.14)-(3.16) can be integrated by the Hamilton-Jacobi method [5] [6] and the constants of integration:

$$\alpha_{1i}, \quad \alpha_{2i}, \quad \alpha_{3i}, \quad \beta_{1i}, \quad \beta_{2i}, \quad \beta_{3i}, \quad i = 1,2, \tag{3.18}$$

there are analogues of the Jacobi elements in the classical problem of two bodies (points) with constant masses. Each system of equations (3.14)-(3.16) defines an aperiodic motion on quasiconic section:

$$\rho_i = \rho_i(t) = \frac{p_i}{1 + e_i\cos\upsilon_i}, \quad \upsilon_i = u_i - \omega_i, \quad p_i = a_i(1 - e_i^2), \quad i = 1,2, \tag{3.19}$$

$$\dot{\rho}_i = \dot{\rho}_i(t) = \frac{1}{\mu_{i0}\gamma_2^2(t)} \cdot \frac{\tilde{\beta}_i}{\sqrt{p_i}} e_i \sin\upsilon_i, \quad \dot{u}_i = \frac{1}{\mu_{i0}\gamma_2^2(t)} \cdot \frac{\tilde{\beta}_i\sqrt{p_i}}{\rho_i^2}, \quad i = 1,2, \tag{3.20}$$

$$\begin{aligned}\tilde{\beta}_1^2 &= f \cdot \mu_1(t_0) m_1(t_0) m_0(t_0) = const, \\ \tilde{\beta}_2^2 &= f \cdot \mu_2(t_0) m_2(t_0)\left[m_0(t_0) + m_1(t_0)\right] = const,\end{aligned} \tag{3.21}$$

$$\begin{aligned}\operatorname{tg}\frac{\upsilon_i}{2} &= \frac{\sqrt{1+e_i}}{\sqrt{1-e_i}}\operatorname{tg}\frac{E_i}{2}, \quad e_i < 1, \quad i = 1,2, \\ E_i - e_i\sin E_i &= M_i, \quad i = 1,2,\end{aligned} \tag{3.22}$$

$$M_i = n_i\left[\phi_i(t) - \phi_i(\tau_i)\right], \quad n_i = \frac{\tilde{\beta}_i}{\mu_{i0}a_i^{3/2}}, \quad \mu_{i0} = \mu_i(t_0), \quad i = 1,2, \tag{3.23}$$

where $\phi_1(t)$, $\phi_2(t)$ the primitive functions, respectively, $\gamma_1^{-2}(t)$ and $\gamma_2^{-2}(t)$,

$$a_i, \quad e_i, \quad \omega_i, \quad \Omega_i, \quad i_i, \quad \phi_i(\tau_i), \quad i = 1,2, \tag{3.24}$$

are orbital elements, that is analogues of the Kepler elements, and

$$\begin{aligned}-2\alpha_{1i} &= \frac{\tilde{\beta}_i^2}{\mu_{i0}a_i}, \quad \alpha_{2i} = \tilde{\beta}_i\sqrt{p_i}, \quad \alpha_{3i} = \tilde{\beta}_i\sqrt{p_i}\cos i_i, \\ \beta_{1i} &= -\phi(\tau_i), \qquad \beta_{2i} = \omega_i, \qquad \beta_{3i} = \Omega_i, \qquad i = 1,2,\end{aligned} \tag{3.25}$$

$$\left.\begin{aligned} x_i &= \gamma_i\rho_i\left[\cos u_i\cdot\cos\Omega_i - \sin u_i\cdot\sin\Omega_i\cdot\cos i_i\right],\\ y_i &= \gamma_i\rho_i\left[\cos u_i\cdot\sin\Omega_i + \sin u_i\cdot\cos\Omega_i\cdot\cos i_i\right],\\ z_i &= \gamma_i\rho_i\left[\sin u_i\cdot\sin i_i\right],\quad r_i = \gamma_i\rho_i,\quad i=1,2,\end{aligned}\right\} \tag{3.26}$$

$$\left.\begin{aligned} \dot{x}_i &= \left(\frac{\dot\gamma_i}{\gamma_i}+\frac{\dot\rho_i}{\rho_i}\right)x_i + \gamma_i\rho_i\dot{u}_i\cdot\left[-\sin u_i\cdot\cos\Omega_i - \cos u_i\cdot\sin\Omega_i\cdot\cos i_i\right],\\ \dot{y}_i &= \left(\frac{\dot\gamma_i}{\gamma_i}+\frac{\dot\rho_i}{\rho_i}\right)y_i + \gamma_i\rho_i\dot{u}_i\cdot\left[-\sin u_i\cdot\sin\Omega_i + \cos u_i\cdot\cos\Omega_i\cdot\cos i_i\right],\\ \dot{z}_i &= \left(\frac{\dot\gamma_i}{\gamma_i}+\frac{\dot\rho_i}{\rho_i}\right)z_i + \gamma_i\rho_i\dot{u}_i\cdot\left[\cos u_i\cdot\sin i_i\right],\quad i=1,2.\end{aligned}\right\} \tag{3.27}$$

Equations (3.14)-(3.17) as equations of perturbed motion in the variables (3.18) have the form:

$$\begin{aligned} \dot\alpha_{k1} &= \frac{\partial\tilde{R}_1}{\partial\beta_{k1}},\quad \dot\beta_{k1} = -\frac{\partial\tilde{R}_1}{\partial\alpha_{k1}},\quad k=1,2,3,\\ \dot\alpha_{k2} &= \frac{\partial\tilde{R}_2}{\partial\beta_{k2}},\quad \dot\beta_{k2} = -\frac{\partial\tilde{R}_2}{\partial\alpha_{k2}},\quad k=1,2,3,\end{aligned} \tag{3.28}$$

where

$$\tilde{R}_i = \frac{1}{\psi_i}R_i(t,\alpha_{k1},\beta_{k1},\alpha_{k2},\beta_{k2}),\quad i=1,2. \tag{3.29}$$

4 The equations of perturbed motion in analogues of the Delaunay elements

Introducing analogues of the canonical Delaunay elements:

$$L_i,\quad G_i,\quad H_i,\quad l_i,\quad g_i,\quad h_i,\quad i=1,2, \tag{4.1}$$

by the formulas:

$$\begin{aligned} -2\alpha_{1i} &= \frac{\tilde\beta_i^4}{\mu_{i0}L_i^2},\qquad \alpha_{2i} = G_i,\qquad \alpha_{3i} = H_i,\\ \beta_{1i} &= \frac{l_i}{n_i} - \phi_i(t),\qquad \beta_{2i} = g_i,\qquad \beta_{3i} = h_i,\qquad i=1,2,\end{aligned} \tag{4.2}$$

equations of perturbed motion in the analogues of the Delaunay elements (4.1) have the form:

$$\left.\begin{aligned} \dot{L}_i &= \frac{\partial R_i^*}{\partial l_i},\qquad \dot{G}_i = \frac{\partial R_i^*}{\partial g_i},\qquad \dot{H}_i = \frac{\partial R_i^*}{\partial h_i},\\ \dot{l}_i &= -\frac{\partial R_i^*}{\partial L_i},\qquad \dot{g}_i = -\frac{\partial R_i^*}{\partial G_i},\qquad \dot{h}_i = -\frac{\partial R_i^*}{\partial H_i},\quad i=1,2,\end{aligned}\right\} \tag{4.3}$$

where

$$R_i^* = \frac{1}{\gamma_i^2(t)}\cdot\frac{\tilde\beta_i^4}{2\mu_{i0}L_i^2} + \tilde{R}_i,\quad i=1,2. \tag{4.4}$$

Explicitly, using equations (4.4), (3.27), (3.5)-(3.7), we obtain:

$$R_1^* = \frac{1}{\gamma_1^2(t)} \cdot \frac{\tilde{\beta}_1^4}{2\mu_{10}L_1^2} + \frac{1}{\psi_1}\left[-\frac{b_1}{2}r_1^2 + f\left(\frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}} - \frac{m_2(m_0+m_1)}{r_2}\right)\right], \quad (4.5)$$

$$\begin{aligned} R_2^* = &\frac{1}{\gamma_2^2(t)} \cdot \frac{\tilde{\beta}_2^4}{2\mu_{20}L_2^2} + \frac{1}{\psi_2}\left[-\frac{b_2}{2}r_2^2 + f\left(\frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}} - \frac{m_2(m_0+m_1)}{r_2}\right)\right] - \\ &- \frac{\mu_2}{\psi_2}\left[(2\dot{\nu}_1\dot{x}_1 + \ddot{\nu}_1 x_1)\,x_2 + (2\dot{\nu}_1\dot{y}_1 + \ddot{\nu}_1 y_1)\,y_2 + (2\dot{\nu}_1\dot{z}_1 + \ddot{\nu}_1 z_1)\,z_2\right]. \end{aligned} \quad (4.6)$$

In the non-resonant case secular perturbations are determined by the equations (4.3), if the disturbing functions (4.5)-(4.6) are averaged with respect to mean anomalies l_1, l_2 and are given by:

$$R_{1sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} R_1^* dl_1 dl_2, \quad R_{2sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} R_2^* dl_1 dl_2, \quad (4.7)$$

then we have:

$$\dot{L}_1 = 0, \quad \dot{L}_2 = 0, \quad (4.8)$$

and, therefore, secular perturbations are determined by the following system of eighth differential equations:

$$\left.\begin{aligned} \dot{G}_i &= \frac{\partial R_{isec}}{\partial g_i}, & \dot{g}_i &= -\frac{\partial R_{isec}}{\partial G_i}, & \\ \dot{H}_i &= \frac{\partial R_{isec}}{\partial h_i}, & \dot{h}_i &= -\frac{\partial R_{isec}}{\partial H_i}, & i = 1,2. \end{aligned}\right\} \quad (4.9)$$

5 The equations of perturbed motion in the analogues of the second system of the Poincare elements

Note that, as in the classical case, analogues of the second system of the Poincare elements [6], [9]:

$$\Lambda_i, \quad \lambda_i, \quad \xi_i, \quad \eta_i, \quad p_i, \quad q_i, \quad i = 1,2, \quad (5.1)$$

are preferable in our problem [5]. They are defined as follows:

$$\left.\begin{aligned} &\Lambda_i = L_i, & &\lambda_i = l_i + g_i + h_i, & \\ &\xi_i = \sqrt{2(L_i - G_i)}\cos(g_i + h_i), & &\eta_i = -\sqrt{2(L_i - G_i)}\sin(g_i + h_i), & \\ &p_i = \sqrt{2(G_i - H_i)}\cos h_i, & &q_i = -\sqrt{2(G_i - H_i)}\sin h_i, & i = 1,2. \end{aligned}\right\} \quad (5.2)$$

The equations of perturbed motion have the form:

$$\left.\begin{aligned} \dot{\Lambda}_i &= \frac{\partial R_i^*}{\partial \lambda_i}, & \dot{\xi}_i &= \frac{\partial R_i^*}{\partial \eta_i}, & \dot{p}_i &= \frac{\partial R_i^*}{\partial q_i}, & \\ \dot{\lambda}_i &= -\frac{\partial R_i^*}{\partial \Lambda_i}, & \dot{\eta}_i &= -\frac{\partial R_i^*}{\partial \xi_i}, & \dot{q}_i &= -\frac{\partial R_i^*}{\partial p_i}, & i = 1,2. \end{aligned}\right\} \quad (5.3)$$

Then the secular perturbations are determined by the equations:

$$\dot{\Lambda}_1 = 0, \quad \dot{\Lambda}_2 = 0, \tag{5.4}$$

$$\left.\begin{aligned} \dot{\xi}_i &= \frac{\partial R_{isec}}{\partial \eta_i}, & \dot{p}_i &= \frac{\partial R_{isec}}{\partial q_i}, \\ \dot{\eta}_i &= -\frac{\partial R_{isec}}{\partial \xi_i}, & \dot{q}_i &= -\frac{\partial R_{isec}}{\partial p_i}, \quad i = 1, 2, \end{aligned}\right\} \tag{5.5}$$

where R_{1sec}, R_{2sec} are conforming secular part of the following expressions:

$$R_{1sec} = \frac{1}{\gamma_1^2(t)} \cdot \frac{\tilde{\beta}_1^4}{2\mu_{10} L_1^2} + \frac{1}{\psi_1}\left[\frac{b_1}{2}\gamma_1^2\rho_1^2 + f\left(\frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}} - \frac{m_2(m_0+m_1)}{\gamma_2 \rho_2}\right)\right], \tag{5.6}$$

$$\begin{aligned} R_{2sec} = \frac{1}{\gamma_2^2(t)} \cdot \frac{\tilde{\beta}_2^4}{2\mu_{20} L_2^2} + \frac{1}{\psi_2}\left[\frac{b_2}{2}\gamma_2^2\rho_2^2 + f\left(\frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}} - \frac{m_2(m_0+m_1)}{\gamma_2 \rho_2}\right)\right] - \\ -\frac{\mu_2}{\psi_2}\left[(2\dot{\nu}_1 \dot{x}_1 + \ddot{\nu}_1 x_1) x_2 + (2\dot{\nu}_1 \dot{y}_1 + \ddot{\nu}_1 y_1) y_2 + (2\dot{\nu}_1 \dot{z}_1 + \ddot{\nu}_1 z_1) z_2\right]. \end{aligned} \tag{5.7}$$

6 Expansion of the perturbing function

To calculate the secular parts of the perturbing functions necessary to calculate the secular part of the following quantities:

$$F_{sec} = \left[\frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}} - \frac{m_2(m_0 + m_1)}{\gamma_2 \rho_2}\right]_{sec} = \left[\frac{m_1 m_2}{r_{12}}\right]_{sec}, \tag{6.1}$$

$$F_{\rho sec} = \left[\frac{b_1 \gamma_1^2}{2\psi_1}\rho_1^2 + \frac{b_2 \gamma_2^2}{2\psi_2}\rho_2^2\right]_{sec}, \tag{6.2}$$

$$V_{sec} = \frac{\mu_2}{\psi_2}\left[(2\dot{\nu}_1 \dot{x}_1 + \ddot{\nu}_1 x_1) x_2 + (2\dot{\nu}_1 \dot{y}_1 + \ddot{\nu}_1 y_1) y_2 + (2\dot{\nu}_1 \dot{z}_1 + \ddot{\nu}_1 z_1) z_2\right]_{sec}. \tag{6.3}$$

In the analogues of the second system of the Poincare elements are simplified the secular expressions for R_{1sec}, R_{2sec}:

$$R_{1sec} = \frac{1}{\gamma_1^2(t)} \cdot \frac{\tilde{\beta}_1^4}{2\mu_{10}\Lambda_1^2} + \frac{1}{\psi_1}\left[-\frac{b_1}{2}\frac{\gamma_1^2 \mu_0^2}{\Lambda_1^4}\left(1 + \frac{3}{2}\frac{\xi_1^2 + \eta_1^2}{\Lambda_1}\right) + F_{sec}\right], \tag{6.4}$$

$$R_{2sec} = \frac{1}{\gamma_2^2(t)} \cdot \frac{\tilde{\beta}_2^4}{2\mu_{20}\Lambda_2^2} + \frac{1}{\psi_2}\left[-\frac{b_2}{2}\frac{\gamma_2^2 \mu_0^2}{\Lambda_2^4}\left(1 + \frac{3}{2}\frac{\xi_2^2 + \eta_2^2}{\Lambda_2}\right) + F_{sec}\right] - V_{sec}, \tag{6.5}$$

$$F_{sec} = \left[\frac{m_1 m_2}{r_{12}}\right]_{sec}, \tag{6.6}$$

$$\begin{aligned} V_{sec} = \frac{9}{16}\frac{\Lambda_1 \Lambda_2}{\mu_{10}\mu_{20}}\frac{\mu_2 \gamma_2}{\psi_2}(2\dot{\gamma}_1 \dot{\nu}_1 + \gamma_1 \ddot{\nu}_1)\frac{(p_1 p_2 + q_1 q_2)}{\sqrt{p_1^2 + q_1^2}}\sqrt{p_2^2 + q_2^2}\sqrt{\eta_1^2 + \xi_1^2} \times \\ \times \sqrt{\eta_2^2 + \xi_2^2}\sqrt{4\Lambda_1 - \eta_1^2 - \xi_1^2}\sqrt{4\Lambda_2 - \eta_2^2 - \xi_2^2}. \end{aligned} \tag{6.7}$$

Expressions (6.6) obtained with the computer algebra system MATHEMATICA [7] [8] can be written as:

$$F_{sec} = \sum_{i=1}^{334} \Pi_i^*(t) P_i(\ni_k) + \sum_{j=1}^{3} \tilde{\Pi}_j(\Lambda_1, \Lambda_2, t), \qquad (6.8)$$

where $P_i(\ni_k) = P_i(\xi_1, \eta_1, p_1, q_1, \xi_2, \eta_2, p_2, q_2)$ is very cumbersome.

7 Conclusion

Solutions of the equations (5.5) by the Picard method in the first approximation have the structure:

$$\ni_k(t) = \ni_k(t_0) + \sum_j P_j(\ni_j(t_0)) \int_{t_0}^{t} \Pi_j^*(t)dt, \quad \ni_k(t_0) = \ni_k(t)|_{t=t_0} = const,$$

where $\ni_k$ are elements ξ_i, η_i, p_i, q_i, and $\ni_{k0}$= *const*.

Solutions of these equations make possible to analyze changes of eccentricities e_1, e_2, inclinations of orbits i_1, i_2, argument of pericentres ω_1, ω_2 and motions of ascending nodes longitude Ω_1, Ω_2, longitude of pericentres π_1, π_2 [5]–[6]. In particular, limiting second-order terms inclusively, we obtain:

$$e_i^2 = \frac{\xi_i^2 + \eta_i^2}{\Lambda_i}, \quad \sin^2 i_i = \frac{p_i^2 + q_i^2}{\Lambda_i},$$

$$\Omega_i = -\operatorname{arctg}\frac{q_i}{p_i}, \quad \pi_i = -\operatorname{arctg}\frac{\eta_i}{\xi_i}, \quad \omega_i = \pi_i - \Omega_i, \quad i = 1, 2.$$

References

[1] *Omarov T.B.* The dynamics of gravitating systems of Metagalaxy Almaty, Nauka (1975) (in Russian)

[2] *Omarov T.B.* (Editor). Non-Stationary Dynamical Problems in Astronomy New-York: Nova Science Publ. Inc. (2002)

[3] *Bekov A.A., Omarov T.B.* The theory of Orbits in Non-Stationary Stellar Systems. Astron. and Astrophys. Transactions, **22** (2003) 145.

[4] *Lukyanov L.G.* The dynamic evolution of the orbits of stars in close binary systems with conservative mass exchange. Astron. Journal **85**, No. 8 (2008) 755–768 (in Russian)

[5] *Minglibayev M.Zh.* Dynamics of nonstationary gravitating systems. Almaty: KazNU Publishing (2009) 209 p. (in Russian)

[6] *Charlier C.L.* Celestial mechanics. Moscow (1966) 628 p. (in Russian)

[7] *Grebenikov E.A.* Mathematical problems homographic dynamics. Moscow: MAKS Press (2010) 256 p. (in Russian)

[8] *Prokopenya A.N.* Solving physical problems with the use of MATHEMATICA. Brest: BSTU Publishing (2005) 260p. (in Russian)

[9] *Minglibayev M.Zh, Mayemerova G.M.* Secular perturbations in the three-body problem with variable masses. In: "Computer Algebra Systems in Teaching and Research. Differential Equations, Dynamical Systems and Celestial Mechanics", L. Gadomski, M. Jakubiak, A.N. Prokopenya (Eds.), Wydawnictwo Collegium Mazovia, Siedlce (2011) 198–204.