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Asymptotic behavior of solutions of singularly integro-differential equations

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Consider the following integro-differential equation with small parameter $\varepsilon > 0$:

$$L_\varepsilon y(t) \equiv \varepsilon^2 y''' + \varepsilon A_0(t) y'' + A_1(t) y' + A_2(t) y = F(t) + \int_0^1 [H_0(t,x) y(x) + H_1(t,x) y'(x)] dx \tag{1}$$

with boundary conditions

$$h_1 y(t) \equiv y(0,\varepsilon) = \alpha, \quad h_2 y(t) \equiv y(1,\varepsilon) = \beta, \quad h_3 y(t) \equiv y'(1,\varepsilon) = \gamma. \tag{2}$$

Assume that the following conditions hold:

I. $A_i(t), i=\overline{0,2}$, $F(t)$ are sufficiently smooth functions on the interval [0,1], i. e. differentiable as many times as needed for our consideration. Functions $H_i(t,x), i=0,1$ are sufficiently smooth in the domain $D=\{0\le t\le 1, 0\le x\le 1\}$ and $H_1(t,0)\ne 0$, $0\le t\le 1$.

II. $A_1(t)\ne 0$, $0\le t\le 1$.

III. The roots of this equation $\mu^2+A_0(t)\mu+A_1(t)=0$ satisfies the conditions

$$\operatorname{Re}\mu_1(t)<-\gamma_1<0, \operatorname{Re}\mu_2(t)>\gamma_2>0$$

IV. 1 is not an eigenvalue of the kernel

$$H(t,s)=\int_s^1\frac{1}{A_1(s)}\left(H_0(t,x)-H_1(t,x)\frac{A_2(x)}{A_1(x)}\right)e^{-\int_s^x\frac{A_2(p)}{A_1(p)}dp}dx+\frac{H_1(t,s)}{A_1(s)}$$

V.

$$\Delta_0=1+\int_0^1\frac{1}{A_1(s)}\left[\int_0^1\left(H_0(s,x)-H_1(s,x)\frac{A_2(x)}{A_1(x)}\right)e^{-\int_s^x\frac{A_2(p)}{A_1(p)}dp}dx+\left(H_1(s,0)+\int_0^1\overline{R}(s,p)H_1(p,0)dp\right)e^{-\int_0^s\frac{A_2(p)}{A_1(p)}dp}\right]ds\ne 0.$$

In this case the fundamental system of solutions of the homogeneous equation $L_\varepsilon y(t)=0$ has the following form:

$$y_1^{(q)}(t,\varepsilon)=\frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx\right)(\mu_1^q(t)y_{10}(t)+O(\varepsilon)),$$

$$y_2^{(q)}(t,\varepsilon)=\frac{1}{\varepsilon^q}\exp\left(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx\right)(\mu_2^q(t)y_{20}(t)+O(\varepsilon)),\tag{3}$$

$$y_3^{(q)}(t,\varepsilon)=y_{30}^{(q)}(t)+O(\varepsilon),\quad q=\overline{0,2}.$$

We introduce the functions

$$K_0(t,s,\varepsilon)=\frac{P_0(t,s,\varepsilon)}{W(s,\varepsilon)};\qquad K_1(t,s,\varepsilon)=\frac{P_1(t,s,\varepsilon)}{W(s,\varepsilon)},\tag{4}$$

where $P_0(t,s,\varepsilon)$, $P_1(t,s,\varepsilon)-$ are the determinants obtained from the Wronskian $W(s,\varepsilon)$ by replacing the third row with the $y_1(t,\varepsilon),0,y_3(t,\varepsilon)$ и $0,y_2(t,\varepsilon),0$ row. The functions $K_0(t,s,\varepsilon),K_1(t,s,\varepsilon)$ satisfies the homogeneous equation $L_\varepsilon y(t)=0$. Function $K(t,s,\varepsilon)=K_0(t,s,\varepsilon)+K_1(t,s,\varepsilon)$ is the Cauchy function. For the functions $K_0(t,s,\varepsilon)$, $K_1(t,s,\varepsilon)$ with regard to (3), (4) we get the following asymptotic representation:

$$K_0^{(q)}(t,s,\varepsilon)=\varepsilon^2\left(\frac{y_{30}^{(q)}(t)}{y_{30}(s)A_1(s)}-\frac{\mu_1^q(t)y_{10}(t)}{\varepsilon^q y_{10}(s)\mu_1(s)(\mu_2(s)-\mu_1(s))}e^{\frac{1}{\varepsilon}\int_s^t \mu_1(x)dx}+O(\varepsilon)\right),\ s\le t,$$

$$K_1^{(q)}(t,s,\varepsilon)=\varepsilon^2\left(\frac{\mu_2^q(t)y_{20}(t)}{\varepsilon^q y_{20}(s)\mu_2(s)(\mu_2(s)-\mu_1(s))}e^{-\frac{1}{\varepsilon}\int_t^s \mu_2(x)dx}+O(\varepsilon)\right), t\le s,\quad q=\overline{0,2}. \tag{5}$$

Let the functions $\Phi_i(t,\varepsilon), i=1,2,3$ are solutions of the following problem:

$$L_\varepsilon\Phi_i(t,\varepsilon)=0,\quad h_k\Phi_i(t,\varepsilon)=\delta_{ki},\qquad k=1,2,3$$

For the boundary functions $\Phi_i(t,\varepsilon), i=1,2,3$ with regard to (3), we get the following asymptotic estimates:

$$\Phi_1^{(q)}(t,\varepsilon)=\frac{1}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}(\mu_1^q(t)y_{10}(t)+O(\varepsilon)),\quad q=\overline{0,2},$$

$$\Phi_2^{(q)}(t,\varepsilon)=\frac{y_{30}^{(q)}(t)}{y_{30}(1)}-\frac{\mu_1^q(t)y_{10}(t)}{\varepsilon^q y_{30}(1)}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}-\frac{\mu_2^q(t)y_{20}(t)y_{30}'(1)}{\varepsilon^{q-1}\mu_2(1)y_{30}(1)}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx}+$$

$$+O(\varepsilon+\varepsilon^{1-q}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}+\varepsilon^{2-q}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx})$$

(6)

$$\Phi_3^{(q)}(t,\varepsilon)=-\varepsilon\frac{y_{30}^{(q)}(t)}{\mu_2(1)y_{30}(1)}+\frac{\mu_1^q(t)y_{10}(t)}{\varepsilon^{q-1}\mu_2(1)y_{30}(1)}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}+\frac{\mu_2^q(t)y_{20}(t)}{\varepsilon^{q-1}\mu_2(1)}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx}+$$

$$+O(\varepsilon^2+\varepsilon^{2-q}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}+\varepsilon^{2-q}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx})$$

Theorem 1. If conditions I-V are satisfied, then the unique solution of the boundary value problem (1), (2) on the interval $0\le t\le 1$ exists and can be presented in the following form

$$y(t,\varepsilon)=\sum_{i=1}^{3} C_i(\varepsilon)Q_i(t,\varepsilon)+P(t,\varepsilon), \tag{7}$$

where

$$Q_i(t,\varepsilon)=\Phi_i(t,\varepsilon)+\frac{1}{\varepsilon^2}\int_0^t K_0(t,s,\varepsilon)\overline{\varphi}_i(s,\varepsilon)ds-\frac{1}{\varepsilon^2}\int_t^1 K_1(t,s,\varepsilon)\overline{\varphi}_i(s,\varepsilon)ds,$$

$$P(t,\varepsilon)=\frac{1}{\varepsilon^2}\int_0^t K_0(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds-\frac{1}{\varepsilon^2}\int_t^1 K_1(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds, \tag{8}$$

$$\overline{\varphi}_i(s,\varepsilon)=\int_0^1\sum_{j=0}^{1}\overline{H}_j(s,x,\varepsilon)\Phi_i^{(j)}(x,\varepsilon)dx, \quad \overline{F}(s,\varepsilon)\equiv F(s)+\int_0^1 R(s,p,\varepsilon)F(p)dp,$$

$$\overline{H}_j(s,x,\varepsilon)\equiv H_j(s,x)+\int_0^1 R(s,p,\varepsilon)H_i(p,x)dp,$$

$$C_i(\varepsilon)=O(1),\ i=1,2,3 \text{ as } \varepsilon\to 0.$$

Theorem 2. Assume conditions I-V are fulfilled, then for the solution $y(t,\varepsilon)$ of the boundary value problem (1) and (2), we obtain the following asymptotic estimations

$$\left|y^{(i)}(t,\varepsilon)\right| \le C(\max_{0\le t\le 1}\left|F(t)-\alpha H_1(t,0)\right| + |\beta| + \varepsilon|\gamma|) + \frac{C}{\varepsilon^i} e^{-\gamma_1 \frac{t}{\varepsilon}} (|\alpha| \max_{0\le t\le 1}|H_1(t,0)| + |\beta| + \varepsilon|\gamma| + \varepsilon \max_{0\le t\le 1}|F(t)|) +$$

$$+\frac{C}{\varepsilon^{i-1}} e^{-\gamma_2 \frac{1-t}{\varepsilon}} (|\alpha| \max_{0\le t\le 1}|H_1(t,0)| + |\beta| + |\gamma| + \max_{0\le t\le 1}|F(t)|),\ i=\overline{0,2}, \tag{9}$$

where $C>0, \gamma_i>0, i=1,2-$ are constants independent of ε.

Proof of the Theorem follows from (5), (6), (7), (8) and estimations:

$$\left|Q_1^{(q)}(t,\varepsilon)\right| \le C + \frac{C}{\varepsilon^q} e^{-\gamma_1 \frac{t}{\varepsilon}} + \frac{C}{\varepsilon^{q-1}} e^{-\gamma_2 \frac{1-t}{\varepsilon}}, \qquad \left|Q_2^{(q)}(t,\varepsilon)\right| \le C\varepsilon + \frac{C}{\varepsilon^q} e^{-\gamma_1 \frac{t}{\varepsilon}} + \frac{C}{\varepsilon^{q-1}} e^{-\gamma_2 \frac{1-t}{\varepsilon}},$$

$$\left|Q_3^{(q)}(t,\varepsilon)\right| \le C\varepsilon + \frac{C}{\varepsilon^{q-1}} e^{-\gamma_1 \frac{t}{\varepsilon}} + \frac{C}{\varepsilon^{q-1}} e^{-\gamma_2 \frac{1-t}{\varepsilon}}, \qquad \left|P^{(q)}(t,\varepsilon)\right| \le \max_{0\le t\le 1}|F(t)| \left(C + \frac{C}{\varepsilon^{q-1}} e^{-\gamma_1 \frac{t}{\varepsilon}} + \frac{C}{\varepsilon^{q-1}} e^{-\gamma_2 \frac{1-t}{\varepsilon}} \right)$$

By Theorem 2, we have that

$$y(0,\varepsilon)=O(1),\quad y'(0,\varepsilon)=O\left(\frac{1}{\varepsilon}\right),\quad y''(0,\varepsilon)=O\left(\frac{1}{\varepsilon^2}\right),\quad \varepsilon\to 0 \quad \text{at the point} \quad t=0,$$

$$y(1,\varepsilon)=O(1),\quad y'(1,\varepsilon)=O(1),\quad y''(1,\varepsilon)=O\left(\frac{1}{\varepsilon}\right),\quad \varepsilon\to 0 \quad \text{at the point} \quad t=1.$$

One can see that the solution of the problem (1) and (2) has the phenomena of *boundary jumps* i.e. it has jumps at the left and at the right ends of the segment simultaneously. Namely, the solution of the problem (1) and (2) has the jump of the zero order at the point $t=0$, and the first order at the point $t=1$.

Obtained growth of derivatives allows us the boundary value problem reduce to the Cauchy problem with initial jump that turn serves as the basis for constructing asymptotic expansions of some singularly perturbed boundary value problems with initial jumps.

Consider the *modified* degenerate equation

$$L_0\bar{y}\equiv A_1(t)\bar{y}'+A_2(t)\bar{y}=F(t)+\int_0^1[H_0(t,x)\bar{y}(x)+H_1(t,x)\bar{y}'(x)]dx+\Delta(t) \tag{10}$$

with boundary conditions

$$\bar{y}(0)=\alpha+\Delta_0,\quad \bar{y}(1)=\beta, \tag{11}$$

where $\Delta(t)$, Δ_0 – are called the initial jumps of the integral term and the solution.

Theorem 3. If conditions I- V are satisfied, then the following asymptotic estimation is valid

$$\left|y^{(i)}(t,\varepsilon)-\bar{y}^{(i)}(t)\right| \le C(\max_{0\le t\le 1}\left|\Delta(t)-\Delta_0 H_1(t,0)\right|+\varepsilon|\gamma|+\varepsilon)+\frac{C}{\varepsilon^i}e^{-\gamma_1\frac{t}{\varepsilon}}(|\Delta_0|\max_{0\le t\le 1}|H_1(t,0)|+\varepsilon|\gamma|+$$
$$+\varepsilon\max_{0\le t\le 1}|\Delta(t)|+\varepsilon)+\frac{C}{\varepsilon^{i-1}}e^{-\gamma_2\frac{1-t}{\varepsilon}}(|\Delta_0|\max_{0\le t\le 1}|H_1(t,0)|+|\gamma|+\max_{0\le t\le 1}|\Delta(t)|+\varepsilon),\ i=\overline{0,2}, \tag{12}$$

where $C>0, \gamma_i>0$ – are constants independent of ε.

Proof. We substitute the variable $y(t,\varepsilon)=u(t,\varepsilon)+\bar{y}(t)$ into (1), (2) to obtain the following equation

$$L_\varepsilon u \equiv \varepsilon^2 u''' + \varepsilon A_0(t)u'' + A_1(t)u' + A_2(t)u = \int_0^1 [H_0(t,x)u(x)+H_1(t,x)u'(x)]dx - \Delta(t) - \varepsilon^2\overline{y'''} - \varepsilon A_0(t)\bar{y}'', \tag{13}$$

with boundary conditions

$$u(0,\varepsilon)=-\Delta_0, \quad u(1,\varepsilon)=0, \quad u'(1,\varepsilon)=\gamma-\bar{y}'(1). \tag{14}$$

The last problem (13), (14) is of the same type as the problem (1) and (2). By applying estimates (9) for the solution $u(t,\varepsilon)$ of the problem (13), (14), we obtain the asymptotic estimates (12).

From the estimation (12) it follows that the solution $y(t,\varepsilon)$ of the boundary value problem (1) and (2) converges to the solution $\bar{y}(t)$ of the modified degenerate problem (10) and (11) as $\varepsilon \to 0$ when the initial jump of the integral term has the form $\Delta(t) = H_1(t,0)\Delta_0$. Thus, we obtain the following relations

$$\lim_{\varepsilon \to 0} y(t,\varepsilon) = \bar{y}(t), \quad 0 < t \le 1,$$

$$\lim_{\varepsilon \to 0} y^{(i)}(t,\varepsilon) = \bar{y}^{(i)}(t), \; i = 1,2, \; 0 < t < 1,$$

where $\bar{y}(t)$ is solution of the problem (10), (11) with additional terms $\Delta(t) = H_1(t,0)\Delta_0$.

Thank you for attention!