# Secular Perturbations of Translational-Rotational Motion of a Non-Stationary Axisymmetric Body in the Central Gravitational Field

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**Abstract.** In the present paper, we study the long-term evolution of the translational-rotational motion of a non-stationary axisymmetric body with constant dynamic shape in the central gravitational field. Equations of motion of the body are obtained in terms of the canonical osculating Delaunay-Andoyer elements. Averaging these equations over the "fast" variables, we derive the differential equations determining the secular perturbations of the translational-rotational motion and solve these equations numerically for some given laws of the masses and principal moments of inertia variation. All the relevant symbolic and numerical computations and visualization of the results are performed with the aid of the computer algebra system Wolfram Mathematica.

**Keywords:** Non-stationary two-body problem · translational-rotational motion · secular perturbations · evolution equations · Wolfram Mathematica.

## 1 Introduction

The classical two-body problem describes the motion of two points of constant masses interacting according to Newton's law of gravitation, and its general solution is well known. Since this solution describes translational motion of two finite bodies with spherically symmetric density distribution, as well, such a model is usually used as the first approximation in describing the orbital motion of real celestial bodies, for example, a planet around the Sun or a satellite around a planet (see [1, 2]). If at least one of the bodies is not spherically symmetric, the problem becomes much more complicated because a mutual gravitational interaction depends on the geometrical shape and mass distribution of the bodies.

Besides, translational and rotational motions depend on each other, and the corresponding equations of motion should be integrated together (see [3–5]).

On the other hand, real celestial bodies are non-stationary, their characteristics, such as mass, size, and shape may vary with time (see [6–10]). Such changes occur especially intensively in double and multiple systems [11]. So it is quite natural to consider the problem of many bodies of variable mass and to investigate an influence of the mass variation on the dynamic evolution of the system (see, for example, [12–17]). It should be noted that dependence of masses on time significantly complicates the problem, and even in case of two interacting bodies of variable mass, a general solution to the equations of motion can be written only in some special cases (see [9, 10, 18]).

In the present paper, we consider a generalized case of the two-body problem when the first body of variable mass $m_1(t)$ is spherically symmetric while the second one has an axis of symmetry (axisymmetric body). The laws of the bodies' masses variation $m_1(t)$, $m_2(t)$ are assumed to be known arbitrary functions of time. Although the mass and size of the second body change with time the dynamic shape of the bodies is preserved and the variation of the body mass and sizes do not result in the appearance of reactive forces and their torques (see [10]). In spite of these simplifying assumptions, the problem is not integrable and the perturbation theory is applied for its investigation. Note that quite tedious symbolic computations should be done in order to derive the evolution equations which are investigated then numerically. All the relevant computations are performed with the computer algebra system Wolfram Mathematica (see [19]).

## 2 Equations of Motion

Let us consider a system of two finite-size bodies $P_1$ and $P_2$ attracting each other according to Newton's law of gravitation. The central body $P_1$ of variable mass $m_1\,(t)$ is a sphere with a spherically symmetric mass distribution. Due to spherical symmetry of $P_1$ its rotational and translational degrees of freedom are not coupled and only its mass change may affect on the motion of the system. The second body $P_2$ of mass $m_2\,(t) = m_2(t_0)\nu(t)$ has an axisymmetric dynamical structure and moves around the central body $P_1$. We assume that the sizes of each body change in a homothetic way and so its dynamic shape is preserved. Therefore, the principal central moments of inertia $A, C$ of the body $P_2$ depend on time and may be represented in the form

$$A(t) = A_0\nu(t)\chi^2(t), \qquad C(t) = C_0\nu(t)\chi^2(t), \tag{1}$$

where $A_0 = A(t_0)$, $C_0 = C(t_0)$, $t_0$ is an initial instant of time, and $\nu = \nu(t)$, $\chi = \chi(t)$ are given functions of time satisfying the conditions $\nu(t_0) = 1$, $\chi(t_0) = 1$ and $\nu(t) > 0, \chi(t) > 0$ for $t > t_0$. The functions $m_1(t)$ and $\nu(t)$ determine the mass variation of the bodies and may be chosen according to the Eddington–Jeans law, for example (see [6, 7]). Remind that the moment of inertia of a rigid body is proportional to its mass and a square of its geometric sizes (see, for example, [21]). As the body $P_2$ is assumed to retain its initial dynamic structure, the function $\chi(t)$ in (1) determining its characteristic size variation is the same for the two principal moments of inertia $A(t)$ and $C(t)$.

It is assumed also that the masses of the bodies vary isotropically at different rates,

$$\frac{\dot{m}_1}{m_1} \neq \frac{\dot{m}_2}{m_2}, \tag{2}$$

and additional reactive forces and the corresponding torques do not arise. Note that a dot over a symbol denotes the total derivative of the corresponding function with respect to time.

Under the assumptions above, the differential equations describing the translational motion of the body $P_2$ around $P_1$ may be written in the form [10]

$$\tilde{m}\ddot{x} = \frac{\partial U}{\partial x}, \qquad \tilde{m}\ddot{y} = \frac{\partial U}{\partial y}, \qquad \tilde{m}\ddot{z} = \frac{\partial U}{\partial z}, \tag{3}$$

where $x,\ y,\ z$ are the Cartesian coordinates of the center of mass of $P_2$ in the relative coordinate system $O_1xyz$ with the origin at the center of body $P_1$ the axes of which are parallel to the axes of the inertial frame, and $\tilde{m} = m_1m_2/(m_1+m_2)$ is the reduced mass. The force function of Newton's interaction of the two bodies is a series in powers of the inverse distance $R$ between the centers of mass of the bodies accurate to the third order (see [1, 3])

$$U = U_1 + U_2, \;\; U_1 = \frac{fm_1m_2}{R}, \;\; R = \left(x^2 + y^2 + z^2\right)^{1/2}, \tag{4}$$

$$U_2 = fm_1\frac{2A + C - 3J}{2R^3}, \tag{5}$$

where $f$ is a gravitational constant,

$$J = A\left(\alpha^2 + \beta^2\right) + C\gamma^2$$

is the moment of inertia of the body $P_2$ relative to the axis $O_1O_2$ connecting the centers of mass of the two bodies, and $\alpha,\ \beta,\ \gamma$ are the direction cosines of the radius-vector $\boldsymbol{R} = (x, y, z)$ relative to the axes of the body $P_2$ fixed frame $O_2xyz$ whose axes coincide with the principal axes.

Note that the force function (5) depends on orientation of the body $P_2$ and so equations (3) must be solved together with the equations describing its rotational motion. As a general solution of such a system cannot be found and we apply the perturbation theory to its investigation, it is expedient to rewrite equations (3) in the form [10, 20]

$$\ddot{\boldsymbol{R}} + f\frac{m_1 + m_2}{R^3}\boldsymbol{R} - b\boldsymbol{R} = grad_{\boldsymbol{R}}W, \tag{6}$$

where the perturbing function $W$ is given by

$$W = -\frac{1}{2}bR^2 + \frac{m_1 + m_2}{m_1m_2}\tilde{U}, \;\; b = b(t) = \frac{\ddot{\sigma}}{\sigma}, \;\; \sigma = \frac{m_1(t_0) + m_2(t_0)}{m_1(t) + m_2(t)}. \tag{7}$$

To describe the rotational motion of the body $P_2$ we introduce the second Cartesian coordinate system $O_2XYZ$ with the origin located at its center of mass whose axes are parallel to the axes of the inertial frame. Then orientation of the body $P_2$ relative to the $O_2XYZ$ frame can be specified in terms of the three Euler angles $\psi$, $\theta$, and $\varphi$. The projections of the angular velocity vector onto the axes $O_2x$, $O_2y$, and $O_2z$ are given by (see [1, 3])

$$p = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi,\;\; q = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi,\;\; r = \dot{\psi}\cos\theta + \dot{\varphi}. \tag{8}$$

The rotational motion of the body $P_2$ about its center of mass is determined by the equations [10, 20]

$$\begin{aligned}
\frac{d}{dt}\left(A(t)p\right) + \left(C(t) - A(t)\right)qr &= \frac{\sin\varphi}{\sin\theta}\left[\frac{\partial U}{\partial\psi} - \cos\theta\frac{\partial U}{\partial\varphi}\right] + \cos\varphi\frac{\partial U}{\partial\theta},\\
\frac{d}{dt}\left(A(t)q\right) + \left(A(t) - C(t)\right)rp &= \frac{\cos\varphi}{\sin\theta}\left[\frac{\partial U}{\partial\psi} - \cos\theta\frac{\partial U}{\partial\varphi}\right] - \sin\varphi\frac{\partial U}{\partial\theta},\\
\frac{d}{dt}\left(C(t)r\right) &= \frac{\partial U}{\partial\varphi}.
\end{aligned} \tag{9}$$

Since equations (6), (9) are not integrable, we can apply a perturbation theory to the investigation of the system dynamics (see, for instance, [22]). This assumes that equations (6), (9) are reduced to two perturbed problems of which each is integrable in the case when there are no perturbations.

## 3 Canonical Form of Equations of Motion

In case of $W = 0$ equations of motion (6) are integrable and their general solution may be written in terms of the Delaunay variables $l, L, g, G, h, H$ which are related to the analogs of the Keplerian orbital elements (see [10]). To apply the perturbation theory it is convenient to rewrite the equations of motion (6) in the canonical form

$$\dot{l} = -\frac{\partial W}{\partial L},\;\; \dot{L} = \frac{\partial W}{\partial l},\;\; \dot{g} = -\frac{\partial W}{\partial G},\;\; \dot{G} = \frac{\partial W}{\partial g},\;\; \dot{h} = -\frac{\partial W}{\partial H},\;\; \dot{H} = \frac{\partial W}{\partial h}, \tag{10}$$

where

$$W = \frac{1}{\sigma^2(t)}\frac{\mu_0^2}{2L^2} + W^*,\quad \mu_0 = f(m_1(t_0) + m_2(t_0)), \tag{11}$$

$$W^* = \left(\frac{m_1 + m_2}{m_1 m_2}U_2 - \frac{1}{2}bR^2\right). \tag{12}$$

In view of the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, formulas (5) and (12) yield

$$W^* = \frac{f(m_1 + m_2)(C - A)}{2m_2}\left(\left[\frac{1}{R^3}\right] - 3\left[\frac{\gamma^2}{R^3}\right]\right) - \frac{1}{2}b\left[R^2\right]. \tag{13}$$

Differential equations (9) describing the rotational motion of the axisymmetric body $P_2$ around its own center of mass also may be written in canonical form as

$$\dot{l}' = \frac{\partial F}{\partial L'},\ \dot{L}' = -\frac{\partial F}{\partial l'},\ \dot{g}' = \frac{\partial F}{\partial G'},\ \dot{G}' = -\frac{\partial F}{\partial g'},\ \dot{h}' = \frac{\partial F}{\partial H'},\ \dot{H}' = -\frac{\partial F}{\partial h'}, \quad (14)$$

where $(l', L')$, $(g', G')$, $(h', H')$ are the three pairs of canonically conjugate variables known as the Andoyer osculating elements (see [10, 20]).

The Hamiltonian $F$ of the rotational motion of the axisymmetric body in (14) is given by

$$F = F_{unper} + F_{per}, \quad (15)$$

where

$$F_{unper} = \frac{1}{2A}\left(G'^2 - L'^2\right) + \frac{L'^2}{2C} = \frac{1}{2}\frac{G'^2}{A} + \frac{1}{2}\left(\frac{1}{C} - \frac{1}{A}\right)L'^2, \quad (16)$$

$$F_{per} = U_2 - \frac{1}{2}bR^2. \quad (17)$$

Note that the distance $R$ between the centers of the bodies $P_1$ and $P_2$ is determined by the exact solution of the corresponding problem of two bodies of variable masses (see [10]). Therefore, equations (11)–(13) may be written in the form

$$W = \frac{1}{\sigma^2(t)} \cdot \frac{\mu_0^2}{2L^2} + \frac{f(m_1+m_2)(C-A)}{2m_2\sigma^3}\left(\left[\frac{1}{\rho^3}\right] - 3\left[\frac{\gamma^2}{\rho^3}\right]\right) - \frac{1}{2}b\sigma^2\left[\rho^2\right]. \quad (18)$$

In a similar way we rewrite (15)–(17) in the form

$$F = \frac{1}{2}\frac{G'^2}{A} + \frac{1}{2}\left(\frac{1}{C} - \frac{1}{A}\right)L'^2 + \frac{fm_1(C-A)}{2\sigma^3}\left[\frac{1}{\rho^3}\right] -$$

$$- \frac{3fm_1(C-A)}{2\sigma^3}\left[\frac{\gamma^2}{\rho^3}\right] - \frac{1}{2}b\sigma^2\left[\rho^2\right] \quad (19)$$

Here,

$$\rho = \frac{a(1-e^2)}{1+e\cos\nu}, \quad (20)$$

$$\gamma = c_{13}\frac{x}{R} + c_{23}\frac{y}{R} + c_{33}\frac{z}{R}, \quad (21)$$

where $a$ and $e$ are the semi-major axis and the eccentricity of the body $P_2$ orbit, $\nu$ is the true anomaly, $x, y, z$ are the Cartesian coordinates of the body $P_2$ center with respect to the relative coordinate system $O_1xyz$, and $c_{13}, c_{23}, c_{33}$ are the direction cosines of the axis $O_2z$ of the body $P_2$ fixed frame with respect to the relative coordinate system $O_1xyz$.

Using the exact solution of the two-body problem (see [10]), we obtain

$$\frac{x}{R} = \tau_{11}\sin\nu + \tau_{12}\cos\nu, \quad \frac{y}{R} = \tau_{21}\sin\nu + \tau_{22}\cos\nu,$$

$$\frac{z}{R} = \tau_{31}\sin\nu + \tau_{32}\cos\nu, \tag{22}$$

where

$$\tau_{11} = -\cos h\sin g - \frac{H}{G}\sin h\cos g, \quad \tau_{12} = \cos h\cos g - \frac{H}{G}\sin h\sin g,$$

$$\tau_{21} = -\sin h\sin g + \frac{H}{G}\cos h\cos g, \quad \tau_{22} = \sin h\cos g + \frac{H}{G}\cos h\sin g,$$

$$\tau_{31} = \cos g\sqrt{1-\frac{H^2}{G^2}}, \quad \tau_{32} = \sin g\sqrt{1-\frac{H^2}{G^2}}. \tag{23}$$

Coefficients $c_{13}, c_{23}, c_{33}$ in (21) are expressed via the Andoyer osculating elements as

$$c_{13} = \varepsilon_{11} + \varepsilon_{12}\sin g' + \varepsilon_{13}\cos g', \quad c_{23} = \varepsilon_{21} + \varepsilon_{22}\sin g' + \varepsilon_{23}\cos g',$$

$$c_{33} = \varepsilon_{31} + \varepsilon_{33}\cos g'. \tag{24}$$

Using (20)-(24), we obtain

$$\frac{\gamma^2}{\rho^3} = \frac{(1+e\cos\nu)^3}{a^3(1-e^2)^3}\cdot\left[(\varepsilon_{11}+\varepsilon_{12}\sin g'+\varepsilon_{13}\cos g')\,(\tau_{11}\sin\nu+\tau_{12}\cos\nu)+\right.$$

$$+(\varepsilon_{21}+\varepsilon_{22}\sin g'+\varepsilon_{23}\cos g')\,(\tau_{21}\sin\nu+\tau_{22}\cos\nu)+$$

$$\left.(\varepsilon_{31}+\varepsilon_{33}\cos g')\,(\tau_{31}\sin\nu+\tau_{32}\cos\nu)\right]^2, \tag{25}$$

where

$$\varepsilon_{11} = \frac{L'\sqrt{G'^2-H'^2}}{G'^2}\sin h', \quad \varepsilon_{12} = \frac{\sqrt{G'^2-L'^2}}{G'}\cos h',$$

$$\varepsilon_{13} = \frac{H'\sqrt{G'^2-L'^2}}{G'^2}\sin h', \quad \varepsilon_{21} = \frac{L'\sqrt{G'^2-H'^2}}{G'^2}\cos h',$$

$$\varepsilon_{22} = -\frac{\sqrt{G'^2-L'^2}}{G'}\sin h', \quad \varepsilon_{23} = \frac{H'\sqrt{G'^2-L'^2}}{G'^2}\cos h',$$

$$\varepsilon_{31} = \frac{L'H'}{G'^2}, \quad \varepsilon_{33} = -\frac{\sqrt{(G'^2-H'^2)\,(G'^2-L'^2)}}{G'^2}. \tag{26}$$

Due to the above formulas, the analytical expressions in the square brackets in (18) and (19) can be expressed in terms of Delaunay–Andoyer elements. Therefore, the right-hand sides in equations (10) and (14) can be expressed in terms of Delaunay–Andoyer elements, as well. These equations completely determine the translational-rotational motion of the nonstationary axisymmetric body $P_2$.

## 4 Evolution Equations

Assuming the resonances are absent in the system and averaging the right-hand sides of (10) and (14) over the fast variables $g$' and $l$ , we obtain equations for the secular perturbations of the translational-rotational motion of the nonstationary axisymmetric body in the problem under consideration. We denote the secular parts of the perturbing functions $W$ and $F$ by $W_{sec}$ and $F_{sec}$, respectively; according to the standard Gauss scheme, we have

$$W_{sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} W dl dg', \quad F_{sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F dl dg'. \tag{27}$$

Accordingly, we can write

$$W_{sec} = \frac{\mu_0^2}{2\sigma^2(t)}\left(\frac{1}{L^2}\right)_{sec} + \frac{f(m_1+m_2)(C-A)}{2m_2\sigma^3}\left[\frac{1}{\rho^3}\right]_{sec} - $$
$$-\frac{3f(m_1+m_2)(C-A)}{2m_2\sigma^3}\left[\frac{\gamma^2}{\rho^3}\right]_{sec} - \frac{1}{2}b\sigma^2\left[\rho^2\right]_{sec}, \tag{28}$$

$$F_{sec} = \frac{1}{2A}\left(G'^2\right)_{sec} + \frac{1}{2}\left(\frac{1}{C}-\frac{1}{A}\right)\left(L'^2\right)_{sec} + \frac{fm_1(C-A)}{2\sigma^3}\left[\frac{1}{\rho^3}\right]_{sec} - $$
$$-\frac{3fm_1(C-A)}{2\sigma^3}\left[\frac{\gamma^2}{\rho^3}\right]_{sec} - \frac{1}{2}b\sigma^2\left[\rho^2\right]_{sec}, \tag{29}$$

where

$$\left[\rho^2\right]_{sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\rho^2 dl dg' = a^2\left(1+\frac{3}{2}e^2\right), \tag{30}$$

When we compute secular perturbations of the quantities

$$\left[\frac{1}{\rho^3}\right]_{sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{dl dg'}{\rho^3}, \tag{31}$$

$$\left[\frac{\gamma^2}{\rho^3}\right]_{sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\gamma^2}{\rho^3}\right) dl dg', \tag{32}$$

it is convenient to use the well-known relation (see [10])

$$\frac{d\nu}{(1+e\cos\nu)^2} = \frac{dl}{(1-e^2)^{3/2}}. \tag{33}$$

Using (33), we compute the right-hand side of (31):

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{dl dg'}{\rho^3} = \frac{1}{4\pi^2 a^3(1-e^2)^{3/2}}\int_0^{2\pi}\int_0^{2\pi}(1+e\cos\nu)\, d\nu dg' = $$
$$= \frac{1}{a^3(1-e^2)^{3/2}}. \tag{34}$$

Using (25) and (33), we rewrite the right-hand side of (32) in the form

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\gamma^2}{\rho^3}\right)dldg' = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\gamma^2\frac{(1+e\cos\nu)}{a^3\left(1-e^2\right)^{3/2}}d\nu dg' =$$

$$= \frac{1}{4\pi^2 a^3\left(1-e^2\right)^{3/2}}\int_0^{2\pi}\int_0^{2\pi}\left[(\varepsilon_{11}+\varepsilon_{12}\sin g'+\varepsilon_{13}\cos g')\times\right.$$

$$(\tau_{11}\sin\nu+\tau_{12}\cos\nu)+(\varepsilon_{21}+\varepsilon_{22}\sin g'+\varepsilon_{23}\cos g')(\tau_{21}\sin\nu+\tau_{22}\cos\nu)+$$

$$\left.+(\varepsilon_{31}+\varepsilon_{33}\cos g')(\tau_{31}\sin\nu+\tau_{32}\cos\nu)\right]^2(1+e\cos\nu)\,d\nu dg'. \qquad (35)$$

Upon computing the integrals in (35), we finally obtain

$$\left[\frac{\gamma^2}{\rho^3}\right]_{sec} = \frac{I}{4a^3\left(1-e^2\right)^{3/2}}, \qquad (36)$$

where

$$I = (\tau_{11}^2+\tau_{12}^2)(2\varepsilon_{11}^2+\varepsilon_{12}^2+\varepsilon_{13}^2)+(\tau_{21}^2+\tau_{22}^2)(2\varepsilon_{21}^2+\varepsilon_{22}^2+\varepsilon_{23}^2)+$$

$$+(\tau_{31}^2+\tau_{32}^2)(2\varepsilon_{31}^2+\varepsilon_{33}^2)+(\tau_{11}\tau_{21}+\tau_{12}\tau_{22})\left(4\varepsilon_{11}\varepsilon_{21}+2\varepsilon_{12}\varepsilon_{22}+2\varepsilon_{13}\varepsilon_{23}\right)+$$

$$+(\tau_{11}\tau_{31}+\tau_{12}\tau_{32})(4\varepsilon_{11}\varepsilon_{31}+2\varepsilon_{13}\varepsilon_{33})+$$

$$+(\tau_{21}\tau_{31}+\tau_{22}\tau_{32})(4\varepsilon_{21}\varepsilon_{31}+2\varepsilon_{23}\varepsilon_{33}) = I\left(G,h,H,h',H',G',L'\right). \qquad (37)$$

The equations for secular perturbations now have the form

$$\dot{L} = \frac{\partial W_{sec}}{\partial l} = 0, \quad \dot{G} = \frac{\partial W_{sec}}{\partial g} = 0, \quad \dot{H} = \frac{\partial W_{sec}}{\partial h},$$

$$\dot{l} = -\frac{\partial W_{sec}}{\partial L}, \quad \dot{g} = -\frac{\partial W_{sec}}{\partial G}, \quad \dot{h} = -\frac{\partial W_{sec}}{\partial H}, \qquad (38)$$

$$\dot{L}' = -\frac{\partial F_{sec}}{\partial l'} = 0, \quad \dot{G}' = -\frac{\partial F_{sec}}{\partial g'} = 0, \quad \dot{H}' = -\frac{\partial F_{sec}}{\partial h'},$$

$$\dot{l}' = \frac{\partial F_{sec}}{\partial L'}, \quad \dot{g}' = \frac{\partial F_{sec}}{\partial G'}, \quad \dot{h}'_{sec} = \frac{\partial F_{sec}}{\partial H'}, \qquad (39)$$

where

$$W_{sec} = \frac{\mu_0^2}{2\sigma^2(t)}\left(\frac{1}{L^2}\right)+\frac{f(m_1+m_2)(C-A)}{2m_2\sigma^3a^3\left(1-e^2\right)^{3/2}}\left(1-\frac{3}{4}I\right)-$$

$$-\frac{1}{2}b\sigma^2a^2\left(1+\frac{3}{2}e^2\right), \qquad (40)$$

$$F_{sec} = \frac{1}{2A}G'^2+\frac{1}{2}\left(\frac{1}{C}-\frac{1}{A}\right)L'^2+\frac{fm_1\left(C-A\right)}{2\sigma^3a^3\left(1-e^2\right)^{3/2}}\left(1-\frac{3}{4}I\right)-$$

$$-\frac{1}{2}b\sigma^2a^2\left(1+\frac{3}{2}e^2\right), \qquad (41)$$

and $a = L^2/\mu_0$, $1-e^2 = G^2/\mu_0 a$, $I = I\,(G, h, H, h', H', G', L')$. Thus, computing secular perturbations reduces to the fourth-order system

$$\dot{H} = \frac{\partial W_{sec}}{\partial h}, \quad \dot{h} = -\frac{\partial W_{sec}}{\partial H}, \tag{42}$$

$$\dot{H}' = -\frac{\partial F_{sec}}{\partial h'}, \quad \dot{h}' = \frac{\partial F_{sec}}{\partial H'}. \tag{43}$$

Upon solving system (42)–(43) we integrate the remaining equations

$$\dot{L} = 0, \quad \dot{G} = 0, \quad \dot{L}' = 0, \quad \dot{G}' = 0, \tag{44}$$

$$\dot{l} = -\frac{\partial W_{sec}}{\partial L'}, \quad \dot{g} = -\frac{\partial W_{sec}}{\partial G}, \quad \dot{l}' = \frac{\partial F_{sec}}{\partial L'}, \quad \dot{g}' = \frac{\partial F_{sec}}{\partial G'}. \tag{45}$$

In view of (40) and (41), system (42), (43) takes the form [13]

$$\dot{H} = \left(\frac{E(t)}{\tilde{m}(t)}\right)\frac{\partial I}{\partial h}, \quad \dot{h} = -\left(\frac{E(t)}{\tilde{m}(t)}\right)\frac{\partial I}{\partial H},$$

$$\dot{H}' = -E(t)\frac{\partial I}{\partial h'}, \quad \dot{h}' = E(t)\frac{\partial I}{\partial H'}, \tag{46}$$

$$E(t) = -\frac{3fm_1(C-A)}{8\sigma^3 a^3(1-e)^{3/2}}, \quad \tilde{m}(t) = \frac{m_1 m_2}{m_1+m_2},$$

$$I = \frac{3}{2} - \frac{L'^2}{2G'^2} - \frac{1}{2}\left(1-\frac{3L'^2}{G'^2}\right)\left(\frac{H^2}{G^2}+\frac{H'^2}{G'^2}-\frac{3H^2H'^2}{G^2G'^2}\right)+$$

$$+\frac{1}{2}\left(1-\frac{H^2}{G^2}\right)\left(1-\frac{L'^2}{G'^2}\right)\cos\left(2(h-h')\right)-$$

$$-\frac{1}{2}\left(1-\frac{H^2}{G^2}\right)\left(\frac{2L'^2}{G'^2}+\frac{H'^2}{G'^2}-\frac{3L'^2H'^2}{G'^4}\right)\cos\left(2(h+h')\right)-$$

$$-\frac{2HH'}{GG'}\left(1-\frac{H^2}{G^2}\right)^{1/2}\left(1-\frac{H'^2}{G'^2}\right)^{1/2}\left(1-\frac{3L'^2}{G'^2}\right)\cos(h+h'). \tag{47}$$

Note that, due to equalities (44), we obtain

$$L = L_0 = const,\ (a = const)\,,\ G = G_0 = const,\ (e = const)\,,$$

$$L' = L'_0 = const, \quad G' = G'_0 = const. \tag{48}$$

## 5 Numerical Solutions

One can readily check that a general solution of the system (46) cannot be found in symbolic form. But we can choose some realistic values for the system parameters and solve equations (46) numerically. To simplify the calculations we use the dimensionless variables. For example, the semi-major axis $a$ is used as a unit of distance and dimensionless time $t^*$ is defined by

$$t^* = \omega_0 t, \quad \omega_0 = \frac{\sqrt{\mu_0}}{a^{3/2}}.$$

The Delaunay-Andoyer elements are replaced by the corresponding dimensionless quantities

$$L^* = \frac{L}{\sqrt{\mu_0 a}}, \quad G^* = \frac{G}{\sqrt{\mu_0 a}}, \quad H^* = \frac{H}{\sqrt{\mu_0 a}},$$

$$L'^* = \frac{L'}{m_{10} a^2 \omega_0}, \quad G'^* = \frac{G'}{m_{10} a^2 \omega_0}, \quad H'^* = \frac{H'}{m_{10} a^2 \omega_0}.$$

Dimensionless masses and moments of inertia are given by

$$m_1(t) = m_{10} m_1^*(t^*), \quad m_2(t) = m_{10} m_2^*(t^*), \quad m_1^*(t_0^*) = 1, \quad m_2^*(t_0^*) = \frac{m_{20}}{m_{10}},$$

$$C(t) = m_{10} a^2 m_2^*(t^*) \chi^2(t^*) C^*, \quad A(t) = m_{10} a^2 m_2^*(t^*) \chi^2(t^*) A^*,$$

where $m_{10} = m_1(t_0)$, $m_{20} = m_2(t_0)$.

Then we can rewrite the equations (45), (46) in dimensionless variables

$$\dot{H}^* = -\frac{3\left(C^* - A^*\right)\chi^2(t^*)}{8\sigma^4\left(1 - e^2\right)^{3/2}}\left(\frac{\partial I^*}{\partial h^*}\right), \quad \dot{h}^* = \frac{3\left(C^* - A^*\right)\chi^2(t^*)}{8\sigma^4\left(1 - e^2\right)^{3/2}}\left(\frac{\partial I^*}{\partial H^*}\right),$$

$$\dot{H}'^* = \frac{3 m_{10}\left(C^* - A^*\right) m_1^*(t^*) m_2^*(t^*)\chi^2(t^*)}{8(m_{10} + m_{20})\sigma^3\left(1 - e^2\right)^{3/2}}\left(\frac{\partial I^*}{\partial h'^*}\right),$$

$$\dot{h}'^* = -\frac{3 m_{10}\left(C^* - A^*\right) m_1^*(t^*) m_2^*(t^*)\chi^2(t^*)}{8(m_{10} + m_{20})\sigma^3\left(1 - e^2\right)^{3/2}}\left(\frac{\partial I^*}{\partial H'^*}\right),$$

$$\dot{l}^* = \frac{1}{\sigma^2} + \frac{1}{2} b\left(7 + 3e^2\right) + \frac{3\left(C^* - A^*\right)\chi^2(t^*)}{8\sigma^4\left(1 - e^2\right)^{3/2}}\left(4 - 3I^*\right),$$

$$\dot{g}^* = -\frac{3}{2} b\sqrt{1 - e^2} + \frac{3\left(C^* - A^*\right)\chi^2(t^*)}{8\sigma^4\left(1 - e^2\right)^2}\left(4 - 3I^* + G^*\frac{\partial I^*}{\partial G^*}\right),$$

$$\dot{l}'^* = -\frac{C^* - A^*}{C^* A^*}\frac{L'}{m_2^*(t^*)\chi^2(t^*)} +$$

$$+\frac{3 m_{10}\left(C^* - A^*\right) m_1^*(t^*) m_2^*(t^*)\chi^2(t^*)}{8(m_{10} + m_{20})\sigma^3\left(1 - e^2\right)^{3/2}}\left(\frac{\partial I^*}{\partial L'^*}\right), \tag{49}$$

$$\dot{g}'^{*} = \frac{G'}{A^{*} m_2^{*}(t^{*}) \chi^2(t^{*})} - \frac{3 m_{10} \left(C^{*} - A^{*}\right) m_1^{*}(t^{*}) m_2^{*}(t^{*}) \chi^2(t^{*})}{8 (m_{10} + m_{20}) \sigma^3 \left(1 - e^2\right)^{3/2}} \left(\frac{\partial I^{*}}{\partial G'^{*}}\right),$$

where $I^{*} = I\left(G^{*}, h^{*}, H^{*}, h'^{*}, H'^{*}, G'^{*}, L'^{*}\right)$ is defined by (47) and

$$\sigma = \frac{m_1^{*}(t_0^{*}) + m_2^{*}(t_0^{*})}{m_1^{*}(t^{*}) + m_2^{*}(t^{*})}.$$

Let the laws of variation of the masses are described by the Eddington-Jeans law

$$m_i^{*}(t^{*}) = \left(m_i^{*}(t_0^{*})^{1-n_i} - \alpha_i \left(1 - n_i\right) t^{*}\right)^{\frac{1}{1-n_i}}, \quad i = 1, 2, \tag{50}$$

where

$$n_1 = 2, \quad n_2 = 3, \quad \alpha_1 = \frac{1}{100000}, \quad \alpha_2 = \frac{1}{200000}.$$

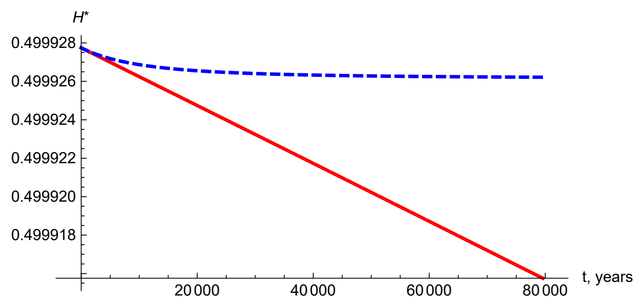


**Fig. 1.** Element $H^{*}(t)$: Red - constant masses, Blue - variable masses.

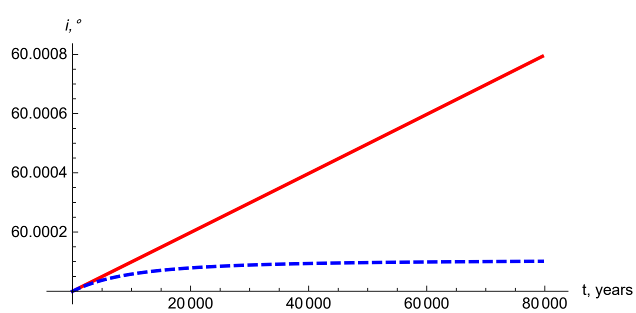


**Fig. 2.** Inclination $i(t) = \arccos(H/G)$: Red - constant masses, Blue - variable masses.

We use here the following values of the physical parameters

$$m_{10} = m_1\left(t_0\right) = 1 M_{\Theta}, \quad m_{20} = m_2\left(t_0\right) = 3 \cdot 10^{-6} M_{\Theta}, \quad e = 0.017,$$

$$A_0 = 0.3295\,(M_\oplus/M_\Theta)\,, \quad C^* = 0.3306\,(M_\oplus/M_\Theta)\,, \quad \chi^2\,(t) = 1 + 2\cdot 10^{-5}t,$$

where $M_\Theta$ is the mass of the Sun, $M_\oplus$ is the mass of the Earth. The dimensionless initial conditions are given by

$$L^* = 1, \quad G^* = 0.999, \quad H^* = 0.499, \quad l^* = \frac{\pi}{4}, \quad g^* = \frac{\pi}{18}, \quad h^* = \frac{\pi}{9},$$

$$L'^* = 3.786, \quad G'^* = 4.372, \quad H'^* = 4.108, \quad l'^* = \frac{\pi}{9}, \quad g'^* = \frac{\pi}{3}, \quad h'^* = \frac{\pi}{18}.$$

Using the system “Mathematica” [19], we obtain numerical solutions of differential equations (49) which are shown in Fig. 1 - 6. Comparison with the case of stationary bodies shows that varying of the masses and sizes of the bodies modify noticeably the evolution of the system.

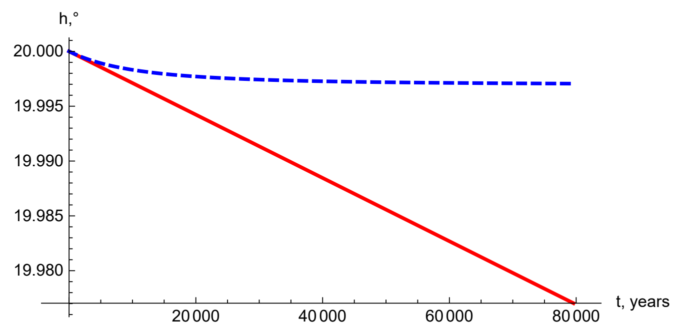


**Fig. 3.** Element $h(t)$: Red - constant masses, Blue - variable masses.

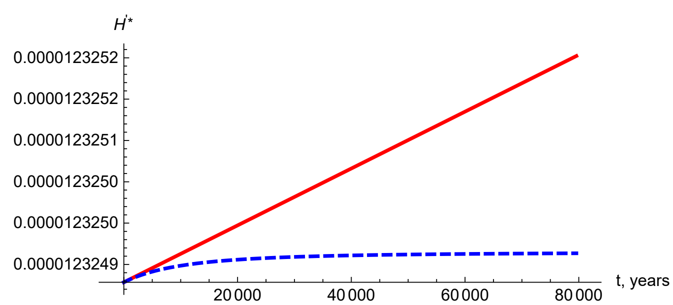


**Fig. 4.** Element $H'^*(t)$: Red - constant masses, Blue - variable masses.

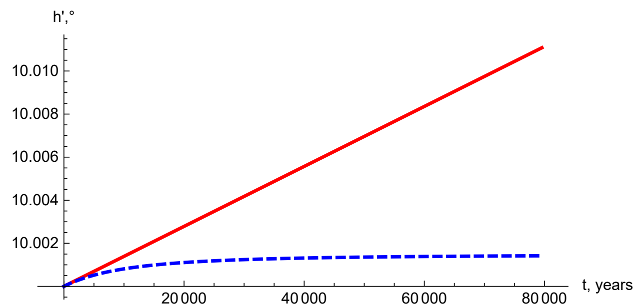


**Fig. 5.** Element $h'(t)$: Red - constant masses, Blue - variable masses.

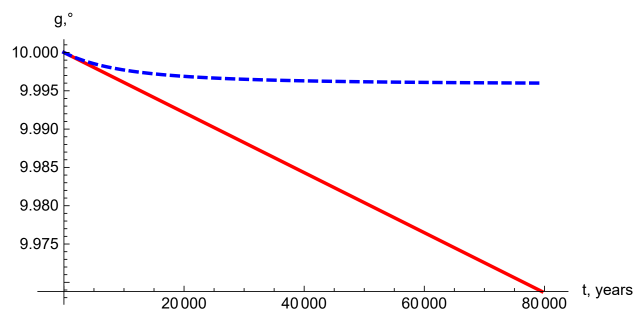


**Fig. 6.** Element $g(t)$: Red - constant masses, Blue - variable masses.

## 6 Conclusions

In the present paper we have considered the classical problem of two non-stationary bodies of variable masses and sizes. Due to the finite size of the bodies, their rotational motion as well as the interaction between translational and rotational degrees of freedom should be taken into account. We have obtained the differential equations describing the translational-rotational motion of the second body around the first one in terms of the Delaunay-Andoyer variables. Averaging these equations gives the evolutionary equations describing long-term behaviour of the system.

Note that in case of $m_1(t) = \text{const}$, $\nu(t) = 1$, $\chi(t) = 1$, $\gamma(t) = 1$, equations (45), (46) describe translational-rotational motion of a stationary triaxial rigid body in the central gravitational field. Non-stationarity of the bodies complicates the problem substantially and solutions to the evolution equations (45), (46) cannot be found in symbolic form. These equations were solved numerically for some realistic values of the system parameters.

All the relevant symbolic and numerical calculations and visualization of the results are performed with the aid of the computer algebra system Wolfram Mathematica.

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