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METHOD FOR SOLVING OF A BOUNDARY VALUE PROBLEM AT PHASE AND INTEGRAL CONSTRAINTS

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Abstract. Method for solving of a boundary value problem of ordinary differential equations with boundary conditions at phase and integral constraints is supposed. The base of the method is an immersion principle based on the general solution of the first order Fredholm integral equation which allows to reduce the initial boundary value problem to the special problem of the optimal equation.

Keywords: boundary value problem of ordinary differential equations, the first order Fredholm integral equation, the principle of immersion, optimal control problem, optimization problem.

Problem statement. We consider the following boundary value problem

$$\dot{x} = A(t)x + B(t)f(x,t) + \mu(t), \quad t \in I = [t_0, t_1] \tag{1}$$

with boundary conditions

$$(x(t_0)) = x_0, x(t_1) = x_1) \in S \subset R^{2n} \tag{2}$$

at phase constraints

$$x(t) \in G(t) : G(t) = \{x \in R^n | \gamma(t) \leq F(x,t) \leq \delta(t), \quad t \in I\}, \tag{3}$$

and integral constraints

$$g_j(x) \leq c_j, \quad j = \overline{1, m_1}; \tag{4}$$

$$g_j(x) = c_j, \quad j = \overline{m_1 + 1, m_2}; \tag{5}$$

$$g_j(x) = \int\limits_{t_0}^{t_1} f_{0j}(x(t),t)dt, \ l; j = \overline{1, m_2}; \tag{6}$$

Here $A(t)$, $B(t)$ are prescribed matrices with piecewise continuous elements of $n \times n$, $n \times m$ order, respectively, $\mu(t)$, $t \in I$ is given n - dimensional vector-function with piecewise continuous elements, m - dimensional vector-function $f(x,t)$ is defined and continuous in the variables $(x,t) = R^n \times I$ and satisfies the following conditions:

$$|f(x,t) - f(y,t)| \leq l|x - y|, \quad \forall (x,t), \quad (y,t) \in R^n \times I, \quad l = const > 0,$$

$$|f(x,t)| \leq c_0|x| + c_1(t), \quad c_0 = const \geq 0, \quad c_1(t) \in L_1(I, R^1),$$

S is the convex closed set. Function $F(x,t) = (F_1(x,t), \ldots, F_r(x,t))$, $t \in I$ is r - dimensional vector-function which is continuous in arguments, $\gamma(t) = (\gamma_1(t), \ldots, \gamma_r(t))$, $\delta(t) = (\delta_1(t), \ldots, \delta_r(t))$, $t \in I$ are prescribed continuous functions.

The values c_j, $j = \overline{1, m_2}$ are prescribed constants, $f_{0j}(x,t)$, $j = \overline{1, m_2}$ are given continuous functions by set of arguments satisfying to the conditions

$$|f_{0j}(x,t) - f_{0j}(y,t)| \le l_j|x-y|, \quad \forall (x,t), \quad (y,t) \in R^n \times I, \quad j = \overline{1, m_2};$$

$$|f_{0j}(x,t)| \le c_{0j}|x| + c_{1j}(t), \quad c_{0j} = const, \quad c_{1j} \in L_1(I, R^1), \quad j = \overline{1, m_2}.$$

Note, that: 1) if $A(t) \equiv 0, \quad m = n, \quad B(t) = I_n$, then the equation (1) can be written as

$$\dot{x} = f(x,t) + \mu(t) = \overline{f}(x,t), \quad t \in I. \tag{7}$$

Therefore, the results obtained below remain valid for the equation (7) at conditions (2)-(6);

2) if $f(x,t) = x + \mu_1(t)$ (or $f(x,t) = C(t)x + \mu_1(t)$), then the equation (1) can be written in form

$$\dot{x} = A(t)x + B(t)x + B(t)\mu_1(t) + \mu(t) = \overline{A}(t)x + \overline{\mu}(t), \quad t \in I, \tag{8}$$

where $\overline{A}(t) = A(t) + B(t)$, $\overline{\mu}(t) = B(t)\mu_1(t) + \mu(t)$. It follows that the equation (8) is a partial case of equation (1).

The following problems are stated:

Problem 1. To find necessary and sufficient conditions for the existence of solutions of boundary value problem (1)-(6).

Problem 2. To construct solution of boundary value problem (1)-(6).

As it follows of problem statement necessary to prove the existence of the pair $(x_0, x_1) \in S$ such that the solution of (1) proceeded from the point x_0 at the time t_0 passes through the point x_1 at the time t_1, along with the solution of the system (1) for each time the phase constraint is satisfied (3), and integrals (6) satisfy (4), (5). In particular, the set S is defined by the relation

$$S = \{(x_0, x_1) \in R^{2n} | H_j(x_0, x_1) \le 0, \quad j = \overline{1, p};$$

$$< a_j, x_0 > + < b_j, x_1 > - d_j = 0, \quad j = \overline{p+1, s}\},$$

where $H_j(x_0, x_1)$, $j = \overline{1, p}$ are convex functions in the variables (x_0, x_1), $x_0 = x(t_0)$, $x_1 = x(t_1)$, $a_j \in R^n$, $b_j \in R^n$, $d_j \in R^1$, $j = \overline{p+1, s}$ are given vectors and the numbers, $< \cdot, \cdot >$ is scalar product.

In many cases, in practice the process under study is described by the equation of the form (1) in the phase space of the system defined by the phase constraint of the form (3). Outside this domain the process is described by completely different equations or process under investigation does not exist. In particular, such phenomena take place in the research of dynamics of the nuclear and chemical reactors (outside the domain (3) reactors do not exist.) Integral constraints of the form (4) characterize the total load experienced by the elements and nodes in the system (for example, total overload cosmonauts), which should not exceed the specified values and equations of the form (5) correspond to the total limits for the system (for example, fuel consumption is equal to a predetermined value).

The essence of the method consists in the fact that at the first stage of research by transformation and introducing a fictitious control the initial problem is immersed in the control problem. Further, the existence of solutions of the original problem and the construction of its solution is carried out by solving the problem of optimal control of a special kind. With this approach, the necessary and sufficient conditions for the existence of solution of boundary value problem (1)-(6) can be obtained from the condition to achieve the lower bound of the functional on a given set, and the solution of the original boundary problem are the limit points of minimizing sequences.

We assume that $f_0(x,t) = (f_{01}(x,t), \dots, f_{0m_2}(x,t))$, where

$$f_0(x,t) = C(t)x + \overline{f_0}(x,t), \quad t \in I, \tag{9}$$

$C(t)$, $t \in I$ is known matrix of $m_2 \times n$ order with piecewise continuous elements, $\overline{f}_{\underline{0}}(x,t) = (\overline{f}_{01}(x,t), \dots, \overline{f}_{0m}(x,t))$. If j-th row of the matrix $C(t)$ is zero, then $f_{0j}(x,t) = \overline{f}_{0j}(x,t)$. Thus, without loss of generality, we can assume the function $f_0(x,t)$ is defined by (9). By introducing additional variables $d = (d_1, \dots, d_{m_1}) \in R^{m_1}$, $d \geq 0$, the relations (4), (6) can be represented as

$$g_j(x) = \int_{t_0}^{t_1} f_{0j}(x(t),t)dt = c_j - d_j, \quad j = \overline{1, m_1},$$

where

$$d \in \Gamma = \{d \in R^{m_1} | \;\; d \geq 0\}.$$

Let the vector $\overline{c} = (\overline{c}_1, \dots, \overline{c}_{m_2})$, where $\overline{c}_j = c_j - d_j$, $j = \overline{1, m_1}$, $\overline{c}_j = c_j$, $j = \overline{m_1 + 1, m_2}$. We introduce vector-function $\eta(t) = (\eta_1(t), \dots, \eta_{m_2}(t))$, $t \in I$, where

$$\eta(t) = \int_{t_0}^{t} f_0(x(\tau),\tau)d\tau, \quad t \in [t_0, t_1].$$

Then

$$\dot{\eta} = f_0(x(t),t) = C(t)x + \overline{f}_0(x,t), \quad t \in I$$
$$\eta(t_0) = 0, \quad \eta(t_1) = \overline{c}, \quad d \in \Gamma.$$

Now the initial boundary value problem (1)-(6) can be written as

$$\dot{\xi} = A_1(t)\xi + B_1(t)f(P\xi,t) + B_2\overline{f}_0(P\xi,t) + B_3\mu(t), \quad t \in I, \tag{10}$$

$$\xi(t_0) = \xi_0 = (x_0, O_{m_2}), \quad \xi(t_1) = \xi_1 = (x_1, \overline{c}), \tag{11}$$

$$(x_0, x_1) \in S, \quad d \in \Gamma, \quad P\xi(t) \in G(t), \quad t \in I, \tag{12}$$

where

$$\xi(t) = \begin{pmatrix} x(t) \\ \eta(t) \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} A(t) & O_{n,m_2} \\ C(t) & O_{m_2,m_2} \end{pmatrix}, \quad B_1(t) = \begin{pmatrix} B(t) \\ O_{m_2,m} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} I_n \\ O_{m_2,n} \end{pmatrix}, \quad B_3 = \begin{pmatrix} O_{n,m_2} \\ I_{m_2} \end{pmatrix}, \quad P = \begin{pmatrix} I_n, & O_{n,m_2} \end{pmatrix}, \quad P\xi = x,$$

$O_{j,k}$ is matrix of $j \times k$ order with zero elements, $O_q \in R^q$ is vector $q \times 1$ with zero elements, $\xi = (\xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_{n+m_2})$.

The basis of the proposed method of solving problems 1 and 2 are the following theorems about the properties of solution of the first order Fredholm integral equation:

$$Ku = \int_{t_0}^{t_1} K(t_0, t)u(t)dt = a, \tag{13}$$

where $K(t_0, t) = \|K_{ij}(t_0, t)\|$, $i = \overline{1, n}$, $j = \overline{1, m}$ is known matrix of $n \times m$ order with piecewise continuous elements in t at fixed t_0, $u(\cdot) \in L_2[I, R^m]$ is source function, $I = [t_0, t_1]$, $a \in R^n$ is given n-dimensional vector.

Theorem 1 it Integral equation (13) for any fixed $a \in R^n$ has a solution if and only if the matrix

$$C(t_0, t_1) = \int_{t_0}^{t_1} K(t_0, t)K^*(t_0, t)dt, \tag{14}$$

$n \times n$ order is positive definited, where (*) is a sign of transposition.

Theorem 2 *Let the matrix $C(t_0, t_1)$ is positive definited. Then the general solution of the integral equation (13) has the form*

$$u(t) = K^*(t_0, t)C^{-1}(t_0, t_1)a + v(t) - K^*(t_0, t)C^{-1}(t_0, t_1) \int_{t_0}^{t_1} K(t_0, t)v(t)dt, \quad t \in I, \tag{15}$$

where $v(\cdot) \in L_2(I, R^m)$ is an arbitrary function, $a \in R^n$ is an arbitrary vector.

Proofs of Theorems 1 and 2 are given in [2, 3]. Application of Theorems 1 and 2 to solve the controllability and optimal control problem are presented in [4-7].

Along with the differential equation (10) with boundary conditions (11) we consider the linear control system

$$\dot{y} = A_1(t)y + B_1(t)w_1(t) + B_2(t)w_2(t) + \mu_2(t), \quad t \in I, \tag{16}$$

$$y(t_0) = \xi_0 = (x_0, O_{m_2}), \quad y(t_1) = \xi_1 = (x_1, \bar{c}), \tag{17}$$

$$(x_0, x_1) \in S, \quad d \in \Gamma, \quad w_1(\cdot) \in L_2(I, R^m), \quad w_2(\cdot) \in L_2(I, R^{m_2}), \tag{18}$$

where $\mu_2(t) = B_3\mu(t)$, $t \in I$.

Let the matrix $\overline{B}(t) = (B_1(t), B_2(t))$ of $(n + m_2) \times (m_2 + m)$ order, and the vector-function $w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \in L_2(I, R^{m+m_2})$. It is easy to see that the control $w(\cdot) \in L_2(I, R^{m+m_2})$ which transfers the trajectory of system (16) from any initial state ξ_0 to any desired state ξ_1 is a solution of the integral equation

$$\int_{t_0}^{t_1} \Phi(t_0, t)\overline{B}(t)w(t)dt = a, \tag{19}$$

where $\Phi(t,\tau) = \theta(t)\theta^{-1}(\tau)$, $\theta(t)$ is the fundamental matrix of solutions of the linear homogeneous system $\dot{\omega} = A_1(t)\omega$, vector

$$a = a(\xi_0, \xi_1) = \Phi(t_0, t_1)[\xi_1 - \Phi(t_1, t_0)\xi_0] - \int\limits_{t_0}^{t_1} \Phi(t_0, t)\mu_2(t)dt.$$

As follows from (13), (19), the matrix $K(t_0, t) = (t_0, t)\overline{B}(t)$.

Further, we consider an immersion principle for boundary value problem (1)-(6) and prove several theorems about solution of the optimization problem.

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