УДК 517.938

CONSTRUCTION OF SURFACE CORRESPONDING TO DOMAIN WALL SOLUTION

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Abstract. Some generalizations of Landau-Lifschitz equation are integrable, admit physically interesting exact solutions and these integrable equations are solvable by the inverse scattering method [1]. Investigating of the integrable spin equations in (1+1)-, (2+1)-dimensions are topical both from the mathematical and physical points of view [2]-[5]. Integrable equations admit different kinds of physically interesting as domain wall solutions [2]. We consider an integrable spin equation [3]. There is a corresponding Lax representation. Moreover the equation allows an infinite number of integrals of motion. We construct a surface corresponding to domain wall solution of the equation. Further, we investigate some geometrical features of the surface.
Keywords: surface, domain wall solution, integrable equation, integrals of motion, nonlinear equation.

We use the geometric approach to one of the generalized Landau-Lifschitz equation [3]

$$\mathbf{S}_t = (\mathbf{S} \times \mathbf{S}_y + u\mathbf{S})_x, \tag{1a}$$

$$u_x = -(\mathbf{S}, (\mathbf{S}_x \times \mathbf{S}_y)), \tag{1b}$$

where $\mathbf{S}$ is spin vector, $S_1^2 + S_2^2 + S_3^2 = 1$, $\times$ is vector product, u is a scalar function. The equation allows an infinite number of motion integrals and has several exact solutions. One of them is the domain wall solution. We identify the spin vector $\mathbf{S}$ and vector $\mathbf{r}_x$ according to [3] the geometric approach

$$\mathbf{S} \equiv \mathbf{r}_x \tag{2}$$

Then (1a), (1b) take the form

$$\mathbf{r}_{xt} = (\mathbf{r}_x \times \mathbf{r}_{xy} + u\mathbf{r}_x)_x \tag{3a}$$

$$u_x = -(\mathbf{r}_x, (\mathbf{r}_{xx} \times \mathbf{r}_{xy})). \tag{3b}$$

If we integrate (3a) by x, then it takes the form

$$\mathbf{r}_t = \mathbf{r}_x \times \mathbf{r}_{xy} + u\mathbf{r}_x.$$

Taking into account Gauss-Weingarten equation and $E = \mathbf{r}_x^2 = 1$ the system is defined as

$$\mathbf{r}_t = (u + \frac{MF}{\sqrt{\Lambda}})\mathbf{r}_x - \frac{M}{\sqrt{\Lambda}}\mathbf{r}_y + \Gamma_{12}^2\sqrt{\Lambda}\mathbf{n},$$

$$u_x = \sqrt{\Lambda}(L\Gamma^2_{12} - M\Gamma^2_{11}),$$

where

$$\Gamma^2_{11} = \frac{2EF_x - EE_t - FE_x}{2\Lambda},$$

$$\Gamma^2_{12} = \frac{EG_x - FE_t}{2\Lambda},$$

$\Lambda = EG - F^2$. Equation (1a), (1b) is integrable equation and has soliton solutions.

Here we present the domain wall solution of the equation (1a), (1b) [3],

$$S^+(x,y,t) = \frac{expiby}{cosh[a(x-bt-x_0)]}, \tag{4a}$$

$$S_3(x,y,t) = -tanh[a(x-bt-x_0)], \tag{4b}$$

where a, b are real constants.

T h e o r e m. *Domain wall solution (4a)-(4b) of the spin system (1a), (1b) can be represented as components of the vector* $\mathbf{r}_x$, *where*

$$r_1 = \frac{1}{a}cos(by)arctg(sh[a(x-bt-x_0)]) + c_1, \tag{5a}$$

$$r_2 = \frac{1}{a}sin(by)arctg(sh[a(x-bt-x_0)]) + c_2, \tag{5b}$$

$$r_3 = -\frac{1}{a}ln|ch[a(x-bt-x_0)]| + c_3, \tag{5c}$$

where c_1, c_2, c_3 *are constants. Solution of the form (5a)-(5c) corresponds to the surface with the following coefficients of the first and second fundamental forms*

$$E = \frac{2 + sh^2[a(x-bt-x_0)]}{(1+sh^2[a(x-bt-x_0)])^2}, \quad F = 0, \tag{6a}$$

$$G = \frac{b^2}{a^2}arctg^2(sh[a(x-bt-x_0)]), \quad L = 0, \tag{6b}$$

$$M = 0, \quad N = -\frac{b^3 arctg^2(sh[a(x-bt-x_0)])}{\sqrt{\Lambda}a^2 ch[a(x-bt-x_0)]}. \tag{6c}$$

P r o o f. From (2) we have

$$(S_1, S_2, S_3) = (r_{1x}, r_{2x}, r_{3x}), \tag{7}$$

i.e.

$$r_{1x} = S_1, \quad r_{2x} = S_2, \quad r_{3x} = S_3. \tag{8}$$

Hence

$$r_1 = \int S_1 dx + c_1, \tag{9a}$$

$$r_2 = \int S_2 dx + c_2, \tag{9b}$$

$$r_3 = \int S_3 dx + c_3, \tag{9c}$$

where c_1, c_2, c_3 are constants of integration. Note

$$S^+ = S_1 + iS_2 = r_x^+,$$

then

$$r^+ = r_1 + ir_2 = \int S^+ dx + c^+, \tag{10}$$

where c^+ is constant of integration. Substituting (4b) to the equation (9c) we have

$$r_3 = \int S_3 dx + c_3 = -\int [tanh[a(x - bt - x_0)]dx + c_3 =$$

$$= -\frac{1}{a} ln|ch[a(x - bt - x_0)]| + c_3, \tag{11}$$

where c_3 is constant. Thus

$$r_3 = -\frac{1}{a} ln|ch[a(x - bt - x_0)]| + c_3, \tag{12}$$

Substituting (4a) to (10) we have

$$r^+ = r_1 + ir_2 = \int S^+ dx + c^+ =$$

$$= \int \frac{expiby}{cosh[a(x - bt - x_0)]} dx + c^+,$$

then

$$r^+ = \frac{1}{a} cos(by) arctg(sh[a(x - bt - x_0)]) + c_1 +$$

$$+ i(\frac{1}{a} sin(by) arctg(sh[a(x - bt - x_0)]) + c_2),$$

i.e. we have obtained

$$r_1 = \frac{1}{a} cos(by) arctg(sh[a(x - bt - x_0)]) + c_1,$$

$$r_2 = \frac{1}{a} sin(by) arctg(sh[a(x - bt - x_0)]) + c_2. \tag{13}$$

Thus, (12), (13) give us (5a)-(5c).

We proceed to prove the second part of the theorem. From (12) and (13) we have

$$r_{1x} = \frac{cos(by)}{1 + sh^2[a(x - bt - x_0)]}, \quad r_{2x} = \frac{sin(by)}{1 + sh^2[a(x - bt - x_0)]}, \tag{14a}$$

$$r_{3x} = -\frac{1}{ch^2[a(x - bt - x_0)]}, \quad r_{1y} = -\frac{b}{a} sin(by) arctg(sh[a(x - bt - x_0)]), \tag{14b}$$

$$r_{2y} = \frac{b}{a} cos(by) arctg(sh[a(x - bt - x_0)]), \quad r_{3y} = 0. \tag{14c}$$

Then we can calculate

$$E = \mathbf{r}_x^2 = r_{1x}^2 + r_{2x}^2 + r_{3x}^2 =$$

$$= \frac{cos^2(by)}{(1 + sh^2[a(x - bt - x_0)])^2} +$$

$$+ \frac{sin^2(by)}{(1 + sh^2[a(x - bt - x_0)])^2} + \frac{1}{ch^2[a(x - bt - x_0)]} = \frac{2 + sh^2[a(x - bt - x_0)]}{(1 + sh^2[a(x - bt - x_0)])^2}. \tag{15}$$

Similarly, using (13) and (14c) we obtain

$$G = \mathbf{r}_y^2 = r_{1y}^2 + r_{2y}^2 + r_{3y}^2 = \frac{b^2}{a^2} arctg^2(sh[a(x - bt - x_0)]). \tag{16}$$

$$F = (\mathbf{r}_x, \mathbf{r}_y) = r_{1x} r_{1y} + r_{2x} r_{2y} + r_{3x} r_{3y} = 0. \tag{17}$$

Formulas (15) - (17) give us the first three equations (6a) - (6c). Using (15) - (17) we compute

$$\Lambda = EG - F^2 = \frac{b^2(2 + sh^2[a(x - bt - x_0)])}{a^2(1 + sh^2[a(x - bt - x_0)])^2} arctg^2(sh[a(x - bt - x_0)]).$$

We calculate the components of the vector $\mathbf{n}$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\sqrt{\Lambda}} = \frac{1}{\sqrt{\Lambda}}(n_1, n_2, n_3),$$

$$n_1 = \frac{1}{\sqrt{\Lambda}}(r_{2x} r_{3y} - r_{3x} r_{2y}) = \frac{b cos(by) arctg(sh[a(x - bt - x_0)])}{\sqrt{\Lambda} a ch[a(x - bt - x_0)]}. \tag{18}$$

Similarly, for the components

$$n_2 = \frac{1}{\sqrt{\Lambda}}(r_{3x} r_{1y} - r_{1x} r_{3y}) = \frac{b sin(by) arctg(sh[a(x - bt - x_0)])}{\sqrt{\Lambda} a ch[a(x - bt - x_0)]}, \tag{19a}$$

$$n_3 = \frac{1}{\sqrt{\Lambda}}(r_{1x} r_{2y} - r_{2x} r_{1y}) = \frac{b arctg(sh[a(x - bt - x_0)])}{\sqrt{\Lambda} a(1 + sh^2[a(x - bt - x_0)])}. \tag{19b}$$

Now, from (14a), (14b) we have

$$r_{1xx} = -\frac{2a cos(by) sh[a(x - bt - x_0)] ch[a(x - bt - x_0)]}{(1 + sh^2[a(x - bt - x_0)])^2}, \tag{20a}$$

$$r_{2xx} = -\frac{2a sin(by) sh[a(x - bt - x_0)] ch[a(x - bt - x_0)]}{(1 + sh^2[a(x - bt - x_0)])^2}. \tag{20b}$$

$$r_{3xx} = \frac{a sh[a(x - bt - x_0)]}{ch^2[a(x - bt - x_0)]}. \tag{20c}$$

Thus, using (18), (19a), (19b), (20a) - (20c) we can compute

$$L = (\mathbf{n}, \mathbf{r}_{xx}) = n_1 r_{1xx} + n_2 r_{2xx} + n_3 r_{3xx}.$$

It is followed

$$L = 0. \tag{21}$$

Similarly, we calculate other coefficients of the second fundamental form

$$M = 0, \tag{22}$$

$$N = -\frac{b^3 arctg^2(sh[a(x - bt - x_0)])}{\sqrt{\Lambda} a^2 ch[a(x - bt - x_0)]}. \tag{23}$$

The formulas (21) - (23) give us the last three equations (6a) - (6c). Finally, Theorem is proved.

Based on the results of work [3], where Gauss-Codazzi-Mainardi equation considered in multidimensional space, we have studied generalized Landau-Lifschitz equation and built the surface corresponding to domain wall solution. Thus, this work fully reveals the meaning of the geometric approach [3] in (2+1) - dimensions.

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