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SDRE based stabilization of the affine control system with the stationary linear part

Mikhail Dmitriev
Institute for Systems Analysis
FRC CSC RAS
Moscow, Russia
mdmitriev@mail.ru

Zainelkhriet Murzabekov
Faculty of Information Technology
Al-Farabi Kazakh National University
Almaty, Kazakhstan
murzabekov-zein@mail.ru

Dmitry Makarov
Institute for Systems Analysis, FRC CSC RAS
Moscow Institute of Physics and Technology
Moscow, Russia
makarov@isa.ru

Gulbanu Mirzakhmedova
Faculty of Information Technology
Al-Farabi Kazakh National University
Almaty, Kazakhstan
gulbanu.myrzahmedova@mail.ru

***Abstract*—A stabilizing regulator is constructed for a nonlinear control system with a constant state matrix and a state-dependent input matrix. A regulator is designed by means of the State-Dependent Riccati Equation (SDRE) technique. The novelty of the work is reducing SDRE to the algebraic matrix Riccati equation with constant coefficients. That simplifies the proof of the asymptotic stability of a closed-loop system. The results are illustrated by the example of a mathematical model of a three-sector economic system, where the stabilization in the equilibrium neighborhood is made by means of investment strategies in the form of feedback control.**

Keywords — three-sector economic cluster, nonlinear system, quadratic functional, SDRE, matrix Riccati equation

I. Introduction

Much attention is paid to the problem of stabilization of nonlinear systems in control theory. In particular, a plenty of new SDRE based algorithms for constructing stabilizing regulators for affine control systems have appeared recently [1]-[4]. A regulator state-dependence leads to recalculation of a control for every new value of a state. It significantly reduces the efficiency of computational procedures for real time applications. The ambiguity of representing a nonlinear system as a system with a linear structure and the lack of sufficiently universal algorithms for solving the Riccati equation, whose coefficients depend on the state, generate a set of possible suboptimal solutions. In applied problems, there are many different types of nonlinearities; therefore, different nonlinear control construction approaches, that are rational with respect to one or another given quality criterion, arise.

In this paper, we consider the problem of constructing a stabilizing control for one class of nonlinear systems in which a state matrix is constant and an input matrix is state-dependent. Such models are often found in applications, in particular, in mathematical economics (see [5],[6]). Here, using the specifics of nonlinearities in the mathematical model of a three-sector economic system (nonlinearities present only in an input matrix)[5], an application of the SDRE approach is used. Stabilizing regulator design is based on a solution of the matrix algebraic Riccati equation that appears like the Kalman's method solving control problems with a quadratic functional to search for a stabilizing control on an infinite time interval.

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II. Simplification the SDRE approach in a case of the stationary linear part

In the literature, the SDRE approach is mainly focused on the construction of feedback laws, as well as on the justification of it stabilizing properties. Let us show that, solving some inverse problem, for the class of affine control systems with the stationary linear part

$$\frac{dy}{dt} = \tilde{A}y + \tilde{B}(y)u, \quad y(t_0) = y_0, \quad t \in [t_0, \infty), \tag{1}$$

where $y \in \mathbb{R}^n$ is a state vector and $u \in \mathbb{R}^r$ is a control, it is possible to build a such quality criteria

$$J(u) = \frac{1}{2}\int_{t_0}^{\infty} [y^T Q(y) y + u^T R u] dt \tag{2}$$

that stabilizing regulator for (1) has the standard Kalman's form $u = -R^{-1}\tilde{B}^T(y)Ky$, where a constant positive-definite gain coefficient matrix K is a solution of some algebraic constant coefficients Riccati equation

$$-K\tilde{A} - \tilde{A}^T K + KCR^{-1}C^T K - Q_1 = 0. \tag{3}$$

Here C is a some constant matrix. Indeed, according to the SDRE technique a control law has the form $u = -R^{-1}\tilde{B}^T(y)K(y)y$, where matrix K is a positive definite solution of the following algebraic matrix Riccati equation

$$-K\tilde{A} - \tilde{A}^T K + K\tilde{B}(y)R^{-1}\tilde{B}^T(y)K - Q(y) = 0.$$

All of it matrices coefficients can be state functions. Therefore, if we choose the weight matrix $Q(y)$ in the criterion (2) so that

$$Q(y) = K\tilde{B}(y)R^{-1}\tilde{B}^T(y)K - KCR^{-1}C^T K + Q_1,$$

where C is the constant matrix, Q_1 is a constant positive definite matrix and for all y, then the SDRE equation turns into a matrix equation (3) with constant coefficients. Thus, in the case of the constant linear part, the SDRE approach in problem (1)-(2) is implemented along the solution of the matrix algebraic Riccati equation with constant coefficients. That is there is some analogy with the simplification of the Pontryagin's maximum principle for a system which is linear by state and nonlinear by control function.

We note here that a similar technique of selecting the quality criterion was used in [7],[8].

III. Stabilizing Regulator for a Three-Sector Economic System

Let's consider the following mathematical model of a three-sector economic system form [5],[9],[10]

$$\frac{dk_i}{dt} = -\lambda_i k_i + (s_i / \theta_i)x_1,\ k_i(0) = k_i^0,\ \lambda_i > 0,\ (i = 0,1,2), \quad (4)$$

where $x_i = A_i\theta_i k_i^{\alpha_i}$ are three functions of the Cobb-Douglas type, $A_i > 0,\ 0 < \alpha_i < 1,\ (i = 0, 1, 2),$

$$\sum_{i=0}^{2} s_0 = 1,\quad s_i \geq 0,\ \sum_{i=0}^{2}\theta_i = 1,\quad \theta_i \geq 0,$$
$$(1-\beta_0)x_0 = \sum_{i=1}^{2}\beta_i x_i = 1,\quad \beta_i \geq 0,\quad (i = 0,1,2). \quad (5)$$

Here $k = (k_0, k_1, k_2)$ is a state vector, k_0 is a capital-production ratio of materials or resources, k_1 is a capital-production ratio of the means of labor, k_2 is the capital-production ratio of the production of consumer goods; $u = (s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$ is a control vector, where (s_0, s_1, s_2) are share of sectors in a distribution of investment resources and $(\theta_0, \theta_1, \theta_2)$ are share of sectors in a distribution of labor resources; x_i is a specific issues by a sector; β_i is a direct material costs in the i-th sector; (k_0^0, k_1^0, k_2^0) are an initial state of the system, where $k_i^0 = k_i(0)$ is a capital-sector ratio of the i-th sector with $t = 0$.

Here we consider the stabilization control problem, where the dynamics equations are the system of three ordinary differential equations (4) with state-dependent control multipliers.

It is required to transfer a nonlinear system from a given initial state (k_0^0, k_1^0, k_2^0) to any sufficiently small neighborhood of the state (k_0^s, k_1^s, k_2^s) on an infinite time interval $[0,\infty)$, where the equilibrium state of the system is chosen as the desired final state.

The control problem can be presented as the stabilization problem for the system [10]

$$\frac{dy}{dt} = Ay + BD(y)u + B(D(y) - D(k^s))v^s,$$
$$y(t_0) = y_0,\quad t \in [t_0, \infty), \quad (6)$$

where $y_1 = k_1 - k_1^s,\ y_2 = k_2 - k_2^s,\ y_3 = k_0 - k_0^s,\ u_1 = s_1 - v_1^s,$ $u_2 = s_2\theta_1 / \theta_2 - v_2^s,\ u_3 = s_0\theta_1 / \theta_0 - v_3^s,\ v_1^s = s_1^s,\ s_2^s\theta_1^s / \theta_2^s = v_2^s,$ $s_0^s\theta_1^s / \theta_0^s = v_3^s,$

$$D(y) = \begin{pmatrix} (y_1 + k_1^s)^{\alpha_1} & 0 & 0 \\ 0 & (y_1 + k_1^s)^{\alpha_1} & 0 \\ 0 & 0 & (y_1 + k_1^s)^{\alpha_1} \end{pmatrix},$$

$D(k^s) = \begin{pmatrix} (k_1^s)^{\alpha_1} & 0 & 0 \\ 0 & (k_1^s)^{\alpha_1} & 0 \\ 0 & 0 & (k_1^s)^{\alpha_1} \end{pmatrix}$. The equilibrium states k^s satisfy the constraints

$$Ak^s + BD(k^s)v^s = 0. \quad (7)$$

Let's introduce the condition

I. The system (7) has a positive solution (k^s, v^s), satisfying the constraints (5).

Let's present (6) in the form of (1). To do this, we introduce a new control vector $w(y) = u + \left(I - D^{-1}(y)D(k^s)\right)v^s$, where I is identity matrix. So system (6) takes the form of (1)

$$\dot{y} = Ay + BD(y)w,\quad y(t_0) = y_0, \quad (8)$$

where $\tilde{A} = A,\ \tilde{B}(y) = BD(y)$. Now we introduce the quality criterion of the form (2)

$$J(w) = \frac{1}{2}\int_{t_0}^{\infty}[y^T Q(y)y + w^T Rw]dt \to \min_w, \quad (9)$$

where $R > 0$ is a constant matrix, $Q(y)$= $KBD(y)R^{-1}D^T(y)B^T K - KBD(k^s)R^{-1}D^T(k^s)B^T K + Q_1,$ $Q_1 > 0$ is a some constant matrix. We use the scheme of the linear-quadratic optimal control algorithm to search for stabilizing control for system (8), i.e. w has the form

$$w = -R^{-1}D^T(y)B^T Ky, \quad (10)$$

where a constant matrix K satisfies the following algebraic matrix Riccati equation with constant coefficients

$$-KA - A^T K + KCR^{-1}C^T K - Q_1 = 0. \quad (11)$$

For solvability of (11) we introduce the condition

II. The triple of constant matrices $\left\{A, C, Q_1^{1/2}\right\}$, where $C = BD(k^s)$, is controllable and observable.

Substituting (10) in (8) yields the closed-loop system

$$\frac{dy}{dt} = A_1(y)y,\qquad y(t_0) = y_0, \quad (12)$$

where $A_1(y) = A - BD(y)R^{-1}D^T(y)B^T K.$ Let's introduce the assumptions

III. $k_1 \geq 0, \quad t \in [0, \infty)$.

IV. $Q(y) = KBD(y)R^{-1}D^T(y)B^TK +$

$-KBD(k^s)R^{-1}D^T(k^s)B^TK + Q_1 > 0, \quad y \in \mathbb{R}^3.$

The condition III means that a capital-labor ratio cannot be negative at each time. Due to the boundedness of matrices in IV the condition IV can be satisfied if a sufficiently large in norm $Q_1 > 0$ is chosen.

Theorem. Let conditions I-IV be satisfied. Then the zero equilibrium point of the closed-loop system (12) is asymptotically stable, i.e. the nonlinear regulator (10), where $K > 0$ is a solution of the Riccati equation (11), is stabilizing.

Sketch of the proof. The conditions of the theorem imply the existence of a positive definite solution K of the matrix equation (11). This solution forms the Lyapunov function $V(y) = \frac{1}{2}y^TKy$. It is shown that the total time derivative of this function along the (12) is negative. Indeed, taking into account the matrix equation (11) and the form of $Q(y)$, we obtain

$$\dot{V} = \frac{1}{2}y^T(KA + A^TK - 2KBD(y)R^{-1}D^T(y)B^TK)y =$$
$$-\frac{1}{2}y^T(Q(y) + KBD(y)R^{-1}D^T(y)B^TK)y < 0.$$

IV. NUMERICAL EXPERIMENTS

Let the initial data for the model (4)-(5) be defined as in Table 1.

TABLE I. PARAMETERS FOR THE THREE-SECTOR ECONOMIC MODEL

i	α_i	β_i	λ_i	A_i	s_i^*	θ_i^*	k_i^*
0	0.46	0.39	0.05	6.19	0.2763	0.3944	966.4430
1	0.68	0.29	0.05	1.35	0.4476	0.2562	2410.1455
2	0.49	0.52	0.05	2.71	0.2761	0.3494	1090.1238

The initial state vector is $y(t_0) = (-400, 100, 200)^T$ and the matrices R, Q_1, K_T have the following form

$$R = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} \frac{1}{640} & \frac{1}{640} & \frac{1}{640} \\ \frac{1}{640} & \frac{1}{160} & \frac{1}{320} \\ \frac{1}{640} & \frac{1}{320} & \frac{1}{160} \end{pmatrix},$$

$$K_T = \begin{pmatrix} 0.4131 \cdot 10^{-3} & 0.1400 \cdot 10^{-3} & 0.1400 \cdot 10^{-3} \\ 0.1400 \cdot 10^{-3} & 0.8814 \cdot 10^{-2} & 02316 \cdot 10^{-3} \\ 0.1400 \cdot 10^{-3} & 0.2316 \cdot 10^{-3} & 0.8814 \cdot 10^{-2} \end{pmatrix}.$$

The closed-loop system trajectories and controls obtained by numerical simulation are presented in Fig. 1 and Fig 2. It can be seen that the trajectories and stabilizing controls tend to the zero equilibrium point. Therefore, a closed-loop control system is asymptotically stable.

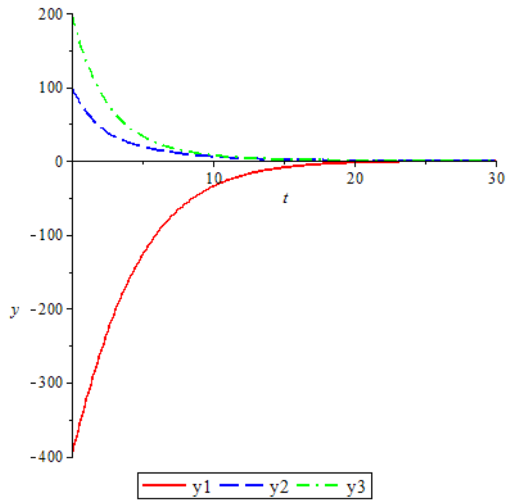

Fig. 1. The trajectories $y(t)$

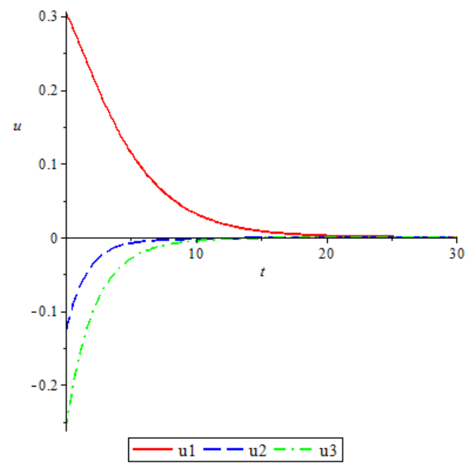

Fig. 2. The control $u(t)$

Using formulas

$$v = \frac{\beta_1 A_1 f_1(y_1) + \beta_2 A_2 f_2(y_2)(1 - u_1 - v_1^s)/(u_2 + v_2^s)}{den},$$
$$den = (1-\beta_0)A_0 f_3(y_3)(1 - u_1 - v_1^s)/(u_3 + v_3^s) +$$
$$\beta_2 A_2 f_2(y_2)(1 - u_1 - v_1^s)/(u_2 + v_2^s),$$
$$s_1 = u_1 + v_1^s, \quad s_2 = (1-v)(1 - u_1 - v_1^s), \quad s_0 = v(1 - u_1 - v_1^s),$$
$$\theta_1 = \frac{1}{1 + s_0/(u_3 + v_3^s) + s_2/(u_2 + v_2^s)},$$
$$\theta_2 = \frac{(1-v)(1-s_1)\theta_1}{(u_2 + v_2^s)}, \quad \theta_0 = \frac{v(1-s_1)\theta_1}{(u_3 + v_3^s)},$$

following from (5), as well as the expression $u = w - \left(I - D^{-1}(y)D(k^s)\right)v^s$, where I is the identity matrix of the corresponding dimension, the distribution of labor $(\theta_0(t), \theta_1(t), \theta_2(t))$ and investment $(s_0(t), s_1(t), s_2(t))$ resources are determined. Fig. 3 and Fig. 4 show the changes in resources for the balance relations (5) without any control constraints. It is seen that the investment resource s_0 does not satisfy the condition of non-negativity of the balance relation (5), i.e. it has invalid value.

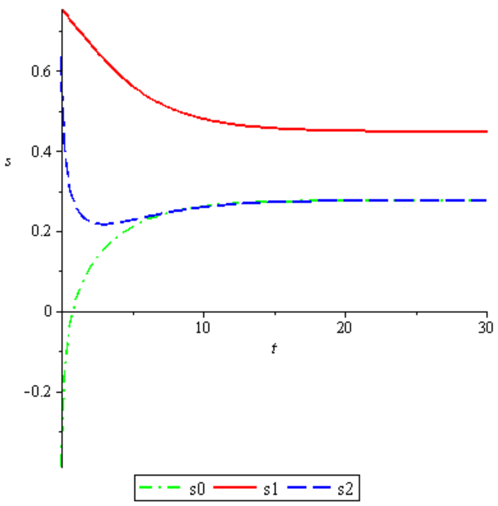

Fig. 3. Distribution of investment

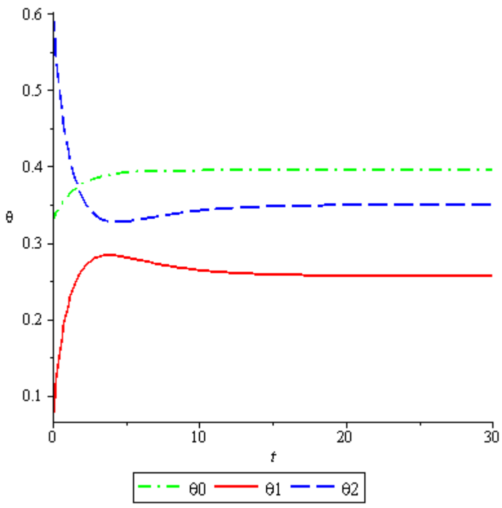

Fig. 4. Distribution oflabor resources

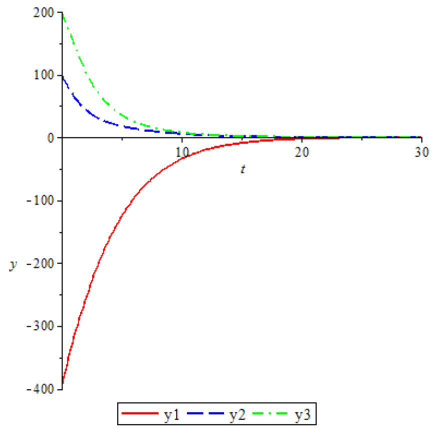

Fig. 5. Trajectories $y(t)$ for contstrained problem

Let us show that the proposed algorithm can generate admissible controls for a number of problems with control constraints. For example, let us have the following constraints

$$-0.65 \le u_1 \le 0.25, \; -0.5 \le u_2 \le 0.4, \; -0.15 \le u_3 \le 0.4.$$

The state trajectories along constrained control are presented in Fig. 5. It may be seen the asymptotic stability of the zero point of the closed-loop system. From Fig 6 one can see that the coordinates u_1 and u_3 of control vector take values partially on the boundary of the admissible region. Fig. 7 and Fig. 8 show distributions of investment and labor resources for balance relations (5) with control constraints. It is obvious that now s_i and θ_i satisfy all the conditions of the balance ratio (5). The investment resource s_0 has values partly at the border, and then tends to an equilibrium value.

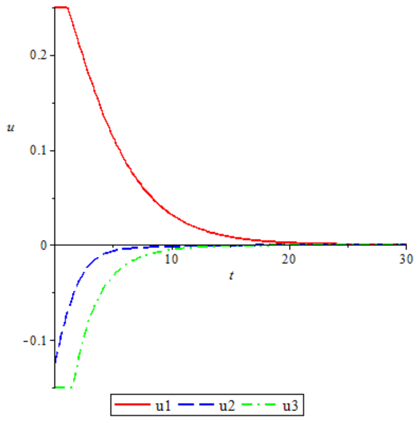

Fig. 6. Contstrained control $u(t)$

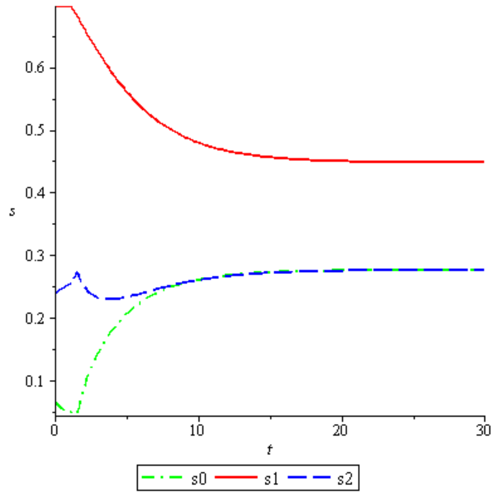

Fig. 7. Distribution of investment for contrained problem

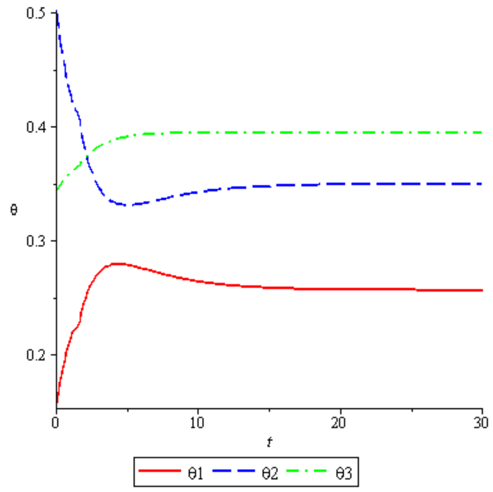

Fig. 8. Distribution of labor resources for contrained problem

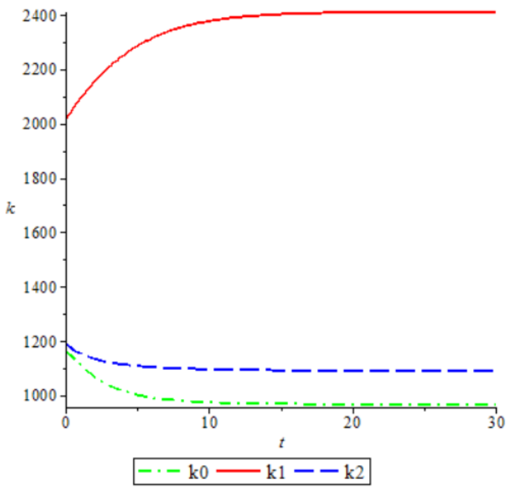

Fig. 9. The capital-labor ratio of the constrained problem

Fig. 9 shows the changes in the capital-labor ratio of sectors that tend to equilibrium values in the interval $[t_0, \infty)$.

V. CONCLUSIONS

The nonlinear affine control system with a stable stationary linear part describing own motions and with a state-dependent input matrix is considered. The algorithm for constructing the nonlinear stabilizing regulator according to the Kalman's algorithm is proposed for this system. The control gain matrix is based on a solution of a specially constructed algebraic matrix Riccati equation with constant coefficients. This algorithm for constructing a stabilizing regulator differs from the ordinary SDRE technique, which uses the solution of the algebraic matrix Riccati equation with coefficients depending on the system state. The asymptotic stability of a closed-loop system along the constructed regulator is established under the certain conditions. The results are illustrated by numerical simulations for a mathematical model of a three-sector economic system. The stabilization of the system in the zero equilibrium point is carried out by means of investment closed-loop control strategies. Computational experiments confirming the efficiency of the algorithm were performed.

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