On Computability in the Hierarchy of Ershov

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For the basic notions of the theory of numberings we refer to the handbook of Yu.L.Ershov, [1].

A lot of known notions of computability, such as computability of the families of computably enumerable sets, constructive models, positive equivalences, are the special cases of the general approach suggested by S.Goncharov and A.Sorbi in [2]. This approach has activated the study of computability at several domains in logic and computer science and, in particular, in the hierarchy of Ershov, [3]–[5], started at the end of 20th century. Besides, it allowed to introduce in [6] direct notion of computable numbering of a family of sets from any fixed level of hierarchy as follows.

Definition 1 *Let Σ^{-1}_n, $n \geqslant 1$, be a class of the hierarchy of Ershov and let $\mathcal{A} \subseteq \Sigma^{-1}_n$. A numbering $\alpha : \omega \to \mathcal{A}$ is called a Σ^{-1}_n-computable numbering of the family $\mathcal{A}$ if the set $\{\langle x, n\rangle : x \in \alpha n\}$ is in Σ^{-1}_n.*

In our considerations we often use some technical criterion of computability in terms of functions which realize the procedures of computations in the Ershov hierarchy.

Definition 2 *We say that total function $h : \omega \to \{0, 1\}$ oscillates at t_1 if there exists t_0 such that $t_0 < t_1$, $h(t_0) = 0$, $h(t) = 1$ for all $t_0 < t < t_1$ and $h(t_1) = 0$.*

Proposition 1 *Let $\mathcal{A} \subseteq \omega$ and $n \geqslant 1$. Then $\mathcal{A} \in \Sigma^{-1}_n$ if and only if there exists a computable function $h(x, t)$ with $\mathrm{range}(h) \subseteq \{0, 1\}$, $h(y, 0) = 0$ for all y, and such that the following conditions hold for every x:*

1) if $x \in \mathcal{A}$ then $\lim_t h(x, t) = 1$ and number of oscillations of function $\lambda t h(x, t)$ does not surpass $(n - 1) \div 2$;

2) if $x \notin \mathcal{A}$ then $\lim_t h(x, t) = 0$ and number of oscillations of function $\lambda t h(x, t)$ does not surpass $n \div 2$ for even n and it does not surpass $(n - 1) \div 2$ for odd n.

Corollary 1 *Let $\mathcal{A} \subseteq \Sigma^{-1}_n$ and $n \geqslant 1$. Then a numbering $\alpha : \omega \to \mathcal{A}$ is Σ^{-1}_n−computable if and only if there exists a computable function $h(n, x, t)$ with $\mathrm{range}(h) \subseteq \{0, 1\}$, $h(n, y, 0) = 0$ for all n, y, and such that the following conditions hold for every n, x:*

1) if $x \in \alpha n$ then $\lim_t h(n, x, t) = 1$ and number of the oscillations of function $\lambda t h(n, x, t)$ does not surpass $(n - 1) \div 2$;

2) if $x \notin \alpha n$ then $\lim_t h(n,x,t) = 0$ and number of the oscillations of function $\lambda t h(n,x,t)$ does not surpass $n \div 2$ for even n and it does not surpass $(n-1) \div 2$ for odd n.

The study of uniform computability of any families of constructive objects is usually done along the results known in the classical case of computable numberings of the families of computably enumerable sets.

Theorem 1 *For every $n \geqslant 1$, there exists a Σ_n^{-1}-computable principal numbering of the family of all Σ_n^{-1} sets.*

Positive and decidable numberings were introduced by A.I.Mal'tsev in [7]. Importance of these notions for the theory of numberings caused by their minimality with respect to reducibility of numberings.

Definition 3 *Let $\Theta_\alpha \rightleftharpoons \{\langle x,y\rangle \mid \alpha x = \alpha y\}$. A numbering α is called positive(or decidable) if Θ_α is a computably enumerable (or, respectively, a decidable) set.*

In [6], S.A.Badaev and S.S.Goncharov suggested conjecture on an existence of a Σ_n^{-1}–computable positive undecidable numbering of the family of all Σ_n^{-1} sets for every $n > 1$.

Theorem 2 (Zh. Talasbaeva, [4]) *There exist infinitely many positive undecidable Σ_n^{-1}– computable numberings of every infinite Σ_n^{-1}– computable family which contains either $\emptyset$ for even n, or N for odd n.*

References

[1] Yu.L. Ershov, *Theory of numberings.* – Nauka, Moscow, 1977 (Russian).

[2] S.S. Goncharov, A. Sorbi, *Generalized computable numberings and nontrivial Rogers semilattices.* Algebra and Logic, 1997, v. 36, no. 6, pp. 359–369.

[3] S.Goncharov, S.Lempp, D.R.Solomon, *Friedberg numberings of families of n-computably enumerable sets.* Algebra and Logic, 2002, v.41, no.2, 81-86.

[4] Zh.Talasbaeva, *Positive Numberings of Families of Sets in the Ershov Hierarchy.* Algebra and Logic, 2003, v.42, no.6, 413-418.

[5] S.Badaev, S.Lempp, *On decomposition of Rogers semilattices.* In preparation.

[6] S.A. Badaev, S.S. Goncharov, *Theory of numberings: open problems.* In: Computability theory and applications. Current trends and open problems. Contemporary Mathematics, 2000, v. 257, pp. 23–38.

[7] A.I. Mal'tsev, *Positive and negative numberings.* Soviet Math. Dokl., 1965, v.160, pp. 75–77.