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Том 18 **№ 3 (69)** **2018**

CONTENTS

Volume 18 **No. 3 (69)** **2018**

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ABOUT ONE COEFFICIENT INVERSE PROBLEM FOR THE HEAT EQUATION IN A DEGENERATING DOMAIN WITH TIME-DEPENDENT BOUNDARIES

M.T. Jenaliyev, M. Yergaliyev

Annotation. In the paper we consider a coefficient inverse problem for the heat equation in a degenerating angular domain with time dependent boundaries. It was shown that with the help of transformations of variables and the desired function, the inverse problem for the heat equation in a degenerate angular domain with two time-dependent boundaries can be reduced to the problem for the heat equation in a degenerate angular domain but with one time-dependent boundary. For the last problem the criterion of existence of the solution was shown.

Keywords. Coefficient inverse problem, heat equation, degenerating domain.

1 Introduction

The inverse problems of the kind which we will consider were investigated in the papers [1], [2]. In that papers it is assumed that the movable boundaries move according to the law obeying Holder class and the domain does not degenerate and the time interval is limited. There uniqueness and existence of the solution of the inverse problem where the required coefficient is a continuous function are established and numerical solutions are obtained.

The peculiarity of our study is that we consider the inverse problem for the heat equation in the degenerating angular domain. For the sake of simplicity and for the purpose of showing the effect of the degeneration of the domain, we consider the problem, where, firstly, the moving part of the boundary changes

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linearly; secondly, the boundary value problem is completely homogeneous; thirdly, the time interval is semi-bounded. In papers [3], [4] we investigated the inverse problem for the heat equation in the angular domain. In this work we consider a coefficient inverse problem for the heat equation in a degenerating domain with time-dependent boundaries.

2 Statement of the problem

In the domain $G_T = \{(x,t)| \ -k_1 t < x < k_2 t, \quad 0 < t < T, \}$, $T < +\infty$, we consider an inverse problem of finding a coefficient $\lambda(t)$ and a function $u(x,t)$ for following heat equation:

$$u_t(x,t) = u_{xx}(x,t) - \lambda(t)u(x,t), \tag{1}$$

with homogeneous boundary conditions

$$u(x,t)|_{x=-k_1 t} = 0, \quad u(x,t)|_{x=k_2 t} = 0, \ 0 < t < T, \tag{2}$$

suspect to the overspecification

$$\int_{-k_1 t}^{k_2 t} u(x,t)dx = E(t), \quad |E(t)| \geq \delta > 0, \ 0 < t < T, \tag{3}$$

where $E(t) \in L_\infty(0,T)$ is a given function and $k_1, k_2 > 0$.

3 Transformations and auxiliary problems

3.1 Transformations

Using the transformations:

$$x_1 = (k_1 + k_2)x + (k_1 + k_2)k_1 t, \quad t_1 = (k_1 + k_2)^2 t, \tag{4}$$

the boundary value problem (1)–(3) in the domain G_T reduces to the boundary value problem in the domain $G_1 = \{(x_1, t_1)| \, 0 < x_1 < t_1, \ 0 < t_1 < T_k = (k_1 + k_2)^2 T\}$:

$$u_{1t_1}(x_1,t_1) + \frac{k_1}{k_1+k_2} u_{1x_1}(x_1,t_1) =$$

$$= u_{1x_1x_1}(x_1,t_1) - \lambda_1(t_1)u_1(x_1,t_1), \{x_1, t_1\} \in G_1, \tag{5}$$

where $\lambda_1(t_1) = \frac{1}{(k_1+k_2)^2}\lambda(t_1)$, with homogeneous boundary conditions

$$u_1(x_1,t_1)|_{x_1=0} = 0, \quad u_1(x_1,t_1)|_{x_1=t_1} = 0,\ 0 < t_1 < T_k, \tag{6}$$

suspect to the overspecification

$$\int_0^{t_1} u_1(x_1,t_1)dx_1 = E_1(t_1), \quad |E_1(t_1)| \geq \delta > 0,\ 0 < t_1 < T_k, \tag{7}$$

where

$$E_1(t_1) = (k_1+k_2)E\left(\frac{t_1}{(k_1+k_2)^2}\right),$$

$$u(x,t) = u\left(\frac{1}{k_1+k_2}x_1 - \frac{k_1}{(k_1+k_2)^2}t_1, \frac{1}{(k_1+k_2)^2}t_1\right) = u_1(x_1,t_1).$$

To transform the equation (5) we introduce new function

$$u_1(x_1,t_1) = \exp\left\{-\frac{k_1^2}{4(k_1+k_2)^2}t_1 + \frac{k_1}{2(k_1+k_2)}x_1\right\} \cdot u_2(x_1,t_1). \tag{8}$$

Then we obtain a linear homogeneous boundary value problem for the heat equation in the angular domain G_1 :

$$u_{2t_1}(x_1,t_1) = u_{2x_1x_1}(x_1,t_1) - \lambda_1(t_1)u_2(x_1,t_1), \{x_1,t_1\} \in G_1, \tag{9}$$

with homogeneous boundary conditions

$$u_2(x_1,t_1)|_{x_1=0} = 0, \quad u_2(x_1,t_1)|_{x_1=t_1} = 0,\ 0 < t_1 < T_k, \tag{10}$$

suspect to the overspecification

$$\int_0^{t_1} u_2(x_1,t_1)dx_1 = E_1(t_1), \quad |E_1(t_1)| \geq \delta > 0,\ 0 < t_1 < T_k. \tag{11}$$

3.2 The auxiliary problem

In accordance to the problem (9)–(11) we will set an auxiliary inverse problem of finding a coefficient $\lambda_2(t_1)$ and a function $v(x_1,t_1)$ in the domain $G_\infty = \{(x_1,t_1)|\ 0 < x_1 < t_1,\ \ t_1 > 0\}$:

$$v_{t_1}(x_1,t_1) = v_{x_1x_1}(x_1,t_1) - \lambda_2(t_1)v(x_1,t_1), \tag{12}$$

with homogeneous boundary conditions

$$v(x_1,t_1)|_{x_1=0}=0,\quad v(x_1,t_1)|_{x_1=t_1}=0,\ t_1>0, \tag{13}$$

suspect to the overspecification

$$\int_0^{t_1} v(x_1,t_1)dx_1=\tilde{E}(t_1),\quad t_1>0, \tag{14}$$

$$\tilde{E}(t_1)=\begin{cases} E_1(t_1), & 0<t_1<T_k,\\ E_2(t_1), & T_k\le t_1<\infty,\end{cases} \tag{15}$$

where $|E_2(t_1)|\ge\delta>0$ is an arbitrary bounded function.

Observation 1. *Solving in G_∞ the problem (12)–(15) and restricting down its solution to the domain G_1, we can find a solution $\{u_2(x_1,t_1),\lambda_1(t_1);\ (x_1,t_1)\in G_1\}$ of the inverse problem (9)–(11).*

3.3 Equivalent problem

In the problem (12)–(15) we replace the required function by the following transformation

$$w(x_1,t_1)=\hat{\lambda}_2(t_1)v(x_1,t_1),\ \text{where}\ \hat{\lambda}_2(t_1)=\exp\left\{\int_0^{t_1}\lambda_2(s)ds\right\}. \tag{16}$$

Then the inverse problem (12)–(15) reduces to a problem for a homogeneous heat equation:

$$w_{t_1}(x_1,t_1)=w_{x_1x_1}(x_1,t_1),\quad \{x_1,t_1\}\in G_\infty, \tag{17}$$

with homogeneous boundary conditions

$$w(x_1,t_1)|_{x=0}=0,\quad w(x_1,t_1)|_{x_1=t_1}=0,\ t_1>0, \tag{18}$$

subject to the overspecification

$$\int_0^{t_1} w(x_1,t_1)dx_1=\hat{\lambda}_2(t_1)\tilde{E}(t_1),\quad |\tilde{E}(t_1)|\ge\delta>0,\ t_1>0. \tag{19}$$

4 Main result

4.1 The solution of the inverse problem (17)–(19)

It follows from our previous results [5], [6], [7], [8] that a homogeneous boundary value problem (17)–(18) along with a trivial solution has a nontrivial solution up to a constant factor defined by formulas:

$$w(x_1,t_1) = w_+(x_1,t_1) + w_-(x_1,t_1), \tag{20}$$

where

$$w_\pm(x_1,t_1) = \frac{1}{4\sqrt{\pi}} \int_0^{t_1} \frac{x_1 \pm \tau}{(t_1-\tau)^{3/2}} \exp\left\{-\frac{(x_1\pm\tau)^2}{4(t_1-\tau)}\right\} \varphi(\tau)d\tau, \tag{21}$$

where function $\varphi(t_1)$ is defined according to the formulas:

$$\varphi(t_1) = C\varphi_0(t_1), \quad C = \text{const} \neq 0, \tag{22}$$

$$\varphi_0(t_1) = \frac{1}{\sqrt{t_1}} \exp\left\{-\frac{t_1}{4}\right\} + \frac{\sqrt{\pi}}{2}\left[1 + \text{erf}\left(\frac{\sqrt{t_1}}{2}\right)\right], \tag{23}$$

moreover, the function $\varphi(t_1)$ belongs to the following class:

$$\theta(t_1)\varphi(t_1) \in L_\infty(R_+), \text{i.e. } \varphi(t_1) \in L_\infty(R_+;\theta(t_1)), \tag{24}$$

where

$$\theta(t_1) = \begin{cases} \sqrt{t_1}\exp\left\{\frac{t_1}{4}\right\}, & \text{if} \quad 0 < t_1 \le T_k, \\ 1, & \text{if} \quad T_k < t_1 < +\infty, \end{cases} \tag{25}$$

and T_k does not necessarily coincide with T.

From (20)–(22) we obtain for the solution $w(x_1,t_1) = Cw_0(x_1,t_1)$ of the homogeneous boundary value problem (17)–(18) the following representation:

$$w_0(x_1,t_1) = w_{0+}(x_1,t_1) + w_{0-}(x_1,t_1), \tag{26}$$

where

$$w_{0\pm}(x_1,t_1) = \frac{1}{4\sqrt{\pi}} \int_0^{t_1} \frac{x_1 \pm \tau}{(t_1-\tau)^{3/2}} \exp\left\{-\frac{(x_1\pm\tau)^2}{4(t_1-\tau)}\right\} \varphi_0(\tau)d\tau. \tag{27}$$

Further using the representation (26)–(27) for the integral condition (19), we get:

$$\int_0^{t_1} w_0(x_1,t_1)dx = \int_0^{t_1} w_{0+}(x_1,t_1)dx_1 + \\ + \int_0^{t_1} w_{0-}(x_1,t_1)dx_1 = \hat{\lambda}_{20}(t_1)\tilde{E}(t_1). \tag{28}$$

It's obvious that $\hat{\lambda}_2(t_1) = C\hat{\lambda}_{20}(t_1)$. By a commutativity property in the integrals of the formula (28), in the sense of the Dirichlet formula, we have

$$\int_0^{t_1} w_{0\pm}(x_1,t_1)dx_1 = \frac{1}{4\sqrt{\pi}}\int_0^{t_1}\varphi_0(\tau)d\tau\int_0^{t_1}\frac{x_1\pm\tau}{(t_1-\tau)^{3/2}}\exp\left\{-\frac{(x_1\pm\tau)^2}{4(t_1-\tau)}\right\}dx_1. \tag{29}$$

Let's calculate the interior integrals from (29). We get

$$\frac{1}{4\sqrt{\pi}}\int_0^{t_1}\frac{x_1\pm\tau}{(t_1-\tau)^{3/2}}\exp\left\{-\frac{(x_1\pm\tau)^2}{4(t_1-\tau)}\right\}dx_1 = \left\| y = \frac{(x_1\pm\tau)^2}{4(t_1-\tau)} \right\| = \\ = \frac{1}{2\sqrt{\pi(t_1-\tau)}}\left(\exp\left\{-\frac{\tau^2}{4(t_1-\tau)}\right\} - \exp\left\{-\frac{(t_1\pm\tau)^2}{4(t_1-\tau)}\right\}\right). \tag{30}$$

Then from (19), (28)–(30) we obtain

$$\int_0^{t_1} w_0(x_1,t_1)dx_1 = \frac{1}{2\sqrt{\pi}}\int_0^{t_1}\frac{1}{\sqrt{t_1-\tau}}\left[2\exp\left\{-\frac{\tau^2}{4(t_1-\tau)}\right\} - \right. \\ \left. - \exp\left\{-\frac{(t_1+\tau)^2}{4(t_1-\tau)}\right\} - \exp\left\{-\frac{t_1-\tau}{4}\right\}\right]\varphi_0(\tau)d\tau = \hat{\lambda}_{20}(t_1)\tilde{E}(t_1). \tag{31}$$

From ratios (16), (19), (31) and $w(x_1,t_1) = Cw_0(x_1,t_1)$ we find the required coefficient

$$\lambda_2(t_1) = \frac{d\,\ln(\hat{\lambda}_2(t_1))}{d\,t_1} = \frac{(\hat{\lambda}_2(t_1))'}{\hat{\lambda}_2(t_1)} = \frac{(C\hat{\lambda}_{20}(t_1))'}{C\hat{\lambda}_{20}(t_1)} = \lambda_{20}(t_1), \tag{32}$$

where we have used the equality

$$\left(\frac{\int\limits_0^{t_1} w(x_1,t_1)d\,x_1}{\tilde{E}(t_1)}\right)' : \frac{\int\limits_0^{t_1} w(x_1,t_1)d\,x_1}{\tilde{E}(t_1)} = \left(\frac{\int\limits_0^{t_1} w_0(x_1,t_1)d\,x_1}{\tilde{E}(t_1)}\right)' : \frac{\int\limits_0^{t_1} w_0(x_1,t_1)d\,x_1}{\tilde{E}(t_1)}. \tag{33}$$

Thus, from (26)–(27), (31)–(33) we obtain the following theorems.

Theorem 1. *The inverse problem (17)–(19) has a solution* $\{w(x_1,t_1),\lambda_2(t_1)\}$ *if and only if the following condition is satisfied*

$$\operatorname{sign}\{E(t_1)\}=\operatorname{sign}\{I[\varphi_0(t_1),t_1]\},\ \forall t_1\in(0,\infty), \tag{34}$$

where

$$I[\varphi_0(t_1),t_1]=\frac{1}{2\sqrt{\pi}}\int_0^{t_1}\frac{1}{\sqrt{t_1-\tau}}\left[2\exp\left\{-\frac{\tau^2}{4(t_1-\tau)}\right\}-\right.$$
$$\left.-\exp\left\{-\frac{t_1-\tau}{4}\right\}\left(\exp\left\{-\frac{t_1\tau}{t_1-\tau}\right\}+1\right)\right]\varphi_0(\tau)d\tau,\ t_1\in(0,\infty), \tag{35}$$

$$\varphi_0(t_1)=\frac{1}{\sqrt{t_1}}\exp\left\{-\frac{t_1}{4}\right\}+\frac{\sqrt{\pi}}{2}\left[1+\operatorname{erf}\left(\frac{\sqrt{t_1}}{2}\right)\right],\ t_1\in(0,\infty). \tag{36}$$

Theorem 2. *Let satisfied conditions of theorem 1. Then the inverse problem (9)–(11) has the following solution* $\{u_2(x_1,t_1),\ \lambda_1(t_1)\}$*: the coefficient* $\lambda_1(t_1)=\lambda_0(t_1)$ *is determined uniquely by the formula (32)–(33) by restricting it down to a finite interval* $(0,T_k)$ *and the solution* $u_2(x_1,t_1)$ *is found by means of the restriction of the function:*

$$v(x_1,t_1)=Cv_0(x_1,t_1),\ \text{where} v_0(x_1,t_1)=[\hat{\lambda}_{10}(t_1)]^{-1}w_0(x_1,t_1),C=\text{const}, \tag{37}$$

on the bounded triangle G_1 *where* $w_0(x_1,t_1)$ *is defined by formulas (26)–(27).*

Theorem 3. *For the solution* $\{u_2(x_1,t_1),\lambda_1(t_1)\}$ *of the inverse problem (9)–(11) using the substitutions (4) and (8) back we get the solution of the inverse problem (1)–(3).*

Observation 2. *According to formulas (23), (26)–(27), the solution* $w_0(x_1,t_1)$ *is a nonnegative function. It should be noted that according to the criterium (34) the function* $\tilde{E}(t_1)$ *from (19) is a variable sign function, since the integral (28) is variable sign function and the coefficient* $\hat{\lambda}_2(t_1)=C\hat{\lambda}_{20}(t_1)$ *(16), (32) is a positive function.*

In work [3] we have showed that the solution of the equivalent problem and additional condition is bounded. Also it have been showed that the solution of the inverse problem (9)–(11) and required function $\lambda_1(t_1)$ have singularity at small values of the variable t. Therefore it should be noted that the solution of the inverse problem $\{u(x,t), \lambda(t)\}$ also have singularity at small values of the variable t.

6 Conclusion

In the paper we consider an inverse problem for the heat equation in a degenerating angular domain with time-dependent boundaries. We have shown that the inverse problem for the homogeneous heat equation with homogeneous boundary conditions has a nontrivial solution $\{u(x,t),\ \lambda(t)\}$ consistent with the integral condition.

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Жиенәлиев М.Т., Ерғалиев М.Ғ. ШЕКАРАЛАРЫ УАҚЫТҚА ТӘУЕЛДІ АЗЫНҒАН ОБЛЫСТАҒЫ ЖЫЛУӨТКІЗГІШТІК ТЕҢДЕУІ ҮШІН ҚОЙЫЛҒАН КОЭФФИЦИЕНТТІ КЕРІ ЕСЕП ТУРАЛЫ

Жұмыста біз шекаралары уақытқа тәуелді азынған облыстағы жылуөткізгіштік теңдеуі үшін қойылған коэффициентті кері есепті қарастырамыз. Жұмыста тәуелсіз айнымалыларды және ізделінді функцияны түрлендіру арқылы екі шекарасы да уақытқа тәуелді азынған облыстағы жылуөткізгіштік теңдеуі үшін қойылған кері есеп шекарасының біреуі ғана уақытқа тәуелді болатын азынған облыстағы жылуөткізгіштік теңдеуі үшін қойылған есепке келтіруге болатындығы көрсетілді. Соңғы есеп үшін шешімділік критерийі алынды.

Кілттік сөздер. Коэффициенттік кері есеп, жылуөткізгіштік теңдеуі, азынған облыс.

Дженалиев М.Т., Ергалиев М.Г. ОБ ОДНОЙ КОЭФФИЦИЕНТНОЙ ОБРАТНОЙ ЗАДАЧЕ ДЛЯ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ В ВЫРОЖДАЮЩЕЙСЯ ОБЛАСТИ С ГРАНИЦАМИ, ЗАВИСЯЩИМИ ОТ ВРЕМЕНИ

В работе мы рассматриваем коэффициентную обратную задачу для уравнения теплопроводности в вырождающейся области с границами, зависящими от времени. Было показано, что при помощи преобразования независимых переменных и искомой функции обратная задача для уравнения теплопроводности в вырождающейся области с двумя границами, зависящими от времени, может быть сведена к задаче для уравнения теплопроводности в вырождающейся угловой области, но уже с одной границей, зависящей от времени. Для последней задачи получен критерий разрешимости.

Ключевые слова. Коэффициентная обратная задача, уравнение теплопроводности, вырождающаяся область.

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