On the Coefficient Inverse Problem of Heat Conduction in a Degenerating Domain

Muvasharkhan Jenaliyev[a)] and Madi Yergaliyev[b)]

Institute of Mathematics and Mathematical Modeling, 050010, Almaty, Kazakhstan

[a)]muvasharkhan@gmail.com
[b)]ergaliev@math.kz

Abstract. In the paper we consider a coefficient inverse problem for the heat equation in a degenerating angular domain. It has been shown that the inverse problem for the homogeneous heat equation with homogeneous initial and boundary conditions has a nontrivial solution up to a constant factor consistent with the integral condition. Moreover, the solution of the considered inverse problem is found in explicit form. It was also proved that the found nontrivial solution is a bounded function for all $t > 0$.

INTRODUCTION AND STATEMENT OF THE PROBLEM

In the domain $G = \{(x,t)|\ \ t > 0,\ 0 < x < t\}$, we consider an inverse problem of finding a coefficient $\lambda(t)$ and the function $u(x,t)$ for following heat equation:

$$u_t(x,t) = u_{xx}(x,t) - \lambda(t)u(x,t), \tag{1}$$

with homogeneous initial condition

$$u(x,0) = 0, \tag{2}$$

and the boundary conditions

$$u(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=t} = 0,\ t > 0, \tag{3}$$

subject to the overspecification

$$\int_0^t u(x,t)dx = E(t), \quad E(0) = 0,\ t > 0, \tag{4}$$

where $E(t) \in L_\infty(\mathbb{R}^1_+)$ is the given function.

The inverse problems of this kind were investigated in the papers [1, 2]. In that papers it is assumed that the movable boundaries move according to the law obeying Holder class and the domain does not degenerate and the time interval is limited. There uniqueness and existence of the solution of the inverse problem where the required coefficient is a continuous function are established and numerical solutions are obtained.

The peculiarity of our study is that we consider the inverse problem for the heat equation in the degenerating angular domain. For the sake of simplicity and for the purpose of showing the effect of the degeneration of the domain, we consider the problem (1)-(3) where, firstly, the moving part of the boundary changes linearly; secondly, the boundary value problem is completely homogeneous; thirdly, the time interval is semi-bounded. It is known that when a domain degenerates at some points, the methods of separation of variables and integral transformations are generally not applicable to this type of problems. In this paper, to prove the existence of a non-trivial solution for the original problem we use the methods and results of our earlier work where solutions are found with help of theory of thermal potentials and the Volterra integral equation of the second kind [3, 4].

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EQUIVALENT PROBLEM

Employing the transformation [1, 2]:

$$w(x,t) = e^{\int_0^t \lambda(s)ds} u(x,t) = \hat{\lambda}(t)u(x,t) \tag{5}$$

the inverse problem (1)-(4) reduces to a problem for the homogeneous heat equation

$$w_t(x,t) = w_{xx}(x,t), \quad 0 < x < t, t > 0, \tag{6}$$

with homogeneous initial condition

$$w(x,0) = 0, \tag{7}$$

and the boundary conditions

$$w(x,t)|_{x=0} = 0, \quad w(x,t)|_{x=t} = 0, \ t > 0, \tag{8}$$

subject to the overspecification

$$\int_0^t w(x,t)dx = \bar{\lambda}(t)E(t), \quad E(0) = 0, \ t > 0. \tag{9}$$

ON A NONTRIVIAL SOLUTION OF THE HOMOGENEOUS BOUNDARY VALUE PROBLEM (6)-(8)

In works [3–5] it was shown that the homogeneous boundary value problem (6)-(8) has a nonzero solution up to a constant factor, which is determined by the formulas:

$$\begin{aligned} w(x,t) &= \frac{1}{4\sqrt{\pi}} \int_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4(t-\tau)}\right\} \nu(\tau)d\tau \\ &\quad + \frac{1}{4\sqrt{\pi}} \int_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\} \varphi(\tau)d\tau, \end{aligned} \tag{10}$$

$$\nu(t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} \varphi(\tau)d\tau, \tag{11}$$

where function $\varphi(t)$ is defined according to the formula:

$$\varphi(t) = C\varphi_0(t), \ \ \varphi_0(t) = \frac{1}{\sqrt{t}} \exp\left\{-\frac{t}{4}\right\} + \frac{\sqrt{\pi}}{2}\left[1 + \operatorname{erf}\left(\frac{\sqrt{t}}{2}\right)\right], \ \ C = const \neq 0, \tag{12}$$

moreover, the function $\varphi(t)$ belongs to the following class:

$$\theta(t)\varphi(t) \in L_\infty(G). \tag{13}$$

Here

$$\theta(t) = \sqrt{t} \cdot e^{-\varepsilon t}, \ \ t > 0, \ \ \varepsilon > 0. \tag{14}$$

THE SOLUTION OF INVERSE PROBLEM (6)-(9)

Substituting the expression of function $\nu(t)$ from the formula (11) into the formula (10), we obtain for the solution $w(x,t) = Cw_0(x,t)$ of the homogeneous problem (6)-(8) the following representation:

$$w_0(x,t) = \frac{1}{4\sqrt{\pi}} \int_0^t \left[\frac{x+\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4(t-\tau)}\right\} + \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\}\right] \varphi_0(\tau)d\tau. \tag{15}$$

Further using the representation (15) for the integral condition (9), we get

$$\begin{aligned} \int_0^t w_0(x,t)dx &= \frac{1}{4\sqrt{\pi}} \int_0^t \int_0^t \frac{x+\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4(t-\tau)}\right\} \varphi_0(\tau)d\tau dx \\ &\quad + \frac{1}{4\sqrt{\pi}} \int_0^t \int_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\} \varphi_0(\tau)d\tau dx. \end{aligned} \tag{16}$$

By the commutativity property in the integrals on the right-hand side of the formula (16), in the sense of the Dirichlet formula, we have

$$\int_0^t w_0(x,t)dx = \frac{1}{4\sqrt{\pi}}\int_0^t \varphi_0(\tau)d\tau \int_0^t \frac{x+\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4(t-\tau)}\right\} dx$$
$$+\frac{1}{4\sqrt{\pi}}\int_0^t \varphi_0(\tau)d\tau \int_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\} dx. \quad (17)$$

Let's calculate the following integrals: Denoting $y = \frac{(x+\tau)^2}{4(t-\tau)}$, then $dy = \frac{(x+\tau)dx}{2(t-\tau)}$, and we get

$$\begin{aligned}\frac{1}{4\sqrt{\pi}}\int_0^t \frac{x+\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4(t-\tau)}\right\} dx &= \frac{1}{2\sqrt{\pi(t-\tau)}}\int_{\frac{\tau^2}{4(t-\tau)}}^{\frac{(t+\tau)^2}{4(t-\tau)}} \exp\{-y\}\, dy \\ &= \frac{1}{2\sqrt{\pi(t-\tau)}}\left(\exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} - \exp\left\{-\frac{(t+\tau)^2}{4(t-\tau)}\right\}\right). \quad (18)\end{aligned}$$

Moreover, denoting $y = \frac{(x-\tau)^2}{4(t-\tau)}$, then $dy = \frac{(x-\tau)dx}{2(t-\tau)}$, and we get

$$\begin{aligned}\frac{1}{4\sqrt{\pi}}\int_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\} dx &= \frac{1}{2\sqrt{\pi(t-\tau)}}\int_{\frac{\tau^2}{4(t-\tau)}}^{\frac{t-\tau}{4}} \exp\{-y\}\, dy \\ &= \frac{1}{2\sqrt{\pi(t-\tau)}}\left(\exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} - \exp\left\{-\frac{t-\tau}{4}\right\}\right). \quad (19)\end{aligned}$$

Then in force (18) and (19) we have

$$\begin{aligned}\int_0^t w_0(x,t)dx &= \frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\left(\exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} - \exp\left\{-\frac{(t+\tau)^2}{4(t-\tau)}\right\}\right)\varphi_0(\tau)d\tau \\ &\quad +\frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\left(\exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} - \exp\left\{-\frac{t-\tau}{4}\right\}\right)\varphi_0(\tau)d\tau \\ &= \frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\left(2\exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} - \exp\left\{-\frac{(t+\tau)^2}{4(t-\tau)}\right\} - \exp\left\{-\frac{t-\tau}{4}\right\}\right)\varphi_0(\tau)d\tau \\ &= \frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\left[2\exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} - \exp\left\{-\frac{t-\tau}{4}\right\}\left(\exp\left\{-\frac{t\tau}{t-\tau}\right\}+1\right)\right]\varphi_0(\tau)d\tau \\ &= \hat{\lambda}_0(t)E(t). \quad (20)\end{aligned}$$

According to formulas (10)-(12) the solution of problem (6)-(8) is a nonnegative function. It should be noted that the function $E(t)$ from (9) also is a nonnegative function, since the integral (17) is nonnegative and the coefficient $\hat{\lambda}_0(t)$ is nonnegative function. Further from (20) we get

$$\int_0^t w(x,t)dx = C\int_0^t w_0(x,t)dx = C\hat{\lambda}_0(t)E(t) = \hat{\lambda}(t)E(t), \quad \hat{\lambda}(t) = C\hat{\lambda}_0(t), \quad C = const \neq 0,$$

find parameter $\hat{\lambda}_0(t)$ and then determine $\lambda(t)$ using the following formula:

$$\lambda(t) = -\frac{d\ln(\hat{\lambda}(t))}{dt} = -\frac{(\hat{\lambda}(t))'}{\hat{\lambda}(t)} = -\frac{(\hat{\lambda}_0(t))'}{\hat{\lambda}_0(t)} = \lambda_0(t).$$

Thus, the coefficient $\lambda_0(t) = \lambda(t)$ is determined uniquely. Then, using the transformation (5), we find the solution of the initial problem (1)-(4). We show, that the solution of the problem (1)-(4) is a bounded function with allowance for the class of function $\varphi(t)$ (13)-(14).

MAIN RESULT

Theorem 1 *Inverse problem (1)-(4) has a solution $\{u(x,t), \lambda(t)\}$, moreover $\lambda(t)$ is uniquely determined and the solution $u(x,t)$ up to a constant factor.*

CONCLUSION

In the paper we consider an inverse problem for the heat equation in a degenerating angular domain. We have shown that the inverse problem for the homogeneous heat equation with homogeneous initial and boundary conditions has a nontrivial solution $\{\lambda(t), u(x,t)\}$ consistent with the integral condition. It was also proved that the found nontrivial solution is a bounded function for $\forall\{t > 0,\ 0 < x < t\}$.

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