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On identities for eigenvalues of a well-posed perturbation of the Laplace operator in a punctured domain

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Abstract. In this paper we consider a class of finite perturbed well-posed problems for the Laplace operator. And, some identities for eigenvalues of the considering problems are found.

Keywords: Laplace operator, Well-posed solvable boundary value problem, Punctured domain, Dirichlet boundary condition, Finite perturbation, Identities for eigenvalues, Resolvent, Nonlocal type boundary value problem, Nuclear operator.

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INTRODUCTION

In 1936 Carleman [1] established the equality

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i(\lambda_i + p)} = \frac{1}{4\pi\sqrt{p}} \int_D \frac{d\omega_x}{\sqrt{\Delta(x)}} [1 + o(p)] \quad (1)$$

for a large positive p , where λ_i are the eigenvalues of the elliptic equation

$$\sum_{i=1}^3 \sum_{j=1}^3 a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^3 a_i \frac{\partial u}{\partial x_i} + au + \lambda u = 0$$

with the boundary condition $u = 0$, and $\Delta(x)$ is the determinant of the coefficients a_{ij} . And Grinberg [2] specified the equality (1) for the equation

$$\sum_{i=1}^3 \frac{\partial^2 u}{\partial x_i^2} + \lambda u = 0$$

with the same boundary condition. The specified equality has the following view

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i(\lambda_i + p)} = \frac{v}{4\pi\sqrt{p}} - \frac{S}{16\pi} \frac{\ln p}{p} + \frac{O(1)}{p},$$

where v is the volume of the domain, S is the square of the boundary.

In this paper, we are going to continue research of the chain of papers [3, 4], where classes of well-posed problems in a Hilbert space for differential operators in a punctured domain were considered and an analogue of the Green formula and classes of self-adjoint problems were obtained by the first author and his co-authors. Also, they derived regularized trace formulas for these operators. For further physical applications of the perturbed Laplace operator on a set of zero measure, see, for instance, the work [5] and references therein. Our aim is to obtain some identities for eigenvalues of well-posed problems for the Laplace operator in a punctured domain.

Let $\Omega_0 = \Omega \setminus \{M_0\}$, where $M_0(x_0, y_0)$ is a fixed inner point of $\Omega \subset \mathbb{R}^2$. It's convenient to consider the class of functions $\tilde{W}_2^1(\Omega)$ (for more details see [3]), which consists of functions $u(x, y) \in W_2^1(\Omega_0)$ and in the neighborhood of point $M_0(x_0, y_0)$ satisfies

$$\begin{aligned} \sup_{0 < \delta < \delta_0} \sup_{y_0 - \delta < \eta < y_0} \delta \left(\left| \frac{\partial u(x_0 + \delta, \eta)}{\partial \xi} \right| + \left| \frac{\partial u(x_0 - \delta, \eta)}{\partial \xi} \right| + |u(x_0 + \delta, \eta)| + |u(x_0 - \delta, \eta)| \right) &\leq C, \\ \sup_{0 < \delta < \delta_0} \sup_{x_0 - \delta < \xi < x_0 + \delta} \delta \left(\left| \frac{\partial u(\xi, y_0 - \delta)}{\partial \eta} \right| + \left| \frac{\partial u(\xi, y_0 + \delta)}{\partial \eta} \right| + |u(\xi, y_0 - \delta)| + |u(\xi, y_0 + \delta)| \right) &\leq C, \end{aligned}$$

and there exist functionals

$$\begin{aligned}\alpha(u) &= -\lim_{\delta \rightarrow +0} \left[\int_{y_0-\delta}^{y_0+\delta} \frac{\partial u(x_0+\delta, \eta)}{\partial \xi} - \frac{\partial u(x_0-\delta, \eta)}{\partial \xi} d\eta \right. \\ &\quad \left. + \int_{x_0-\delta}^{x_0+\delta} \frac{\partial u(\xi, y_0-\delta)}{\partial \eta} - \frac{\partial u(\xi, y_0+\delta)}{\partial \eta} d\xi \right] \\ &< \infty,\end{aligned}\tag{2}$$

$$\beta(u) = -\lim_{\delta \rightarrow +0} \int_{y_0-\delta}^{y_0+\delta} [u(x_0-\delta, \eta) - u(x_0+\delta, \eta)] d\eta < \infty,\tag{3}$$

$$\gamma(u) = -\lim_{\delta \rightarrow +0} \int_{x_0-\delta}^{x_0+\delta} [u(\xi, y_0-\delta) - u(\xi, y_0+\delta)] d\xi < \infty.$$

In this paper we consider the case when $\beta(u) = \gamma(u) = 0$ and we omit these functionals.

Let us consider the Dirichlet problem for the Poisson equation in $\tilde{W}_2^1(\Omega)$.

$$\Delta u(x, y) = f(x, y), (x, y) \in \Omega,\tag{4}$$

$$u(x, y)|_{\partial\Omega} = 0,\tag{5}$$

$$\alpha(u) = \alpha(K\Delta u),\tag{6}$$

where K is continuous operator in $L_2(\Omega)$ sense, which is mapping from $L_2(\Omega)$ to $\tilde{W}_2^1(\Omega)$.

If function $u(x, y)$ continuously depends on $f(x, y)$ by $L_2(\Omega)$ norm, then the linear bounded functional $\alpha(u)$ also will be linear bounded functional of $f(x, y)$ by the norm of $L_2(\Omega)$. Then by the Riesz theorem on general view of a linear bounded functional on the Hilbert space $L_2(\Omega)$ we can write

$$\alpha(Kf) = \int_{\Omega} \int \alpha(\xi, \eta) f(\xi, \eta) d\xi d\eta,\tag{7}$$

where $\alpha(\cdot, \cdot)$ is an arbitrary function from $L_2(\Omega)$ which is depending on the operator K and on the functional $\alpha(\cdot)$.

Using (4) and (7) we can rewrite condition (6) in the following form:

$$\alpha(u) = \int_{\Omega} \int \alpha(\xi, \eta) \Delta u(\xi, \eta) d\xi d\eta.\tag{8}$$

We denote by L_K an operator corresponding to the problem (4)-(5), (8). Then an operator L_0 corresponds to the operator L_K when $\alpha(\xi, \eta) = 0$ in the sense of $L_2(\Omega)$.

THE MAIN RESULT

The following theorem is the main result of the paper.

Theorem 1. *Let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues of the self-adjoint operator L_0 in ascending order. And let $\{\mu_k\}_{k=1}^{\infty}$ be the eigenvalues of operator (4)-(5), (8) arranged in ascending order of their absolute values with regard to multiplicities. Then for all $l \in \mathbb{Z}_+$ we have*

$$\sum_{k=1}^{\infty} \left(\frac{1}{\mu_k^{l+1}} - \frac{1}{\lambda_k^{l+1}} \right) = \sum_{i+j=l} \int_{\Omega} \int \alpha(\xi, \eta) (L_0)^{-i} (L_K)^{-j} G(\xi, \eta, x_0, y_0) d\xi d\eta,$$

where $G(x, y, \xi, \eta)$ is the Green's function of the following Dirichlet problem for the Laplace operator

$$\begin{aligned}\Delta u(x, y) &= f(x, y), (x, y) \in \Omega, \\ u(x, y)|_{\partial\Omega} &= 0.\end{aligned}$$

In particular, for $l = 0$ we get

$$\sum_{k=1}^{\infty} \left(\frac{1}{\mu_k} - \frac{1}{\lambda_k} \right) = \int_{\Omega} \int \alpha(\xi, \eta) G(\xi, \eta, x_0, y_0) d\xi d\eta.$$

Note that in paper [6] some interesting formulas are given for the Green function of the Dirichlet problem for the polyharmonic equation in the different cases.

Now we formulate several well-known results (see [3]).

Theorem 2. Let $f(x, y) \in L_2(\Omega)$, then the function

$$W(x, y, f) = \int_{\Omega} \int G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta - G(x, y, x_0, y_0) \int_{\Omega} \int \alpha(\xi, \eta) f(\xi, \eta) d\xi d\eta$$

is a unique solution in $\tilde{W}_2^1(\Omega)$ of nonlocal boundary value problem (4)-(5), (8).

Theorem 3. If K is a linear continuous operator, then the resolvent of the operator L_K is

$$(L_K - \lambda I)^{-1} f(x, y) = (L_0 - \lambda I)^{-1} f(x, y) - \alpha(KL_0(L_0 - \lambda I)^{-1} f(x, y)) L_K (L_K - \lambda I)^{-1} G(x, y, x_0, y_0). \quad (9)$$

Theorem 3 gives a form of the resolvent of L_K . In what follows, we need the following well-known statement (see [7, p. 96]).

Lemma 4. Let H is a linear bounded non negative operator. Then for any orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ of the space $L_2(\Omega)$ the sum

$$\sum_{k=1}^{\infty} (H\varphi_k, \varphi_k) \quad (10)$$

is the same (infinity or finite). Operator H is a nuclear if and only if the value of (10) is finite.

PROOF OF THE MAIN RESULT

First of all, let us prove the following lemma.

Lemma 5. The following statements are true:

- (i) operator $(L_K - \lambda I)^{-1} - (L_0 - \lambda I)^{-1}$ is a nuclear operator;
- (ii) if $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$ are eigenvalues of the operators L_0 and L_K , respectively, and if they are numbered with the occurring multiplicities in the ascending order, then for any real value $\lambda \in U = \{\lambda : |\lambda| < \min(\lambda_1, |\mu_1|)\}$ the formula

$$\text{Tr}((L_K - \lambda I)^{-1} - (L_0 - \lambda I)^{-1}) = \sum_{k=1}^{\infty} \left[\frac{1}{\mu_k - \lambda} - \frac{1}{\lambda_k - \lambda} \right] < \infty \quad (11)$$

is valid.

Proof. (i) From (9) we have

$$(L_K - \lambda I)^{-1} f(x, y) = (L_0 - \lambda I)^{-1} f(x, y) - \alpha(KL_0(L_0 - \lambda I)^{-1} f(x, y)) L_K (L_K - \lambda I)^{-1} G(x, y, x_0, y_0),$$

and

$$\begin{aligned} & \alpha(KL_0(L_0 - \lambda I)^{-1} f(x, y)) L_K (L_K - \lambda I)^{-1} G(x, y, x_0, y_0) \\ &= L_K (L_K - \lambda I)^{-1} G(x, y, x_0, y_0) \int_{\Omega} \int \alpha(\xi, \eta) L_0 (L_0 - \lambda I)^{-1} f(\xi, \eta) d\xi d\eta \\ &= L_K (L_K - \lambda I)^{-1} G(x, y, x_0, y_0) \left[\int_{\Omega} \int \Delta_{\xi, \eta} \alpha(\xi, \eta) (L_0 - \lambda I)^{-1} f(\xi, \eta) d\xi d\eta \right. \\ & \quad \left. + \int_{\partial\Omega} \int \alpha(\xi, \eta) \frac{\partial}{\partial n_{\xi, \eta}} (L_0 - \lambda I)^{-1} f(\xi, \eta) ds_{\xi, \eta} - \int_{\partial\Omega} \int \frac{\partial \alpha(\xi, \eta)}{\partial n_{\xi, \eta}} (L_0 - \lambda I)^{-1} f(\xi, \eta) ds_{\xi, \eta} \right]. \end{aligned} \quad (12)$$

Compute the trace of (12). It is obvious that

$$(L_0 - \lambda I)^{-1} f(x, y)|_{\partial\Omega} = 0,$$

then

$$\int_{\partial\Omega} \int \alpha(\xi, \eta) \frac{\partial}{\partial n_{\xi, \eta}} (L_0 - \lambda I)^{-1} f(\xi, \eta) d\xi d\eta - \int_{\partial\Omega} \int \frac{\partial \alpha(\xi, \eta)}{\partial n_{\xi, \eta}} (L_0 - \lambda I)^{-1} f(\xi, \eta) d\xi d\eta = 0.$$

Hence

$$\begin{aligned} & Tr(\alpha(KL_0(L_0 - \lambda I)^{-1} f(x, y)) L_K(L_K - \lambda I)^{-1} G(x, y, x_0, y_0)) \\ &= Tr\left(\int_{\Omega} \int \alpha(\xi, \eta) L_K(L_K - \lambda I)^{-1} G(x, y, x_0, y_0) L_0(L_0 - \lambda I)^{-1} f(\xi, \eta) d\xi d\eta\right) \\ &= \int_{\Omega} \int \alpha(\xi, \eta) L_0(L_0 - \lambda I)^{-1} L_K(L_K - \lambda I)^{-1} G(\xi, \eta, x_0, y_0) d\xi d\eta < \infty. \end{aligned}$$

Thus, the operator $(L_K - \lambda I)^{-1} - (L_0 - \lambda I)^{-1}$ is a nuclear operator.

(ii) $(L_0 - \lambda I)^{-1}$ is a linear bounded non-negative operator for real $\lambda \in U = \{\lambda : |\lambda| < \min(\lambda_1), |\mu_1|\}$. Then by Lemma 4 the sum

$$\sum_{k=1}^{\infty} ((L_0 - \lambda I) \varphi_k, \varphi_k)$$

has the same value for any orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ of $L_2(\Omega)$. If we take as an orthonormal basis the orthonormal basis of L_0 operator's eigenfunctions then we get

$$\sum_{k=1}^{\infty} ((L_0 - \lambda I) \varphi_k, \varphi_k) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda}.$$

Since, for any orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ we have

$$Tr((L_K - \lambda I)^{-1} - (L_0 - \lambda I)^{-1}) = \sum_{k=1}^{\infty} ((L_K - \lambda I) \varphi_k, \varphi_k) - \sum_{k=1}^{\infty} ((L_0 - \lambda I) \varphi_k, \varphi_k) = \sum_{k=1}^{\infty} ((L_K - \lambda I) \varphi_k, \varphi_k) - \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda},$$

or

$$\sum_{k=1}^{\infty} ((L_K - \lambda I) \varphi_k, \varphi_k) = Tr((L_K - \lambda I)^{-1} - (L_0 - \lambda I)^{-1}) + \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda}.$$

By using the biorthogonal basis from the root functions of the operator L_K instead of the orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$, we take (11). Lemma 5 is proved. $\square$

Proof of Theorem 1. By Lemma 5 we get

$$\sum_{k=1}^{\infty} \left[\frac{1}{\mu_k - \lambda} - \frac{1}{\lambda_k - \lambda} \right] = - \int_{\Omega} \int \alpha(\xi, \eta) L_0(L_0 - \lambda I)^{-1} L_K(L_K - \lambda I)^{-1} G(\xi, \eta, x_0, y_0) d\xi d\eta. \quad (13)$$

For all real $\lambda \in U = \{\lambda : |\lambda| < \min(\lambda_1), |\mu_1|\}$ we take

$$\frac{1}{\lambda_k - \lambda} = \frac{1}{\lambda_k} \left(\frac{1}{1 - \frac{\lambda}{\lambda_k}} \right) = \frac{1}{\lambda_k} \left(1 + \frac{\lambda}{\lambda_k} + \left(\frac{\lambda}{\lambda_k} \right)^2 + \dots + \left(\frac{\lambda}{\lambda_k} \right)^n + \dots \right), \quad (14)$$

$$\frac{1}{\mu_k - \lambda} = \frac{1}{\mu_k} \left(\frac{1}{1 - \frac{\lambda}{\mu_k}} \right) = \frac{1}{\mu_k} \left(1 + \frac{\lambda}{\mu_k} + \left(\frac{\lambda}{\mu_k} \right)^2 + \dots + \left(\frac{\lambda}{\mu_k} \right)^n + \dots \right), \quad (15)$$

$$L_0(L_0 - \lambda I)^{-1} = (I - \lambda(L_0)^{-1})^{-1} = I + \lambda(L_0)^{-1} + \lambda^2(L_0)^{-2} + \dots + \lambda^n(L_0)^{-n} + \dots, \quad (16)$$

$$L_K(L_K - \lambda I)^{-1} = (I - \lambda(L_K)^{-1})^{-1} = I + \lambda(L_K)^{-1} + \lambda^2(L_K)^{-2} + \dots + \lambda^n(L_K)^{-n} + \dots. \quad (17)$$

By using expansions (14)-(17) and (13), we have

$$\sum_{l=0}^{\infty} \left[\sum_{k=1}^{\infty} \left(\frac{1}{\mu_k^{l+1}} - \frac{1}{\lambda_k^{l+1}} \right) \right] \lambda^l = \sum_{l=0}^{\infty} \left[\sum_{i+j=l} \int_{\Omega} \int \alpha(\xi, \eta) (L_0)^{-i} (L_K)^{-j} G(\xi, \eta, x_0, y_0) d\xi d\eta \right] \lambda^l.$$

This proves Theorem 1. $\square$

We note papers [8–14] in which the related topics with the interesting applications are studied.

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REFERENCES

1. T. Carleman, *Berichte der Sachsisch. Akad. der Wiss. zu Leipzig, Math. Phys. Klasse* **LXXXVIII**, 119 (1936).
2. S. I. Grinberg, *Russian Mathematical Surveys* **58**, 97 (1953), (in Russian).
3. B. E. Kanguzhin, and A. A. Aniyarov, *Mathematical Notes* **89**, 819 (2011).
4. B. E. Kanguzhin, D. B. Nurakhmetov, and N. E. Tokmagambetov, *Russian Mathematics* **58**, 6 (2014).
5. D. A. Zubok, and I. Yu. Popov, *Theoretical and Mathematical Physics* **119**, 629-639 (1999).
6. T. Sh. Kal'menov, B. D. Koshanov, and M. Yu. Nemchenko, *Complex Variables and Elliptic Equations* **53**, 177-183 (2008).
7. I. C. Gohberg, and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, American Mathematical Society, 1969.
8. B. Kanguzhin, N. Tokmagambetov, and K. Tulenov, *Complex Variables and Elliptic Equations* **60**, 107 (2015).
9. M. Ruzhansky, and N. Tokmagambetov, *International Mathematics Research Notices* (2016), (DOI: 10.1093/imrn/rnv243).
10. T. Sh. Kal'menov, and N. E. Tokmagambetov, *Siberian Mathematical Journal* **54**, 1023 (2013).
11. D. Suragan, and N. Tokmagambetov, *Sib. Elektron. Mat. Izv.* **10**, 141 (2013).
12. M. A. Sadybekov, and B. T. Torebek, *AIP Conference Proceedings* **1676**, 020073 (2015).
13. N. Tokmagambetov, and G. Nalzhupbayeva, *AIP Conference Proceedings* **1676**, 020098 (2015).
14. B. Kanguzhin, and N. Tokmagambetov, *Trends in Mathematics* **63**, 235 (2014).