Annotation

This tutorial is a summary of the basis of actuarial mathematics. It is made on the basis of a special course, which the author reads on the Mechanics and Mathematics Department of KazNU.

The tutorial contains a theoretical material with the questions for self-examination, a set of problems for self-solving and an exam program.

The tutorial is intended for senior undergraduate students of specialty "Mathematics". It can be used by the students of other specialties of Mechanics and Mathematics Department. The manual is also recommended to the students of economic specialties, wishing to learn the basic methods of insurance mathematics.

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**INTRODUCTION TO LIFE INSURANCE**

Actuaries apply scientific principles and techniques from a range of other disciplines to problems involving risk, uncertainty and finance.

The first actuaries were employed by life insurance companies in the early eighteenth century to provide a scientific basis for mana­ging the companies’assets and liabilities. The liabilities depended on the number of deaths occurring amongst the insured lives each year. The modelling of mortality became a topic of both commercial and ge­neral scientific interest, and it attracted many significant scientists and mathematicians to actuarial problems, with the result that much of the early work in the field of probability was closely connected with the development of solutions to actuarial problems.

The earliest life insurance policies provided that the policyhol­der would pay an amount, called the **premium**, to the insurer. If the named life insured died during the year that the contract was in force, the insurer would pay a predetermined lump sum, the **sum insured**, to the policyholder or his or her estate. So, the first life insurance con­tracts were annual contracts. Each year the premium would in­crea­se as the probability of death increased. If the insured life became very ill at the renewal date, the insurance might not be renewed, in which case no benefit would be paid on the life’s sub­se­quent death. Over a large number of contracts, the premium income each year should approximately match the claims outgo. This method of matching income and outgo annually, with no attempt to smooth or balance the premiums over the years, is called **assessmentism**. This method is still used for group life insurance, where an employer purchases life insurance cover for its employees on a year-to-year basis.

The radical development in the later eighteenth century was the level premium contract. The problem with assessmentism was that the annual increases in premiums discouraged policyholders from rene­wing their contracts. The level premium policy offered the poli­cyholder the option to lock-in a regular premium, payable perhaps weekly, monthly, quarterly or annually, for a number of years. This was much more popular with policyholders, as they would not be pri­ced out of the insurance contract just when it might be most needed. For the insurer, the attraction of the longer contract was a greater likelihood of the policyholder paying premiums for a longer period. However, a problem for the insurer was that the longer contracts were more complex to model, and offered more financial risk. For these contracts then, actuarial techniques had to develop beyond the year-to-year modelling of mortality probabilities. In particular, it be­came necessary to incorporate financial considerations into the mo­del­ling of income and outgo. Over a one-year contract, the time value of money is not a critical aspect. Over, say, a 30-year contract, it becomes a very important part of the modelling and management of risk.

Another development in life insurance in the nineteenth century was the concept of **insurable interest**. This was a requirement in law that the person contracting to pay the life insurance premiums should face a financial loss on the death of the insured life that was no less than the sum insured under the policy. The insurable interest requi­rement disallowed the use of insurance as a form of gambling on the lives of public figures, but more importantly, removed the incentive for a policyholder to hasten the death of the named insured life. Sub­sequently, insurance policies tended to be purchased by the insured life, and we use the convention that the policyholder who pays the premiums is also the life insured, whose survival or death triggers the payment of the sum insured under the conditions of the contract.

The earliest studies of mortality include life tables constructed by John Graunt and Edmund Halley. A life table summarizes a sur­vi­val model by specifying the proportion of lives that are expected to sur­vive to each age. Using London mortality data from the early se­venteenth century, Graunt proposed, for example, that each new life had a probability of 40% of surviving to age 16, and a probability of 1% of surviving to age 76. Edmund Halley, famous for his astro­no­mical calculations, used mortality data from the city of Breslau in the late seventeenth century as the basis for his life table, which, like Graunt’s, was constructed by proposing the average («medium» in Halley’s phrase) proportion of survivors to each age from an arbi­trary number of births. Halley took the work two steps further. First, he used the table to draw inference about the conditional survival probabilities at intermediate ages. That is, given the probability that a newborn life survives to each subsequent age, it is possible to infer the probability that a life aged, say, 20, will survive to each sub­se­quent age, using the condition that a life aged zero survives to age 20. The second major innovation was that Halley combined the mo­r­tality data with an assumption about interest rates to find the value of a whole life annuity at different ages. A whole life annuity is a con­tract paying a level sum at regular intervals while the named life (the annuitant) is still alive. The calculations in Halley’s paper bear a re­mar­kable similarity to some of the work still used by actuaries in pen­sions and life insurance.

**1**

**PROBABILITY FOR THE AGE-AT-DEATH**

**1.1. The Survival Function**

Let us consider a newborn child.

This newborn's age-at-death *X* is a continuous-type random va­riab­le.

Let *F*(*x*) denote the distribution function (d.f.) of *X*:

*F*(*x*) = *P*(*X* ≤ *x*), *x* ≥ 0, (1.1.1)

and let

*s*(*x*) = 1 - *F*(*x*) = *P*(*X* > *x*), *x* ≥ 0. (1.1.2)

We always assume that

*F*(0) = 0,

which implies

*s*(0) = 1.

D e f i n i t i o n 1.1.1. The function *s*(*x*) is called a *survival function* (s.f.)

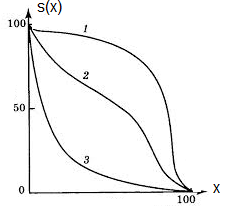
For any positive *x*, *s*(*x*) is the probability a new-born will attain age *x*.

The distribution of *X* can be defined by specifying either the func­tion *F*(*x*) or the function *s*(*x*). Within actuarial science and de­mo­­graphy, the survival function has traditionally been used as a star­ting point for further development. Within probability and statistic, the distribution function usually plays this role. However, from the properties of the distribution function, we can deduce corresponding properties of the survival function.

P r o p e r t i e s o f *s*(*x*):

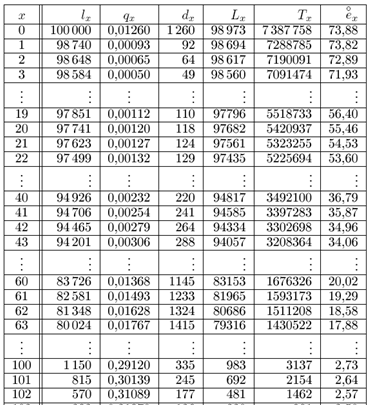
1. *s*(*x*) decreases;
2. *s*(0) = 1, *s*(∞) = 0;
3. *s*(*x*) is continuous.

F i g u r e 1.1.1. Survival Functions (examples)



**1.2. Life Tables**

T a b l e 1.2.1 (Example of Life Tables):



We have presented an excerpt from a typical life table in the table 1.2.1. In such a table the column *lx* denote the number of lives which have survived to age *x*. To make sense of this, we have to assume a starting population *l*0. In our case *l*0 = 100,000, but any value would have sufficed.

Note that in table 1.2.1, *l*1 = 98,740. This means that 100,000 - 98,740 = 1,260 lives have died in the first year of life.

*Relation of Life Table Function lx to the Survival Function.* Consider the large group *l*0 (=10,000; 100,000 etc.) of newborns. Let *Х*1, …, are their ages-аt-death.

Let *I*(*A*) is the indicator of the event *A*. The number of members of this group, who have lived to *x* age, is a random variable

*L*(*x*) = *I*(*Хi* > *x*),

*lx*= *E* *L*(*x*) =  *l*0 *s*(*x*).

Thus,

*lx*= *l*0 *s*(*x*).

It follows that:

1. the value *lx* depends on the age *x* like *s*(*x*) up to a constant *l*0;
2. *s*(x) = is an average proportion of surviving to age *x* in this group.

**1.3. Curve of Death**

We introduce for the intervals (*х*, *х* + *t*) a random variable

*tDx*= *L*(*x*) – *L*(*x* + *t*).

*tDx*is the number of individuals who died between the ages of *x* to *x* + *t* from fixed group of *l*0 newborns. An expectation of this random variable is one of the basic Life Table values

*tdx* = *E tDx*= *lx*- *lx*+*t*= *l*0(*s*(*x*) - *s*(*x* + *t*)),

here *s*(*x*) - *s*(*x* + *t*) = *P*(*x* < *Xi*≤ *x* + *t*) is a probability of death in the interval (*x*; *x* + *t*].

N o t e. Index 1 is normally not written, so

1*dx*= *dx*.

D e f i n i t i o n. The function

*f*(*x*) = *F* '(*x*) = - *s*'(*x*)

(the distribution density of the random variable *X*) is called *the curve of deaths*.

We can show: the value *dx* depends on the age *x* like the curve of deaths *f*(*x*) up to a constant *l*0:

*dx* ≈ *l*0*f*(*x*).

1.4. Force of Mortality

The symbol (*x*) is used to denote a *life-aged-x*.

Find the conditional probability that (*x*) will die between the ages of *x* to *x* + *t*:

*Р*(*х* < *X* ≤ *x* + *t* / *X* > *x*) = =

=  = .

If the value *t* is small or *f*(*ξ*) varies slowly in the interval (*x*, *x* + *t*), then we can show that

*Р*(*х* < *X* ≤ *x* + *t* / *X* > *x*) ≈  ≡ *t* *μx*.

D e f i n i t i o n. The function *μx*=  is called the *force of mortality*.

N o t e. The value *tμx* is approximately equal to the probability of death of a people aged *x* years in the interval (*x*, *x* + *t*).

The approximation of *μx*in a Life Table is the value *qx*:

*qx* =  == *μx*.

P r o p e r t i e s o f *μx*:

1. *μx*≥ 0.
2. .

P r o o f. We have from conditions *s*(∞) = 0, *s*(0) = 1:



1. *s*(*x*) = , *F*(*x*) = 1 - .

P r o o f. The equality

*μu* = - *s*'(*u*)/*s*(*u*)

is the differential equation. We have:

*μudu* = - *s*'(*u*)*du*/*s*(*u*) = - *ds*(*u*)/*s*(*u*).

We obtain 3) by integrating of both sides of this equality.

**1.5. Expectation-of-life**

*Expectation-of-life.* The value *e*00 (expectation-of-life) is the average life expectancy of a randomly selected newborn. We have

*e*00= E*X* = 

Expectation-of-life is one of the most important indicators of the quality of life in different countries.

*Variance-of-life.*The value V*X* (variance-of-life) is the scatter of the life expectancy of a randomly selected newborn around *e*00. We have

V*X* = E(*X* - *e*00)2 = E*X*2 – (*e*00)2 = - (*e*00)2.

*Asymmetry* of the random variable *Х* is the value



If γ > 0 (< 0), we have a long right (left) tail of the distribution of *X*.

*Excess* of the random variable *Х* is the value



If κ ≈ 0, then the distribution of *X* is close to normal (we have κ = 0 for *N*-distribution).

I m p o r t a n t n o t e. It can be hown that the expression of the expectation-of-life in terms of the survival function takes place:

E*X* =  = .

Likewise, we have for E*X*2:

E*X*2 = 2= 2

**⇒** V*X* = 2 - (*e*00)2.

**1.6.** **Analytical laws of mortality**

*De Moivre's model* (1729)*.* It is assumed that a lifetime is dis­tri­buted uniformly on the interval (0, *ω*), where *ω* is the Limiting age. We have for this model (0 ≤ *x* < *ω*):

*f*(*x*) = 1/*ω*,

*F*(*x*) = *x*/*ω*,

*s*(*x*) = 1 - *x*/*ω*,

*μx*= *f*(*x*)/*s*(*x*) = 1/(*ω* - *x*).

De Moivre's law does not take into consideration many cha­rac­teristics associated with lifetime. For example, in this model a curve of death is the segment (0, *ω*) of a horizontal line, but an empirical curves of death have a maximum in the region of 80 years.

*Gompertz's model* (1825)*.* In this model a force of mortality is given by the formula (*x* ≥ 0, *α* > 0, *B* > 0)

*μx*= *f*(*x*)/*s*(*x*) = *Веαх*.

– Survival function is

*s*(*x*) = *exp*(-) = *exp*(-) = *exp*(- *B*(*еαx*– 1)/*α*).

–Curve of death is

*f*(*x*) = *μxs*(*x*) = *B exp*(*αx* - *B*(*еαx*– 1) /*α*).

It has a maximum at *х* = (ln*α* - ln*B*)/*α*. This information can be used to find the estimators of the parameters *α* and *B*. So, if we know that the mortality maximum was observed for the age of 78.3 years, and the quartile of a Gompertz distribution function *х*0.25 = 33.4 years, then the system of equations for the estimation of the parameters *α* and *B* takes the form



This system can be solved by numerical methods.

*Makeham's model* (1860)*.* Here a force of mortality is given by the more general formula (*x* ≥ 0, *B* > 0, *A* ≥ - *B*, *α* > 0)

*μx* = *А* + *Веαx*,

where the constant *А* has been interpreted as a mortality from accidents, and the term *Веαx* as mortality from aging. Here we have

*s*(*x*) = *ехр*(- *Ах* - *В*(*еαx*– 1)/*α*),

*f*(*x*) = (*А* + *Веαх*)*ехр*(- *Ах* - *В*(*еαx*– 1)/*α*).

*Weibull's model* (1939)*.* Here the force of mortality is given by the formula (*x* ≥ 0, *k* > 0, *n* > 0)

*μx* = *kxn*,

*s*(*x*) = *exp*(-) = *exp*(-),

*f*(*x*) = *kxn* *exp*(-),

where *f*(*x*) has the maximum at the point *х* =  .

N o t e. The support for simple analytic survival functions has declined in recent years. Many feel that the belief in universal laws of mortality is naive. With the increasing speed and storage capacity of computers, the advantages of some analytic forms in computation involving more than one life are not longer of great importance. Nevertheless, some interesting research has recently reiterated the biological arguments for analytic laws of mortality.

**Questions for self-test to the Chapter 1**

1. What is a random variable *X*?
2. What is a distribution function of lifetime *F*(*x*)?
3. What is a survival function *s*(*x*)?
4. What is the relationship between *F*(*x*) and *s*(*x*)?
5. What are the properties of *s*(*x*)? Write down.
6. Curve of Death *f*(*x*): as it relates to *F*(*x*)?
7. What is force of Mortality μx? Do a conclusion.
8. Please list the properties of μx.
9. What is the Life Tables?
10. Life Table function *lx*: what is its relation to a survival function *s*(*x*)?
11. Life Table function *dx*: what is its relation to a curve of death *f*(*x*)?
12. Life Table function *qx*: what is its relation to a force of mortality *μx*?
13. Give the definition of an Expectation-of-Life.
14. Please express an expectation-of-life in terms of a survival function.
15. Give the definition of a Variance-of-life.
16. Please express a variance-of-life in terms of a survival function.
17. Please define the de Moivre's model: *f*(*x*), *F*(*x*), *s*(*x*), *μx*.
18. What are the disadvantages of the de Moivre's model?
19. Please define the Gompertz's model: *μx*, *s*(*x*), *f*(*x*).
20. Please define the Makeham's model: *μx*, *s*(*x*), *f*(*x*).

**2**

**TIME-UNTIL-DEATH**

2.1. Time-until-Death for a Person Age *x*

The random variable *T*(*X*) = *X* - *x* is called the *Time-until-Death* for a person age *x*.

The conditional probability that a newborn will die between the ages *x* and *x* + *t*, given survival to age *x*, is

*Fx*(*t*) = P(*T*(*x*) ≤ *t*) = P(*X* – *x* ≤ *t* / *X* > *x*) = P(*X* ≤ *x* + *t* / *X* > *x*) = *tqx*=

===.

If we know the tabulated values *lx*and *lx*+*t*, then *Fx*(*t*) is

*Fx*(*t*) = ==.

The distribution density *fx*(*t*) of random variable *Т*(*х*) is

*fx*(*t*) =  *Fx*(*t*) = , 0 ≤ t < ∞.

2.2. Values associated with *Т*(*х*)

To make probability statements about *T*(*x*), we use the notations:

1. The probability that (*x*) will die within *t* years (the d.f. of *T*(*x*)) is

*tqx* = Р(*Т*(*х*) ≤ *t*) = 

2. The probability that (*x*) will attain age *x* + *t* (the s.f. of *T*(*x*)):

*tрx* = Р(*Т*(*х*) > *t*) = 1 – *tqx* = 

In the special case of a life-at-0, we have *T*(*x*) = *X* and

*xр*0 = Р(*X* > *x*) = *s*(*x*), *x* ≥ 0.

N o t e. We have

1*qx* ≡ *qx* = P{(*x*) will die within 1 year};

1*px* ≡ *px* = P{(*x*) will attain age *x* + 1}.

Thus,the probability that (*x*) will die within 1 years is

*qx* = Р(*Т*(*х*) ≤ 1) **= **

Similarly,

*px* = 1 – *qx* = Р(*Т*(*х*) < 1) **= **

3. There is a special symbol for the more general event that (*x*) will survive *t* years and die within the following *u* years; that is, (*x*) will die between ages *x* + *t* and *x* + *t* + *u*. This special symbol is given by

*t*/*uqx* = Р(*t* < *Т*(*х*) ≤ *t* + *u*) = Р(*Т*(*х*) ≤ *t* + *u*) - Р(*Т*(*х*) ≤ *t*) =

= *t*+*uqx* - *tqx* = (1 - *t+upx*) – (1 –  *tpx*) = *tpx* – *t*+*upx*.

We have

*t*/*uqx* = 

As before, if *u* = 1, the prefix is deleted in *t*/*uqx* and we have

*t*/1*qx* = *t*/ *qx* = *t*+1*qx* - *tqx* = *tpx* – *t*+1*px*.

We have

*t*/ *qx* = ****

**2.3. Complete-expectation-of-life**

The expected value of *T*(*x*), denoted by *е*0*х*, is called the *complete-expectation-of-life*:

*е*0*х*= Е*Т*(*х*).

It can be shown that

*е*0*х*= .

N o t e. The complete-expectation-of-life at various ages is often used to compare levels of public health among different populations.

We have

Е*Т*(0) = Е*Х* = *е*00,

then, *е*00 > *е*0*х*for any *х* > 0.

By transforming the last integral, we obtain the expression

*е*0*х*== = .

Further, it can be shown that

Е(*Т*(*х*))2 = 2= .

This result is useful in the calculation of V*Т*(*х*) by

V*Т*(*х*) = Е(*Т*(*х*))2 - (Е*Т*(*х*))2 = - (*е*0*х*)2.

N o t e. The value

*ex* = 

is called the *curtate expectation* (we will consider it below) and counts only full years of future life.

Other characteristic of the distribution of *Т*(*х*) can be de­ter­mi­ned. The *median future lifetime* of (*x*), to be denoted by *m*(*x*), can be found by solving

P{T(x) > m(x)} = 1/2

for m(x). In particular, *m*(0) is given by solving *s*(*m*(0)) = 1/2.

**2.4. Partly time-until-death**

Let's consider an *n*-year endowment insurance. It pays if the in­su­red dies within n years, or pays if insured survives n years.

The moment of the insurance payment given by the formula min(*T*(*x*), *n*) and is called the *partly time-until-death*, and the corres­pon­ding expectation is called *n-year temporary complete life ex­pe­c­tation* of (*x*) and denoted by :

= E min(*T*(*x*), *n*).

It can show that *n*-year temporary complete life expectation can be calculated as

= .

A variance of partly time-until-death can be calculated as

V min(*T*(*x*), *n*) = - ()2.

**2.5. Curtate-future-lifetime**

A discrete random variable associated with the future lifetime is the number of future years completed by (*x*) prior to death. It is called the *curtate-future-lifetime* of (*x*) and is denoted by *K*(*x*):

*К*(*х*) = [*Т*(*х*)].

Because *K*(*x*) is the greatest integer in *T*(*x*), it is a discrete random variable ant its distribution law is (k = 0. 1, 2,…)

Р(*К*(*х*) = *k*) = Р(*k* ≤ *T*(*х*) < *k*+1) = Р(*k* < *T*(*х*) ≤ *k*+1) =

== *kpx*– *k*+1*px* = *k*/ *qx*.

The switching of inequality is possible since, under our as­sump­tion that *Т*(*х*) is a continuous-type random variable, *P*(*T*(*х*) = *k*) =   
= *Р*(*T*(*х*) = *k*+1) = 0.

N o t e. As *Х* = *Т*(0), then we can determine the *curtate-lifetime*

*К*(0) = [*X*].

Its distribution law is (k = 0, 1, 2,…)

Р(*К*(0) = *k*) = = *s*(*k*) – *s*(*k*+1) = .

The expected value of the random variable *К*(*х*) is denoted by *ex*and called the *curtate-expectation-of-life*:

*ex*≡ E*K*(*x*).

It is calculated by the formula:

*ex*= *k*P(*K*(*x*) = *k*) = *s*(*x* + *k*) = *lx*+*k* = 

Further,

V*K*(*x*) = *ks*(*x* + *k*) - *ex*- *ex*2.

**Questions for self-test to the Chapter 2**

1. What is a random variable *T*(*X*)?
2. Please derive the distribution of *T*(*x*).
3. What is an expression of *Fx*(*t*) in terms of *s*(*x*)?
4. What is an expression of *Fx*(*t*) in terms of *lx*?
5. Please write down the distribution density of a random variable *T*(*x*).
6. Values associated with *Т*(*х*): *tqx*, *tрx*: what do they mean?
7. How the values *tqx* and *tрx* are related to each other?
8. What is an expression of *tqx* through *s*(*x*)?
9. What is an expression of *tрx* through *s*(*x*)?
10. The value t/uqx: what does it mean?
11. What is an expression of t/uqx through *s*(*x*)?
12. Please define a Complete-expectation-of-life.
13. What is its calculation via *tpx* and *s*(*x*).
14. What is the value *К*(*х*)?
15. Please express *К*(*х*) in terms of *Т*(*х*).
16. What is a Curtate-expectation-of-life?

**3**

**FRACTIONAL AGES**

**3.1. Assumptions for the Fractional Ages**

To specify the distribution of *T*(*х*), we must postulate an analytic form or adopt a Life Table and assumption about the distribution bet­ween integers.

We will examine three assumptions that are widely used in actuarial science. These will be stated in term of survival function and in a form to show the nature of interpolation over the interval (*n*, *n* + 1) implied by each assumption. In each statement, *n* is an integer and 0 ≤ *x* ≤ 1. The assumptions are the following:

– *Linear interpolation*:

*s*(*x*) = *s*(*n*)(*n* + 1) - *s*(*n* + 1)*n* + (*s*(*n* + 1) - *s*(*n*))*x*, *n* ≤ *x* ≤ *n* + 1.

This is known as the *uniform distribution* or, perhaps more properly, a uniform distribution of deaths within each year of age. Under this assumption *s*(*x*)is a linear function.

– *Exponential interpolation*, or linear interpolation on ln *s*(*x*):

*s*(*x*) = *s*(*n*)*рnx - n*, *n* ≤ *x* ≤ *n* + 1.

This is consistent with the assumption of a *constant force of mortality* within each year of age. Under this assumption *s*(*x*) is an exponential function.

– *Harmonic interpolation*:

*s*(*x*) = 

This is what is known as the *hyperbolic* assumption, or *Balducci* assumption, for under it *s*(*x*)is a hyperbolic function.

*Uniform distribution of deaths.* We have

*s*(*x*) = *an*+ *bnx* (*n* ≤ *x* ≤ *n* + 1).

We can take *s*(*n*) and *s*(*n* + 1) from the Life Table and construct the equations

*an*+ *bnn* = *s*(*n*),

*an*+ *bn*(*n* + 1) = *s*(*n*+1).

We are finding *an*and *bn* from these equations:

*bn*= *s*(*n* + 1) - *s*(*n*),

*an*= *s*(*n*) - (*s*(*n* + 1) - *s*(*n*))*n* = *s*(*n*) - *s*(*n* + 1)*n* + *s*(*n*)*n* =

= *s*(*n*)(*n* + 1) - *s*(*n* + 1)*n*.

N o t e. *bn* < 0 because *s*(*x*) decreases.

So, the function *s*(*х*) is approximated by the linear spline on the intercept *n* ≤ *x* ≤ *n* + 1:

*s*(*x*) = *s*(*n*)(*n* + 1) - *s*(*n* + 1)*n* + (*s*(*n* + 1) - *s*(*n*))*x*.

The curve of death is

*f*(*x*) = - *s*'(*x*) = *s*(*n*) - *s*(*n* + 1) = - *bn*, *n* < *x* < *n* + 1.

The force of mortality is

*μх*= , *n* < *x* < *n* + 1.

We can write *μх*through a Life Tables values (we divide the numerator and denominator of the last expression by *s*(*n*)):

*μх*= .

N o t e. We have:

a) *f*(*x*) = const;

b) *μ'х*> 0, i.e. the force of mortality increases between the interpolation points (*n* < *x* < *n*+1);

c) both values *f*(*x*) and *μх* are not defined at an integer points.

*Constant force of mortality.* The function *s*(*x*) is approximated on the intercept *n* ≤ *x* ≤ *n*+1 by the decreasing exponential function *an*. We have

*an* = *s*(*n*),

*an* = *s*(*n* + 1).

By help to dividing the second equation by the first one and taking the logarithm of the both sides of this equation we obtain

- *bn*= ln,

then

*an*= *s*(*n*)= *s*(*n*),

i.е.

*bn*= - ln *pn*, *an*= *s*(*n*)

and

*s*(*x*) = *s*(*n*)*рnx - n*, *n* ≤ *x* ≤ *n* + 1,

*f*(*x*) = - *s*'(*x*) = - *s*(*n*)*рnx-n* ln*pn*, *n* < *x* < *n* + 1,

*μх*=  - ln*pn*, *n* < *x* < *n* + 1.

N o t e. We have: *μх*= const between the interpolation points.

*Balducci assumption.* In this case we have the harmonic in­terpolation. We interpolate the function 1/*s*(*x*) by the linear function.

We have in the intercept *n* ≤ *x* ≤ *n* + 1:

*s*(*x*) = 

*f*(*x*) = - *s*′(*x*) = , *n* < *x* < *n* + 1,

*μх*= , *n* < *x* < *n* + 1.

**3.2. Distribution of Fractional Age**

Let *τ* = {*x*} ({⋅} is fractional part). Then a random variable *X* (age-аt-death) can be represented by the sum of integer and fractional parts: *Х* = *К*(0) + *τ*. The value *τ* describes the time of death within a year.

Find the conditional distribution of τ under the condition that the death occurred at the age of *n* years:

Р(*τ* ≤ *t* / *K*(0) = *n*) = Р(*X* - *K*(0) ≤ *t* / *K*(0) = *n*) =

= Р(*X* ≤ *t* + *n* / *n* ≤ *X* < *n*+1) =  =

= = , 0 < t < 1.

The value *s*(*n*+*t*) (more precisely *ln*+*t*)for *n* ∈ Z and 0 < *t* < 1 is absent in a Life Tables. Therefore, we use the approximations for fractional ages to find it.

*Uniform distribution of deaths:*

*F*(*t*⏐.) = Р(*τ* ≤ *t* / *K*(0) = *n*) =   
= =

= 0 < *t* < 1;

= *F* ′(*t*⏐.) = 1;

*s*(*t*⏐.) = 1 – *F* ′(t⏐.) = 1 - *t*;

*μt*⏐. = 

Thus,

а) a death is equiprobable on any day between two human birthdays;

b) the conditional distribution Р(*τ* ≤ *t* / *K*(0) = *n*) doesn't depend on *n* and coincides with the unconditional distribution Р(*τ* ≤ *t*);

c) the random variables *K*(0) and *τ* are independent.

*Constant force of mortality:*

*F*(*t*⏐.) = Р(*τ* ≤ *t* / *K*(0) = *n*) =   
=

 0 < *t* < 1;

*f*(*t*⏐.) =

*s*(*t*⏐.) = 1 - 

*μt*⏐. =

We can get the similar formulas for the Balducci assumption.

**3.3. Expectation and Variance for Fractional Age**

Find an expectation and a variance of the fractional age with the condition that the human died at the age of *n* years.

*a*(*n*) = E(*τ*⏐*K*(0) = *n*) = 

We have

P(*τ* > *t*⏐*K*(0) = *n*) = 1 - P(*τ* ≤ *t*⏐*K*(0) = *n*) = 1 - 



hence

*a*(*n*) = 

*b*(*n*) = V(*τ*⏐*K*(0) = *n*) =  
=

We have

*Uniform distribution of deaths:*

*a*(*n*) = 1/2, *b*(*n*) = 1/12.

*Constant force of mortality:*

*a*(*n*) =  *b*(*n*) ≈ 1/12.

*Balducci assumption:*

 *b*(*n*) ≈ 1/12.

N o t e. In practice, formulas are used (if *K*(0) and *τ* are inde­pendent):

*ex*0 ≈ *ex*+ 1/2,

V*T*(*x*) ≈ V*K*(0) + 1/12.

**Questions for self-test to the Chapter 3**

1. What is the value *T*(*x*)?
2. For which the value *T*(*x*) is needed?
3. Which assumptions about the distribution of *T*(*x*) do we have?
4. Uniform distribution of deaths: which form of survival function do we have in this case?
5. Uniform distribution of deaths: write down a curve of deaths and force of mortality.
6. What does this mean: μ'х > 0 between the interpolation points in the case of uniform distribution of deaths?
7. Why the values f(x) and μх are not defined at an integer points?
8. Constant force of mortality: which form of survival function do we have in this case?
9. Balducci assumption: which form of survival function do we have in this case?
10. What is the conditional distribution of a random variable τ (Fractional Age) under the condition that the death occurred at the age of *n* years (in general case)?
11. Conditional distribution of a random variable τ (Fractional Age) under the condition that the death occurred at the age of *n* years in case of uniform distribution of deaths: *F*(*t*⏐.), *f*(*t*⏐.), *s*(*t*⏐.), *μt*⏐.
12. Expectation of the fractional age under the condition that the human died at the age of *n* years (value *a*(*n*)): please, write down a formula.
13. Variance of the fractional age under the condition that the human died at the age of *n* years (value *b*(*n*)): please, write down a formula.

**4**

**MULTY-LIFE THEORY**

**4.1. Joint-Life Status**

Let (*x*1, ..., *xm*) be *m* independent lives of ages *x*1, ..., *xm* wish to enter into an insurance contract. Denote by

*Т*(*хk*) = *X* - *хk*

a time-until-death of a person № *k*.

Set up a correspondence

*Т*(*х*1),…, *Т*(*хm*) → *U*,

where *U* is the status, to which corresponds its lifetime *T*(*U*).

Two the most common statuses are:

– Joint-Life Status;

– Last-Survivor Status.

*Joint-Life Status.* It is a status that survives as long as all mem­bers of a set of lives survive and fails upon the first death. It is deno­ted by

*U* = (*х*1: … : *хm*),

where *хi* is the age of member № *i* and *m* is the number of members.

We have

*T*(*U*) = min{*Т*(*х*1),…, *Т*(*хm*)}.

The survival function of this status is

Р(*T*(*U*) > *t*) = P(min{*Т*(*х*1),…, *Т*(*хm*)} > *t*) = P(*Т*(*х*1) > *t*,…, *Т*(*хm*)} > t)

and (under the condition of independence of deaths)

Р(*T*(*U*) > *t*) = .

Other probabilistic characteristics of lifetime for *T*(*U*) are:

– Distribution function:

= 1 - = .

– Distribution density:

Р(*T*(*U*) *>* *t*) = = .

**4.2. Last-Survivor Status**

It is a status that exists as long as at least one member of a set of lives is alive and fails upon the last death. It is denotes by

*U* = 

where *хi* is the age of member № *i* and *m* is the number of members.

We have

*T*(*U*) = max{*Т*(*х*1),…, *Т*(*хm*)}.

The distribution function of this status is

= *Р*(*T*(*U*) ≤ *t*) = Р(max{*Т*(*х*1),…, *Т*(*хm*)} ≤ *t*) =

= Р(*Т*(*х*1) ≤ *t*,…, *Т*(*хm*) ≤ *t*)

and (under the condition of independence of deaths)

= 

– Survival function:

= 1 - .

– Distribution density:

 Р(*T*(*U*) ≤ *t*) = .

**Questions for self-test to the Chapter 4**

1. What is a Status?
2. What are two the most common statuses?
3. As long a Joint-Life Status survives?
4. Please deduce the formula for Survival function of a Joint-Life Status.
5. Which is the kind of a distribution function of a Joint-Life Status?
6. Which is the kind of a distribution density of a Joint-Life Status?
7. Distribution of a Joint-Life Status.
8. Please deduce the formula for a distribution function of a Last-Survivor Status.
9. Which is the kind of a survival function of a Joint-Life Status?
10. Which is the kind of a distribution density of a Last-Survivor Status?

**5**

**MULTIPLE DECREMENTS**

Up to now we have been assuming that, except for retirement at a certain age, death is the only cause of decrement acting on a body of lives. Consider the case of an employer, however, where disability or withdrawal would be other reason for terminating employment. The insurance company covering claims for the employer would want to treat death and disability separately, since the amounts of the claims would be different. It might also be true that, in some situa­tions, different causes of death should be analyzed separately.

Multiple decrement theory is the area of mathematics which deals with these kinds of problems; it allows us to study each kind of decrement individually and to draw conclusions from the results.

Let us assume we have *m* causes of decrement acting indepen­dently on a body of lives. Our original *lх* will now be written *lх*(*τ*), the total number of lives attaining age *x*.

Now if (1), (2), ..., (*m*) are the *m* causes of decrement, we denote by *dх*(*k*) the number of decrement from cause (*k*) between ages *x* and *x*+1. Also, *dх*(τ) is the total number of decrements from all causes, so we have



We also have



*qх*(*k*) is the probability that a life aged *x* will leave within one year because (*k*), so we have



*qх*(*τ*) and *рх*(*τ*) are analogous to our old *qх*and *рх*, so we have









The *central rate of decrement from all causes* at age *x*, assuming uniform distribution of decrements, is



The *central rate of decrement from cause* (*k*) is given by



**Questions for self-test to the Chapter 5**

1) What is a Multiple Decrements?

2) What are the values *dх*(*k*), *dх*(*τ*)? Which is the relationship between them?

3) Values *qх*(*k*). Whether there is a value *pх*(*k*)? Why?

4) Central rate of decrement from all causes at age *x* – what is it?

5) Central rate of decrement from cause (*k*) – what is it?

**6**

**LIFE ANNUITIES**

**6.1. Life Annuities. The main types**

*Pure endowment.* We deduce a general formula for this case.

Assume, that a unit of money is to be paid *t* years from now to an individual currently aged *x*, if the individual survives to that time. The value of this payment at the present time is equal to (*i* is the current annual interest rate)

*tEx* = (*tpx*)(1 + *i*)-*t* ≡ *ν t tpx*. (6.1.1)

The present value of a pure endowment is also called the *net single premium* for the pure endowment.

*Life annuities.* A more common type of situation is called a *life annuities.*

We will assume constant payments of 1 per year for as long as the individual is alive, with the first payment due at the end of the year:



The symbol for the present value of this payments to a life aged *x* is *ах*, and the formula is

*ах* = (1 + *i*)-1*px* + (1 + *i*)-2 2*px* + … =

= (6.1.2)

The present value (6.1.2) is also called the *net single premium* for the annuities.

*Temporary life annuity.* Perhaps a life annuity has payments which will end after a certain period *n*:



A *temporary life annuity* which will only continue for a maxi­mum of *n* years is denoted  and the formula is

 (6.1.3)

*n-year deferred life annuity* is one in which the first payment to a person now aged *x* doesn't occur until age *x* + *n* + 1:



This is denoted by *n*⏐*аx*, and the formula is

*n*⏐*аx* =  (6.1.4)

N o t e. Annuity *n*⏐аx can be thought of as omitting the first *n* payments from *аx*, so we have

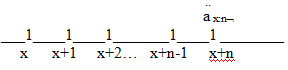
*n*⏐*аx* = *аx* -  . (6.1.5)

As with interest-only annuities, there are *annuities-prenu­me­ran­do* and *annuities-postnumerando*. denotes a life annuity whose first payment occurs immediately:













Expression of annuities prenumerando through annuities postnumerando:

= 1 + *ах*,

= 1+ 

=

**6.2. Life Annuities Payable *m*thly**

*Life Annuities Payable mthly.*In practice life annuities are often payable more frequently than once a year, with monthly being a very common frequently. Consider life annuities payable *m*thly.Here we convert the annual rate in the monthly rate.

Let *ax*(*m*) is the present value of an postnumerando life annuity to a life aged *x* where each early payment of 1 is divided into *m* evenly spaced payments of 1/*m* each, the first due at age *x*+1/*m*:



We have

1) Life annuities:

*ax*(*m*) = 1/*mpx* ν1/*m* +2/*mpx* ν2/*m* + … ≈ *ax*+ ;

1/*m* + *ax*(*m*)  ≈ 1/*m* + *ax*+  = - 1+ = - .

2) Deferred life annuities:

*n*/ *ax*(*m*) = *ax*+*n*(*m*)*nEx*≈ *n*/ *ax* +  *nEx*;

 *nEx* ≈ (- ) *nEx*= *n*/  -  *nEx*.

3) Temporary life annuities:





*Continuous* life annuity:

; 

**6.3. Varying Life Annuities**

*Increasing life annuity* (*Ia*)*x*. Let (*Ia*)*x* denote the present value at age *x* of a life annuity, first payment in one year, having payments of 1, 2, 3,...:



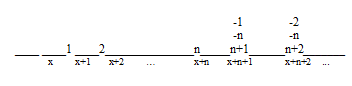
We have:

(*Ia*)*x* = *ах* + 1/ *ах* + 2/ *ах* + … + *n*/ *ах* + …

*Increasing life annuity with a term of n year* (*Ia*)*x*:*n*⎤ :



or



Then

(*Ia*)*x*:*n*⎤ = (*Ia*)*x* – *n**n*/ *ax*- *n*/ (*Ia*)*x*.

*Decreasing life annuity with a term of n yea.* 

We have

(*DA*)*x*:*n*⎤ = (*n* + 1) *ax*:*n*⎤ - (*Ia*)*x*:*n*⎤.

N o t e. The concept of "decreasing annuity" is only used for temporarylife annuity.

**6.4. Annual premiums and premium reserves**

*Annual premiums.*One possible way to pay for a deferred life annuities would be with a *series* of premium payment rather than a net single premium. Most commonly the premium payments would continue for the length of the deferred period, but other situations are possible. Problems of this type should be treated as equation of value: the present value of the two annuities (premiums and benefits) should be set equal to each other.

Let *tP*()be the annual premium, payable at the beginning of each year for *t* years, for *n*-year deferred life annuity-prenumerando, then the equation of value is:

*tP*()=,

hence,

*tP*() =  / . (6.4.1)

*Premium reserves*. This is analogous to the concept of «out­stan­ding principal» (see «Financial Math»), so the reserve is just the pre­sent value of all future benefits minus the present value of all future premiums.

Consider the reserve after *t* years of an *n*-year deferred life an­nuity of 1 which is being paid for by annual premiums for *n* years, of size*nP*() each.

The symbol for the reserve after *t* years is .

Consider two cases.

1. If *t* ≥ *n*, then all premiums have been paid. In this case, the re­serve will just equal the value at age *x* + *t* of all future benefits, which is :

 = , *t* ≥ *n*.

2) If *t* < *n*, then the present value of all future benefits is , and the present value of future premiums is equal to *nP*(). Hence

 =  - *nP*(), *t* < *n*.

*Gross premium.* Let as realized that when calculating premiums for deferred annuities, there are some practical aspects. For example, there may be expenses involved to the insurer in underwriting the risk, in issuing the contract, and in the continuing administration of the account. The premium which is the actually charged in a business transaction, including expenses and other costs, is called the *gross premium*, and the amount by which the gross premium exceeds the net premium is called *loading.*

**Questions for self-test to the Chapter 6**

1. What is pure endowment*tEx*?
2. Please define a Life annuity *ах*: diagram, formula.
3. Please define a Temporary life annuity *ax:n*⎤: diagram, formula.
4. Please define a n-year deferred life annuity *n*⏐*аx*: diagram, formula.
5. What is the relationship between *аx*,*ax:n*⎤ and*n*⏐*аx*?
6. Life annuity payable *m*thly: definition, diagram.
7. What is an expression of a life annuity payable *m*thly (*ax*(*m*)) through a life annuity *ax*?
8. What is an expression of a deferred life annuity payable *m*thly (*n*/*ax*(*m*)) through a life annuity *n*/ *ax*?
9. What is an expression of a temporary life annuity payable *m*thly(*ax:n*⎤(*m*)) through a life annuity *ax:n*⎤?
10. Please define an increasing life annuity (*Ia*)*x*.
11. An increasing life annuity with a term of n year (*IA*)*x:n*⎤ - what is it?
12. A decreasing life annuity with a term of n year (*DA*)*x:n*⎤ - what is it?
13. Why the concept of "decreasing annuity" is only used for temporarylife annuity?

**7**

**LIFE INSURANCE**

**7.1. Life insurance (basic concept)**

*Whole life policy Ах.* It is a policy where a fixed amount, the *face value*, is paid to the insured's beneficiary at the end of the year of death, whenever that may be. The price of such a policy with face value of 1, to an insured aged *x*, is given by the symbol *Ах*. The for­mula is:

*Ах* =  (7.1.1)

N o t e. Eventually *tpx*= 0, so this sum is actually finite.

*Term insurance* *A*1*x*:*n*⎤. In some cases a company will sell *term* insurance, which means that the face value is paid only if death occurs within a prescribed period. If the period is *n* years and the insured is aged *x*, then the price is denoted (for a payment of 1), and the formula is

*A*1*x*:*n*⎤  =  (7.1.2)

*Deferred insurance n/ Ax.* The price for *deferred insurance*, where the policy of face amount 1 is purchased at age *x* but doesn't come into force until age *x* + *n*. Policy price is

*n*/*Ax* = *Ах*+*nν n**npx*. (7.1.3)

N o t e. We have the relationship between (7.1.1) - (7.1.3):

*Ах* = *A*1*x*:*n*⎤  + *n*/ *Ax*.

*n*-year endowment insurance *Ax*:*n*⎤. In this case the face value is paid if death occurs within a prescribed *n*-year period or, if the policyholder is still alive at the end of *n* years, he receives the face value at the time. The price for this benefit, with face value 1, is denoted *Ax*:*n*⎤ and is the sum of *n*-year term insurance and a pure endowment (see (6.1.1)) at age *x* + *n*. Hence we have

*Ax*:*n*⎤ = *A*1*x*:*n*⎤ + *nEx*= ⏐ *nEx*≡ *Ax*:*n*1⎤ ⏐= *A*1*x*:*n*⎤ + *Ax*:*n*1⎤.

N o t e. In this context the symbol  is sometimes used in place of *nEx*.

N o t e (relationship between whole life policy *Ах*and life annuity ). We have (*d* is an annual discount rate)

*А х* =

= ν –  = ν – ( – 1) =

=⏐(1 + *i*)(1 - *d*) = 1⏐= 1 - *d* . (7.1.4)

**7.2. Insurance payable at the moment of death**

Until now, we have always assumed that insurance is payable at the end of the year following death. In practice, however, this is often not the case and it is more common for insurance to be payable at the moment of death.

We first consider the case where a payment of 1 is due at the end of the (1/*m*)th part of year in which death occurs. The net single premium for this insurance to a life aged *x* is given by

*Ax*(*m*) = .

Taken the limit of this expression as *m* approaches infinity, we obtain the net single premium for insurance payable at the moment of death. Denoting this by , we have

.

N o t e. 

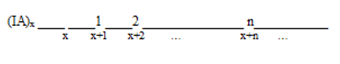
Expressions like (term insurance payable at the moment of death),  (*n*-year endowment insurance payable at the mo­ment of death) and so on, all exist and have the expected meanings. We have





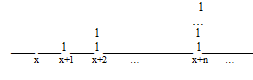
**7.3. Varying insurance**

*Increasing whole life insurance* (*IA*)*x*. This policy takes into account inflation:



This is a policy which provides death benefits of 1 in the first year, 2 - in the second year, and so on, increasing by 1 per year, pa­yable at the end of the year of death. It can be represented in the form of deferred policies:

We have:

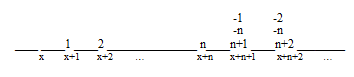


(*Ia*)*x*= *Ax* + 1/ *Ax*+2/ *Ax*+ …

*Increasing term insurance* (*IA*)1*x*:*n*⎤. This police refers to an in­creasing policy with a final benefit of *n* in the *n*th year:



It can be represented in the form



We have

(*IA*)1*x*:*n*⎤ = (*IA*)*x* - *n n/ Ax* - *n /* (*IA*)*x*.

*Decreasing term insurance* (*DA*)1*x*:*n*⎤. The first payment is *n* and payments decrease annually to a final benefit of 1 in the *n*th year. Nothing than

(*DA*)1*x*:*n*⎤ + (*IA*)1*x*:*n*⎤ = (*n* + 1)*A*1*x*:*n*⎤ .

Hence we find (*DA*)1*x*:*n*⎤.

**7.4. Annual premiums and reserves**

*Annual premiums.*The basic idea is that the present value of all future premium payments should equal the present value of all future benefits. Since the first payment will almost always occur at the time the policy is taken out, basic equation of value will have the general form

*Р*= *А*,

where *А* is some insurance symbol (whole life, term, varying, etc), and  is some annuity symbol (life, temporary, etc).

If a whole life policy is to be paid for by a life annuity, the symbol is *Рх* and we have

*Рх* = *Ах*/ x.

If *n*-year term insurance is being purchased with *n* annual payments contingent on survival, we have

*P*1*x*:*n*⎤ = *A*1*x*:*n*⎤ /

*The reserve* is defined to be the excess of the present value of future benefits over the present value of future premiums. For a whole life policy of 1, issued to a person aged *x* and payable by net premiums at the beginning of each year, the net level premium reserve *t* years later is given by

*tVx* = *Ax*+*t* – *Px*.

The reserve given by this formula is called the policy year *terminal reserve*, which means that *t* is assumed to be an integer, so that *tVx* exists at the end of a policy year.

There are a number of useful formulae which can be developed for reserves. For example, using *Ax*+*t*= 1-d*х**x*+*t* we obtain

*tVx* = 1 - (*Px* + *d*)

Using *Px* = *Ax* /*x* = 1/ *x* - *d*, we have the very useful formula

*tVx* = 1 - /.

**Questions for self-test to the Chapter 7**

1. What is a whole life policy Ах?
2. Why the sum in the whole life annuity formula is actually finite?
3. What is a term insurance A1x:n⎤.
4. What is a difference in the formulas of a whole life policy and a term insurance?
5. What is a deferred insurance n/Ax?
6. n-year endowment insurance Ax:n⎤.
7. Please write down an expression of n-year endowment insurance through A1x:n⎤ and a pure endowment.
8. Increasing whole life insurance (IA)x: what is it?
9. What this type of insurance takes into account?
10. Increasing term insurance (IA)1x:n⎤: what is it?
11. Please express (IA)1x:n⎤ through others policies.
12. Decreasing term insurance (DA)1x:n⎤: what is it?
13. What is the relationship between (IA)1x:n⎤ and (DA)1x:n⎤?
14. Is there an endless decreasing term insurance?

**8**

**MULTI-LIFE THEORY**

**8.1. Joint-life status**

We define life annuity and life insurance functions for a joint-life situation.

Let (*x*1), … , (*xm*) are *m* individuals aged *x*1, …, *xm*.

A life annuity paying 1 at the end of each year as long as *m* lives, aged *x*1, … , *xm*, all survive has present value denoted .

Let is the probability of *m* independent lives all surviving for a *t* years. We have

 = .

Hence,

 (8.1.1)

A life insurance policy paying 1 at the end of the year in which the first death occurs in a group of *m* lives, aged *x*1, … , *xm*, has present value denoted . We have

 (8.1.2)

N o t e.  is the present value of an insurance that pays on the death of (*x*), if (*x*) dies before (*y*), and  pays on the death of (*x*), if (*x*) dies after (*y*). Note that

.

Other joint-life functions can be defined analogously to their single-life counterparts. For example, the average lifetime of the group (*x*1), …, (*xm*) is

*ex* = 

The probability that at least one of the group would not live another year

.

Further,



;



N o t e. The probability  that at least one of *m* people will die over the next *n* years, is not equal to the product of the probabilities . Eg., for two people it is (according to the formula of addition of probabilities)

.

**8.2. Last survivor problems**

An annuity may continue as long as at least one person is alive. This is called a *last survivor* situation.

We let  denote the probability that at least one of a group of *m* lives, aged *x*1, …, *xm* will survive for *n* years. We have

 = 1 - .

Then the present value of an annuity on which 1 is paid at the end of year, while at least one of the insured is alive, is

. (8.2.1)

N o t e. The special case of two lives gives



and



In the case of insurance, we have

 (8.2.2)

**8.3. Reversionary annuities**

A reversionary annuity is an annuity which begins at the end of the year of death of given individual, now aged *y*. The payments are made to a second individual, now aged *x*, as long as he is alive.

Note that *x* must be alive when *y* dies for payments to begin. The notation for this annuity is *ах*/*у*, and a formula for its present value is easy derived.

Clearly a payment occurs at time *t* if, and only if, *y* is dead but *x* is still alive.

Hence

*ах*/*у* = = *ах*- *аху*, (8.3.1)

where *ах*is life annuity for (*x*), *аху* is joint-life annuity for (*х*) and (*у*).

Formula (8.3.1) makes sense. Our reversionary annuity is an annuity to *x*, which has present value *ах*, but will not pay during the joint lifetime of *x* and *y*, so we must subtract *аху*.

*Reversionary Annuities Payable m*thly *а*(*m*)*х*/*у*. We have

*а*(*m*)*х*/*у*= *а*(*m*)*х*- *а*(*m*)*ху*≈ *ах*+ (*m*-1)/(2*m*) – (*аху*+ (*m*-1)/(2*m*)) =

= *ах*- *аху*. (8.3.2)

**Questions for self-test to the Chapter 8**

1. Joint-life status: what is it?
2. In what case a life annuity for a joint-life status will be paid?
3. In what case a life insurance policy for a joint-life status will be paid?
4. What is *A*1*xy*?
5. Last survivor status: what is it?
6. In what case a life annuity for a last survivor status will be paid?
7. In what case a life insurance policy for a last survivor status will be paid?
8. Reversionary annuitiy: what is it?
9. Reversionary annuitiy payable *m*thly: what is its expression through a reversionary annuitiy?

**PROBLEMS**

**Chapter 1**

1) Consider the married couple: Jeanette is 30 years old and John is 37 years old. Find the probability that:

a) both Jeanette and John live for 30 more years;

b) at least one of them doesn't live for 30 more years. Use the Life Table.

2) Consider the married couple: Jenny is 35 years old and Bill is 33 years old. Find the probability that they live at least 40 years. Use the Life Table.

3) Pierre is 20 years old, John and Mike are 30-year-old twins. Find the probability that:

a) all three live 30 years;

b) at most one doesn't live for 30 years.

4) Lydia is 22 years old, Jeffrey and John are 28-year-old twins. Find the probability that:

a) at least one doesn't live for 30 years;

b) Lydia reaches age 45 and exactly one of the twins lives for 25 years.

5) Find the probability that a 30-year-old man reaches age 60 years. Use the Life Table.

6) Find the probability that 65-year-old man will die between ages 70 and 75. Use the Life Table.

7) Find the probability that a 50-years-old woman will die between ages 70 and 80. Use the Life Table.

8) Find the probability that a 45-years-old woman dies within 10 years. Use the Life Table.

9) The function

*f*(*x*) = *cx*2exp(-*x*3) (0 ≤ *x* < ∞), *f*(*x*) = 0 (*x* < 0)

is the curve of deaths. Find the constant *c*.

10) Given:

*s*(*x*) = 1/(1+*x*)2.

Find:

а) the force of mortality for a 40-year-olds;

b) the probability that (39) dies within 10 years.

11) The lifetime of the (35) is described by de Moivre's law with the limiting age *ω* = 100. Find the probability that this person lives for at least 25 years.

12) Some population consists of 90 people aged 80 years; *q*80 = 0,95. Find the probability that at least two of them will reach ages 81.

13) Mary, Diana and Saule are 72, 73, 74 years old (respectively). We have:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *x* | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 |
| *lx* | 1 000 | 952 | 879 | 786 | 656 | 612 | 425 | 359 | 286 |

Find the probability that all the women will die within 3 years.

14) Ruslan, Michael and Serik are 72, 73, 74 years old (respectively). We have:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *x* | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 |
| *lx* | 1 000 | 952 | 879 | 786 | 656 | 612 | 425 | 359 | 286 |

Find the probability that all the men live for at least 4 years.

**Chapter 2**

1) The survival function is given by the formula

*s*(*x*) = (1 - *x*/100)1/2, 0 ≤ *x* ≤ 100.

Find:

а) the force of mortality for 30 years;

b) the probability that (40) will die between ages 60 and 65.

2) Given: *μх* = 0,003 for 60 ≤ *х* ≤ 70. Find the probability that (65):

а) reaches ages 66;

b) lives for at least 3 years;

c) dying between age 67 and age 69.

3) Given:

*lx* = *а* + *bx*, *l*100= 0.

Find the complete-expectation-of-life of (50).

4) Let *μх* = 0,001 for 30 ≤ *x* ≤ 33. Find 2/ *q*30.

5) Given: *рх*= 0,9 ∀ *х*. Find the probability that (30) will die between ages 55 and 60.

6) Given: *qх*= 0,1 ∀ *х*. Find the probability that (40) will die between ages 45 and 50.

7) Given: *pх*= 0,8 ∀ *х*. Find the probability that (60) will die between ages 65 and 70.

8) Given: *μх* = 0,001 for 50 ≤ *х* ≤ 60. Find the probability that (50):

а) reaches ages 51;

b) dies within one year;

c) dies between ages 55 and 58.

9) Given: *μх* = 0,002 for 55 ≤ *х* ≤ 60. Find the probability that (55):

а) reaches age 56;

b) dies within one year;

c) dies between ages 56 and 58.

10) The survival function is given by the formula

*s*(*x*) = (1 - *x*/90)1/2, 0 ≤ *х* ≤ 90.

Find:

а) the force of mortality for a 40-year-olds;

b) the probability that (50) will die between ages 60 and 70.

11) Find the probability that a 55-years-old woman will die between ages 68 and 70.

12) The survival function is given by the formula

*s*(*x*) = exp(-*x*2).

Find:

а) the force of mortality for a 50-year-olds;

b) the probability that (50) dies within 5 years.

13) Given:

*lx* = *а* + *bx*, *l*80= 0.

Find the curtate-expectation-of-life for (70).

**Chapter 3**

1) Find the probability that (50) will die between ages 50,5 and 51. Assume the uniform distribution of deaths.

2) Find the probability that (50) will die between ages 50,5 and 51. Assume the constant force of mortality.

3) Find the probability that (50) will die between ages 50,5 and 51. Use the Balducci assumption.

4) Find the probability that (58) will die between ages 58.1 and 58.2. Assume the uniform distribution of deaths.

5) Find the probability that (60) will die between ages 70.25 and 70.75. Assume the constant force of mortality.

6) Find the probability that (72) will die between ages 75.5 and 76.5. Use the Balducci assumption.

7) Find the probability that (70) will die between ages  and . Assume the constant force of mortality.

8) Find the probability that (60) will die between ages  and . Assume the uniform distribution of deaths.

**Chapter 4**

1) Assuming that T(70), T(75) and T(80) are independent, expressed by *lx* the probability that:

a) the first death will occur in the interval from 5 to 8 years;

b) the last death will occur in the interval from 5 to 8 years.

1. Assuming that T(50), T(55) and T(60) are independent, expressed by *lx* the probability that:

a) the first death will occur in the interval from 10 to 15 years;

b) the last death will occur in the interval from 10 to 15 years.

**Chapter 5**

1) Consider a population in which 4 decrements are acting, namely (a), (b), (c), (d). Write an expression for the probability that 65-year-old:

а) will leave the population because (a) between ages 67 and 68;

b) will leave the population because (b), (c) or (d) between ages 65 and 67.

2) Consider a population in which 4 decrements are acting, namely (a), (b), (c), (d). Write an expression for the probability that 56-year-old:

а) will leave the population because (b) between ages 65 and 66;

b) will leave the population because (a), (c) or (d) between ages 60 and 63.

3) Consider a population in which 3 decrements are acting, namely (a), (b), (c). Write an expression for the probability that 50-year-old:

а) will leave the population because (a) between ages 59 and 60;

b) will leave the population because (b) or (c) between ages 55 and 58.

4) Consider a population in which 3 decrements are acting, namely (a), (b), (c). Write an expression for the probability that 60-year-old:

а) will leave the population because (a) between ages 65 and 66;

b) will leave the population because (b) or (c) between ages 65 and 68.

5) Consider a population in which 3 decrements are acting, namely (a), (b), (c). Write an expression for the probability that 66-year-old:

а) will leave the population because (a) within one year;

b) will leave the population because (b) or (c) between ages 69 and 73.

**Chapter 6**

1) Find the net single premium for a 30-year pure endowment of 20 000 sold to a male aged 40, in each of the following cases:

a) *l*40 = 95 000, *l*70 = 25 000;

b) *px* = 0.9 for all *x*;

c) *px* = 0.9 if 0 ≤ *x* < 50 and *px* = 0.8 if 50 ≤ *x* < 90.

Assume *i* = 0.1.

2) Find the net single premium for a 30-year pure endowment of 20 000 sold to a male aged 40, in each of the following cases:

a) *l*x = *l*0(1 - *x*/100);

b) *l*x = (90 - *x*)1/3.

Assume *i* = 0.12.

3) Find the net single premium for a 30-year pure endowment of 200 sold to a female aged 45, in each of the following cases:

a) *l*45 = 94 000, *l*75 = 24 000;

b) *px* = 0.95 for all *x*;

c) *px* = 0.95 if 0 ≤ *x* < 55 and *px* = 0.85 if 55 ≤ *x* < 90.

Assume *i* = 0.11.

4) Find the net single premium for a 30-year pure endowment of 200 sold to a female aged 45, in each of the following cases:

a) *l*x = *l*0(1 - *x*/120);

b) *l*x = (100 - *x*)1/3.

Assume *i* = 0.1.

5) John can have $500 000 if alive at age 50, or $*X* today. Find *X*. Assume: John is 25 years old, *i* = 0.1; *l*25 = 98 000; *l*50 = 85 000.

6) Irene, aged 25, purchases a contract for 3 payments of 10 000 each at ages 35, 40 and 45, if she is alive. Find the net single premium for this contract. Assume *l*x = *l*0(1 - *x*/90), *i* = 8%.

7) Find the net single premium for a life annuity of 10 000 per year, with the first payment due in one year, sold to a 40-year-old. Assume *px* = 0.95 for all *x*, *i* = 0.08.

8) Find the net single premium for a life annuity of 5 000 per year, with the first payment due in one year, sold to a 35-year-old. Assume *l*x = 10 000(1 - *x*/100), *i* = 7.5%.

9) Find the net single premium for a deferred life annuity of

8 000 per year, sold to a 38-year-old, if the first payment is deferred until age 43. Assume *px* = 0.97 for all *x*, *i* = 0.09.

10) Find the net single premium for a deferred life annuity of

7 500 per year, sold to a 32-year-old, if the first payment is deferred until age 38. Assume *l*x = 1000(1 - *x*/120), *i* = 7%.

11) Paul is 55 years old. How much money must be invested to provide Paul with monthly payments of 100 for life? The 1st payment will occur in exactly one month. We have *a*55 = 20.

12) Anna is 45 years old. How much money must be invested to provide Anna with the payments of 200 every 4 months for life? The 1st payment will occur in exactly 4 month. We have *a*45 = 25.

13) Jack is 60 years old. How much money must be invested to provide Jack with monthly payments of 150 for life? The 1st payment occurs immediately. We have *a*60 = 15.

14) Henry is 40 years old. How much money must be invested to provide Henry with the payments of 180 every 4 months for life? The 1st payment occurs immediately. We have *a*40 = 30.

15) Brian is 45 years old. He buys an annuity of *Y* per month, the 1st payment to be made immediately. For the first 48 months, payments will be guaranteed. After that, payments will continue for as long as he is alive. The cost of this annuity is 45 000. Find *Y* (express in terms of a life and financial annuities).

16) Irwin is 55 years old. He buys an annuity of *Z* per month, the 1st payment to be made immediately. The payments will continue for as long as he is alive. The cost of this annuity is 55 000, *i* = 6%. Find *Z* (express in terms of life annuities).

17) Olga is 63 years old, she is about retired. Her salary is

100 000 per year. She will receive a pension of 45% of her final salary at the end of each year as long as she is alive. Find the present value of this benefits (express in terms of a life annuities).

18) Sylvia is 60 years old, she is about retired. Her salary is

120 000 per year. She will receive a pension of 40% of her final salary at the end of each month as long as she is alive. Find the present value of this benefits (express in terms of a life annuities).

19) Laura is 43 years old. Her retirement benefit (1/2 of her salary) beginning at age 58. Laura's current salary is 90 000 per year and her salary will increase at 10% each year for the next 15 years. We assume that Laura will not die before age 58 and *i* = 8%. Find the present value to Laura of her future retirement benefits (express in terms of life annuities).

20) Mike, aged 35, purchases a life annuity which will pay 1500 in one year's time, with annual payments that will increase by 300 per year thereafter. Find the present value of this annuity (express in terms of life annuities).

21) Alice, aged 40, purchases a life annuity which will pay 1500 in one year's time, with annual payments that will increase by 200 per year thereafter, a maximum of 15 payments is to be made. Find the present value of this annuity (express in terms of a life annuities).

**Chapter 7**

1) Find the price of whole life insurance with a face value 120 sold to a person aged 35, if *i* = 0.07; *qx* = 0.05 for all *x*.

2) Find the price of whole life insurance with a face value 500 sold to a person aged 45, if *i* = 0.11; *lx* = 10 000(1 - *x*/100).

3) Find the price of whole life insurance with a face value 150 sold to a person aged 50 is to be made at the end of the 5-year period in which death occurs, if *i* = 0.08; *qx* = 0.06 for all *x*.

4) Find the price of whole life insurance with a face value 150 sold to a person aged 50 is to be made at the end of the 5-year period in which death occurs, if *i* = 0.12; *lx* = 10 000(1 - *x*/110).

5) Find the price of a term policy for a 30 year with a face value 120 sold to a person aged 40, if *i* = 0.07; *qx* = 0.04 for all *x*.

6) Find the price of a term policy for a 30 year with a face value 120 sold to a person aged 42, if *i* = 0.14; *lx* = 1 000(1 - *x*/110).

7) Find the price of a 30-year endowment policy with a face value 250 sold to a person aged 42, if *i* = 0.08; *qx* = 0.05 for all *x*.

8) Find the price of a 30-year endowment policy with a face value 200 sold to a person aged 40, if *i* = 0.12; *lx* = 1 000(1 - *x*/115).

9) Find the price of a 40-year endowment policy with a face value 100 sold to a person aged 30, if *i* = 0.1; *l*x = 1 000(1 - *x*/100).

10) Find the net single premium for a 300 life insurance policy, payable at the moment of death, to a (42), if *i* = 0.095, *tp*42 = 0.95*t* ∀*t*.

11) Find the net single premium for a 500, if the (45) is purchasing 20-year term insurance policy, payable at the moment of death. Given: *i* = 0.09, *tp*40 = 0.92*t* ∀*t*.

12) Find the net single premium for a 200, if the (50) is purchasing 25-year endowment insurance policy, payable at the moment of death. Given: *i* = 0.08, *tp*50 = 0.9t ∀*t*.

**Chapter 8**

1) Two lives are both governed by the formula *l*x = 100 - *x*, 0 ≤ *x* ≤ 100. Find *μ*50+t:60+*t*.

2) Two lives are both governed by the formula *l*x = 1000(1-*x*/110), 0 ≤ *x* ≤ 110. Find *μ*40+*t*:45+*t*.

3) Three lives are governed by the formula *l*x = 10 000(1 - *x*/115), 0 ≤ *x* ≤ 115. Find *μ*35+*t*:40+*t*:45+*t*.

4) A life annuity pays 100 while both lives survive, 30 if the first life only survives, 25 if the second life only survives. If the age are 40 and 50, find the present value in terms of annuity functions.

5) An 20-year temporary annuity pays 1000 while both lives survive, 350 if the first life only survives, 300 if the second life only survives. If the age are 45 and 50, find the present value in terms of annuity functions.

**EXAM PROGRAM**

**1. The Survival Function**

1. Distribution function of lifetime *F*(*x*). Survival function *s*(*x*).
2. Properties of *s*(*x*).
3. Curve of Death *f*(*x*).
4. Force of Mortality *μx*.
5. Properties of *μx*.

**2. Life Tables**

1. Life Tables.
2. Life Table function *lx*. Its relation to a survival function *s*(*x*).
3. Life Table function *dx*. Its relation to a curve of death *f*(*x*).
4. Life Table function *qx*. Its relation to a force of mortality *μx*.

**3. Expectation-of-life**

1. Expectation-of-life.
2. Expression of an expectation-of-life in terms of a survival function.
3. Variance-of-life.
4. Expression of a variance-of-life in terms of a survival function.

4. Analytical laws of mortality

1. De Moivre's model: *f*(*x*), *F*(*x*), *s*(*x*), *μx*.
2. Disadvantages of the de Moivre's model.
3. Gompertz's model: *μx*, *s*(*x*), *f*(*x*).
4. Makeham's model: *μx*, *s*(*x*), *f*(*x*).

5. Time-until-Death for a Person Age *x*

1. Random variable *T*(*x*). Distribution of *T*(*x*). Expression of *Fx*(*t*) in terms of *s*(*x*) and *lx*.
2. Values associated with *Т*(*х*): *tqx*, *tрx*. Its expression through *s*(*x*).
3. Value *t*/*uqx*. Its expression through *s*(*x*).
4. Complete-expectation-of-life. Its calculation via *tpx* and *s*(*x*).
5. Curtate-expectation-of-life.

**6. Assumptions for the Fractional Ages:** **Uniform Distributionof Deaths**

1. Survival function.
2. Curve of death.
3. Force of mortality.
4. What does this mean: *μ* '*х*> 0 between the interpolation points?
5. Why the values *f*(*x*) and *μх* are not defined at an integer points?

**7. Distribution of Fractional Age: Uniform Distributionof Deaths**

1. Conditional distribution of a random variable *τ* (Fractional Age) under the condition that the death occurred at the age of *n* years (in general case).
2. Conditional distribution of a random variable *τ* (Fractional Age) under the condition that the death occurred at the age of n years in case of uniform distribution of deaths: *F*(*t*⏐.), *f*(*t*⏐.), *s*(*t*⏐.), μt⏐.
3. Expectation of the fractional age with the condition that the human died at the age of n years (value *a*(*n*)). Write down a formula.
4. Variance of the fractional age with the condition that the human died at the age of *n* years (value *b*(*n*)). Write down a formula.

**8. Multy-Life Theory (probabilities)**

1. Concept of Status. Two the most common statuses.
2. Joint-Life Status: definition.
3. Survival function of a Joint-Life Status. Distribution function. Distribution density.
4. Last-Survivor Status: definition.
5. Distribution function of a Last-Survivor Status. Survival function. Distri­bution density.

**9. Multiple Decrements**

1. What is a Multiple Decrements?
2. Values *dх*(*k*), *dх*(*τ*), the relationship between them.
3. Values *qх*(*k*). Whether there is a value *pх*(*k*)? Why?
4. Central rate of decrement from all causes at age *x*.
5. Central rate of decrement from cause (*k*).

**10. Life Annuities. The main types**

1. Pure endowment*tEx*.
2. Life annuity *ах*. Diagram, formula.
3. A temporary life annuity *ax*:*n*⎤. Diagram, formula.
4. An *n*-year deferred life annuity *n*⏐*аx*. Diagram, formula.
5. Relationship between *аx*,*ax*:*n*⎤ and*n*⏐*аx*.

**11. Life Annuities Payable *m*thly**

1. Life annuity payable *m*thly. Definition, diagram.
2. Expression of a life annuity payable *m*thly (*ax*(*m*)) through a life annuity *ax*.
3. Expression of a deferred life annuity payable *m*thly (*n*/*ax*(*m*)) through a life annuity *n*/ *ax*.
4. Expression of a temporary life annuity payable *m*thly (*ax*:*n*⎤(*m*)) through a life annuity *ax*:*n*⎤.

**12. Varying Life Annuities**

1. An increasing life annuity (*Ia*)*x*.
2. An increasing life annuity with a term of *n* year (*IA*)*x*:*n*⎤.
3. An decreasing life annuity with a term of *n* year (*DA*)*x*:*n*⎤.
4. Why the concept of "decreasing annuity" is only used for temporarylife annuity?

**13. Life Insurance (basic concept)**

1. Whole life policy *Ах*. Why the sum in the whole life annuity formula is actually finite?
2. Term insurance *A*1*x*:*n*⎤.
3. Deferred insurance *n*/*Ax*.
4. *n*-year endowment insurance *Ax*:*n*⎤. Expression through *A*1*x*:*n*⎤ and pure endowment.

**14. Varying insurance**

1. Increasing whole life insurance (*IA*)*x*.

2.Increasing term insurance (*IA*)1*x*:*n*⎤.

3.Decreasing term insurance (*DA*)1*x*:*n*⎤.

**15. Multi-life Theory (insurance)**

1. Joint-life status. A life annuity *ax*1, …,*xm*. In what case it will be paid?
2. Joint-life status. A life insurance policy *Ax*1, …,*xm*. In what case it will be paid?
3. What is *A*1*xy*?
4. Last survivor status. A life annuity . In what case it will be paid?
5. Last survivor status. A life insurance policy In what case it will be paid?

Life Table

(Russia, 2014, women)

|  |  |  |  |
| --- | --- | --- | --- |
| Age *x* | *lx* | *dx* | *qx* |
| 0 | 100000 | 662 | 0,00662 |
| 1 | 99338 | 62 | 0,00062 |
| 2 | 99276 | 35 | 0,00036 |
| 3 | 99241 | 32 | 0,00032 |
| 4 | 99209 | 27 | 0,00027 |
| 5 | 99183 | 21 | 0,00021 |
| 6 | 99161 | 23 | 0,00023 |
| 7 | 99139 | 19 | 0,00019 |
| 8 | 99120 | 18 | 0,00018 |
| 9 | 99102 | 14 | 0,00014 |
| 10 | 99087 | 18 | 0,00018 |
| 11 | 99069 | 17 | 0,00017 |
| 12 | 99052 | 21 | 0,00021 |
| 13 | 99032 | 22 | 0,00023 |
| 14 | 99009 | 30 | 0,00030 |
| 15 | 98980 | 39 | 0,00039 |
| 16 | 98941 | 42 | 0,00042 |
| 17 | 98899 | 44 | 0,00044 |
| 18 | 98855 | 52 | 0,00052 |
| 19 | 98803 | 59 | 0,00059 |
| 20 | 98745 | 58 | 0,00059 |
| 21 | 98686 | 56 | 0,00057 |
| 22 | 98630 | 58 | 0,00059 |
| 23 | 98572 | 61 | 0,00062 |
| 24 | 98511 | 68 | 0,00069 |
| 25 | 98443 | 85 | 0,00086 |
| 26 | 98358 | 93 | 0,00095 |
| 27 | 98265 | 104 | 0,00106 |
| 28 | 98161 | 108 | 0,00110 |
| 29 | 98053 | 128 | 0,00131 |
| 30 | 97925 | 138 | 0,00140 |
| 31 | 97787 | 165 | 0,00169 |
| 32 | 97622 | 175 | 0,00180 |
| 33 | 97447 | 190 | 0,00195 |
| 34 | 97256 | 194 | 0,00200 |
| 35 | 97062 | 225 | 0,00232 |
| 36 | 96837 | 222 | 0,00229 |
| 37 | 96615 | 225 | 0,00233 |
| 38 | 96390 | 235 | 0,00244 |
| 39 | 96155 | 242 | 0,00251 |
| 40 | 95913 | 249 | 0,00259 |
| 41 | 95665 | 267 | 0,00279 |
| 42 | 95398 | 276 | 0,00290 |
| 43 | 95121 | 279 | 0,00293 |
| 44 | 94843 | 302 | 0,00319 |
| 45 | 94541 | 325 | 0,00344 |
| 46 | 94216 | 334 | 0,00354 |
| 47 | 93882 | 365 | 0,00388 |
| 48 | 93517 | 351 | 0,00375 |
| 49 | 93166 | 380 | 0,00407 |
| 50 | 92787 | 413 | 0,00445 |
| 51 | 92374 | 446 | 0,00483 |
| 52 | 91928 | 471 | 0,00512 |
| 53 | 91457 | 488 | 0,00534 |
| 54 | 90969 | 515 | 0,00566 |
| 55 | 90454 | 589 | 0,00651 |
| 56 | 89865 | 649 | 0,00722 |
| 57 | 89216 | 670 | 0,00751 |
| 58 | 88546 | 717 | 0,00810 |
| 59 | 87829 | 769 | 0,00876 |
| 60 | 87060 | 813 | 0,00934 |
| 61 | 86246 | 879 | 0,01019 |
| 62 | 85368 | 969 | 0,01136 |
| 63 | 84398 | 985 | 0,01167 |
| 64 | 83413 | 1106 | 0,01326 |
| 65 | 82307 | 1217 | 0,01478 |
| 66 | 81090 | 1179 | 0,01453 |
| 67 | 79911 | 1480 | 0,01853 |
| 68 | 78431 | 1339 | 0,01707 |
| 69 | 77092 | 1506 | 0,01953 |
| 70 | 75587 | 1644 | 0,02175 |
| 71 | 73943 | 1576 | 0,02132 |
| 72 | 72366 | 2072 | 0,02863 |
| 73 | 70294 | 1986 | 0,02826 |
| 74 | 68308 | 2224 | 0,03257 |
| 75 | 66083 | 2540 | 0,03844 |
| 76 | 63543 | 2569 | 0,04042 |
| 77 | 60975 | 2881 | 0,04724 |
| 78 | 58094 | 3093 | 0,05323 |
| 79 | 55001 | 3279 | 0,05962 |
| 80 | 51722 | 3285 | 0,06352 |
| 81 | 48437 | 3624 | 0,07482 |
| 82 | 44813 | 3805 | 0,08491 |
| 83 | 41007 | 3762 | 0,09173 |
| 84 | 37246 | 3905 | 0,10484 |
| 85 | 33341 | 3793 | 0,11375 |
| 86 | 29548 | 3811 | 0,12899 |
| 87 | 25737 | 3591 | 0,13952 |
| 88 | 22146 | 3458 | 0,15614 |
| 89 | 18688 | 3117 | 0,16679 |
| 90 | 15571 | 2951 | 0,18952 |
| 91 | 12620 | 2480 | 0,19651 |
| 92 | 10140 | 2241 | 0,22105 |
| 93 | 7899 | 1913 | 0,24217 |
| 94 | 5986 | 1599 | 0,26713 |
| 95 | 4387 | 1222 | 0,27862 |
| 96 | 3165 | 946 | 0,29896 |
| 97 | 2219 | 709 | 0,31955 |
| 98 | 1510 | 514 | 0,34026 |
| 99 | 996 | 359 | 0,36091 |
| 100 | 636 | 243 | 0,38135 |
| 101 | 394 | 158 | 0,40143 |
| 102 | 236 | 99 | 0,42100 |
| 103 | 136 | 60 | 0,43995 |
| 104 | 76 | 35 | 0,45816 |
| 105 | 41 | 20 | 0,47554 |
| 106 | 22 | 11 | 0,49202 |
| 107 | 11 | 6 | 0,50756 |
| 108 | 5 | 3 | 0,52212 |
| 109 | 3 | 1 | 0,53570 |
| 110+ | 1 | 1 | 1,00000 |

Life Table

(Russia, 2014, men)

|  |  |  |  |
| --- | --- | --- | --- |
| Age *x* | *lx* | *dx* | *qx* |
| 0 | 100000 | 820 | 0,00820 |
| 1 | 99180 | 71 | 0,00072 |
| 2 | 99108 | 45 | 0,00045 |
| 3 | 99064 | 39 | 0,00040 |
| 4 | 99024 | 36 | 0,00036 |
| 5 | 98988 | 29 | 0,00029 |
| 6 | 98960 | 30 | 0,00031 |
| 7 | 98929 | 27 | 0,00028 |
| 8 | 98902 | 27 | 0,00027 |
| 9 | 98875 | 25 | 0,00025 |
| 10 | 98851 | 29 | 0,00029 |
| 11 | 98822 | 31 | 0,00032 |
| 12 | 98790 | 34 | 0,00034 |
| 13 | 98757 | 39 | 0,00039 |
| 14 | 98718 | 49 | 0,00050 |
| 15 | 98669 | 67 | 0,00068 |
| 16 | 98602 | 90 | 0,00091 |
| 17 | 98512 | 106 | 0,00107 |
| 18 | 98406 | 124 | 0,00126 |
| 19 | 98282 | 140 | 0,00142 |
| 20 | 98142 | 175 | 0,00178 |
| 21 | 97967 | 203 | 0,00207 |
| 22 | 97765 | 210 | 0,00214 |
| 23 | 97555 | 229 | 0,00235 |
| 24 | 97326 | 254 | 0,00261 |
| 25 | 97072 | 282 | 0,00291 |
| 26 | 96790 | 319 | 0,00330 |
| 27 | 96471 | 353 | 0,00366 |
| 28 | 96118 | 366 | 0,00381 |
| 29 | 95752 | 388 | 0,00405 |
| 30 | 95364 | 469 | 0,00492 |
| 31 | 94894 | 515 | 0,00543 |
| 32 | 94379 | 552 | 0,00585 |
| 33 | 93828 | 579 | 0,00617 |
| 34 | 93249 | 601 | 0,00645 |
| 35 | 92648 | 671 | 0,00724 |
| 36 | 91977 | 713 | 0,00775 |
| 37 | 91264 | 719 | 0,00788 |
| 38 | 90545 | 730 | 0,00807 |
| 39 | 89815 | 713 | 0,00793 |
| 40 | 89102 | 754 | 0,00846 |
| 41 | 88348 | 741 | 0,00839 |
| 42 | 87607 | 764 | 0,00873 |
| 43 | 86843 | 767 | 0,00883 |
| 44 | 86076 | 766 | 0,00890 |
| 45 | 85310 | 847 | 0,00993 |
| 46 | 84463 | 880 | 0,01041 |
| 47 | 83583 | 954 | 0,01141 |
| 48 | 82629 | 941 | 0,01139 |
| 49 | 81689 | 1002 | 0,01227 |
| 50 | 80687 | 1103 | 0,01367 |
| 51 | 79583 | 1140 | 0,01433 |
| 52 | 78443 | 1176 | 0,01499 |
| 53 | 77267 | 1209 | 0,01565 |
| 54 | 76058 | 1299 | 0,01707 |
| 55 | 74759 | 1425 | 0,01907 |
| 56 | 73334 | 1493 | 0,02036 |
| 57 | 71841 | 1545 | 0,02150 |
| 58 | 70296 | 1642 | 0,02335 |
| 59 | 68654 | 1735 | 0,02527 |
| 60 | 66919 | 1886 | 0,02818 |
| 61 | 65034 | 1946 | 0,02993 |
| 62 | 63087 | 2054 | 0,03256 |
| 63 | 61033 | 1997 | 0,03271 |
| 64 | 59036 | 2053 | 0,03477 |
| 65 | 56983 | 2249 | 0,03946 |
| 66 | 54735 | 1984 | 0,03624 |
| 67 | 52751 | 2376 | 0,04504 |
| 68 | 50375 | 2068 | 0,04106 |
| 69 | 48307 | 2103 | 0,04354 |
| 70 | 46203 | 2247 | 0,04863 |
| 71 | 43956 | 2085 | 0,04743 |
| 72 | 41872 | 2611 | 0,06236 |
| 73 | 39261 | 2342 | 0,05966 |
| 74 | 36918 | 2404 | 0,06513 |
| 75 | 34514 | 2573 | 0,07456 |
| 76 | 31941 | 2357 | 0,07381 |
| 77 | 29583 | 2396 | 0,08099 |
| 78 | 27187 | 2368 | 0,08711 |
| 79 | 24819 | 2294 | 0,09243 |
| 80 | 22525 | 2141 | 0,09503 |
| 81 | 20384 | 2127 | 0,10435 |
| 82 | 18257 | 2093 | 0,11463 |
| 83 | 16165 | 1930 | 0,11938 |
| 84 | 14235 | 1837 | 0,12906 |
| 85 | 12398 | 1695 | 0,13671 |
| 86 | 10703 | 1610 | 0,15042 |
| 87 | 9093 | 1459 | 0,16050 |
| 88 | 7634 | 1325 | 0,17352 |
| 89 | 6309 | 1183 | 0,18747 |
| 90 | 5126 | 1063 | 0,20733 |
| 91 | 4063 | 890 | 0,21894 |
| 92 | 3174 | 737 | 0,23230 |
| 93 | 2436 | 590 | 0,24233 |
| 94 | 1846 | 521 | 0,28195 |
| 95 | 1326 | 360 | 0,27142 |
| 96 | 966 | 277 | 0,28675 |
| 97 | 689 | 208 | 0,30228 |
| 98 | 481 | 153 | 0,31795 |
| 99 | 328 | 109 | 0,33368 |
| 100 | 218 | 76 | 0,34942 |
| 101 | 142 | 52 | 0,36508 |
| 102 | 90 | 34 | 0,38060 |
| 103 | 56 | 22 | 0,39592 |
| 104 | 34 | 14 | 0,41097 |
| 105 | 20 | 8 | 0,42569 |
| 106 | 11 | 5 | 0,44003 |
| 107 | 6 | 3 | 0,45394 |
| 108 | 3 | 2 | 0,46739 |
| 109 | 2 | 1 | 0,48033 |
| 110+ | 1 | 1 | 1,00000 |

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