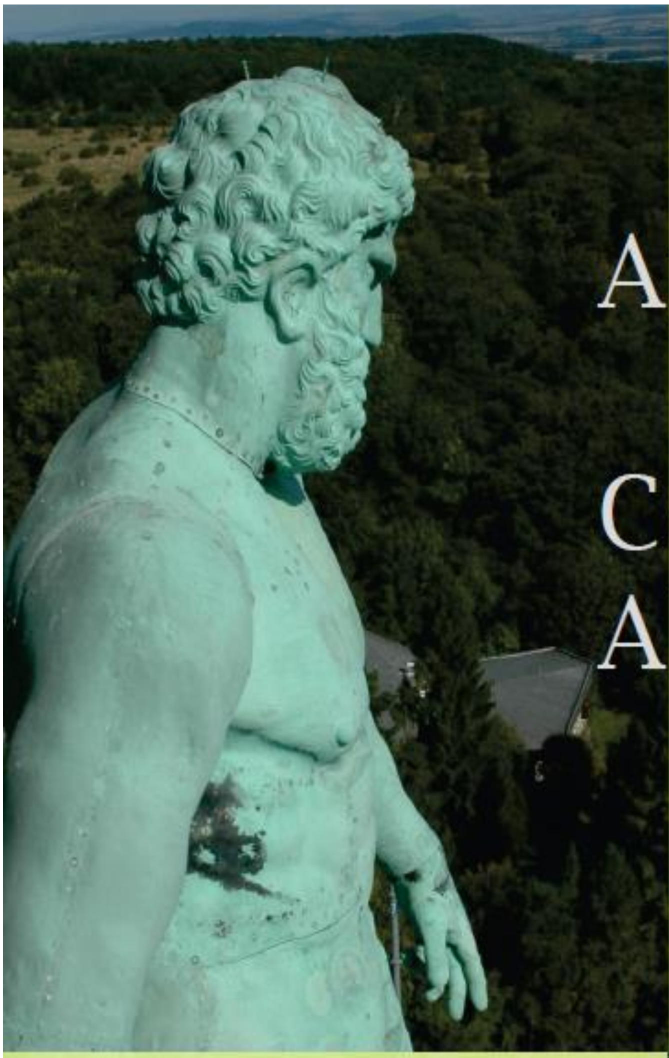

22nd Conference on

Applications of Computer Algebra

ACA 2016
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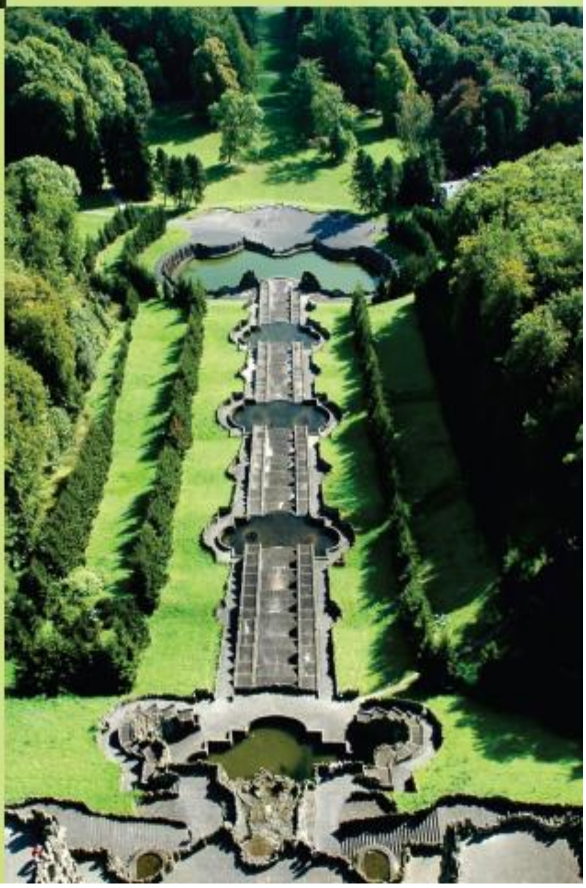

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submission of talks

Notification of acceptance:
July 13th, 2016

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Secular Perturbations in the Two-Planetary Three-Body Problem with the Masses Varying Anisotropically with Different Rates

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1 Statement of the problem

Let us consider a system of three mutually attracting spherical celestial bodies T_0, T_1 and T_2 of masses

$$m_0 = m_0(t), \quad m_1 = m_1(t), \quad m_2 = m_2(t) \tag{1}$$

varying anisotropically with different rates (law of masses variation is arbitrary) (Refs. [1])

$$\frac{\dot{m}_0}{m_0} \neq \frac{\dot{m}_1}{m_1} \neq \frac{\dot{m}_2}{m_2}. \tag{2}$$

On the basis of Meshcherskiy equation (Refs. [2]), we can write the equations of motion of three-body problem with variable masses in the presence of reactive forces in the absolute coordinate system in the form

$$m_j\ddot{\vec{R}}_j = grad_{\vec{R}_j} U + \dot{m}_j\vec{V}_j, \quad U = f\left(\frac{m_0 m_1}{R^*_{01}} + \frac{m_0 m_2}{R^*_{02}} + \frac{m_1 m_2}{R^*_{12}}\right),$$

where $\vec{u}_j$ are the absolute velocities of the separating particles,

$$\vec{V}_j = \vec{u}_j - \dot{\vec{R}}_j \neq 0, \quad j = 0, 1, 2, \tag{3}$$

are the relative velocities of the separating particles, $\vec{R}_j$ are the radius vectors of the center of the spherical bodies, $\vec{R}_{ij}$ are the distances between the centers of the spherical bodies, f is the gravitational constant. Following L.G. Lukyanov (Refs. [3]), we assume that the reactive forces are applied to the center of the respective spherical bodies. Usually in the observational astronomy the laws of the mass change (see Eqs. (1, 2)) and the relative velocities of the separating particles (see Eqs. (3)) can be determined experimentally for specific celestial bodies. Therefore, we assume that the values of $m_j(t)$, $\vec{V}_j$, $j = 0, 1, 2$, are known (see Eqs. (1, 3)).

It should be noted that in general case of the three-body problem with variable masses changing anisotropically in the different rates there is no any integral of motion. Therefore, the problem under consideration is investigated by methods of the perturbation theory (Refs. [1], [4, 5, 6]), and with the use of analytical calculations system Mathematica (Refs. [7]).

2 Equations of Motion in Terms of the Delaunay Elements

Using the Jacobi coordinates, we can rewrite the equations of motion in the form

$$\mu_1\ddot{\vec{r}}_1 = grad_{\vec{r}_1}U + \vec{F}_1, \quad \mu_2\ddot{\vec{r}}_2 = grad_{\vec{r}_2}U - (2\dot{\nu}_1\dot{\vec{r}}_1 + \ddot{\nu}_1\vec{r}_1) + \vec{F}_2.$$

where reduced masses are given by

$$\mu_1 = \frac{m_1 m_0}{m_0+m_1} \neq const, \quad \mu_2 = \frac{m_2(m_0+m_1)}{m_0+m_1+m_2} \neq const, \quad \nu_1 = \frac{m_1}{m_0+m_1} \neq const.$$

The functions

$$\vec{F}_1 = \vec{F}_1(F_{1x}, F_{1y}, F_{1z}) = \vec{F}_1(t) = \frac{\dot{m}_1}{m_1}\vec{V}_1 - \frac{\dot{m}_0}{m_0}\vec{V}_0 \neq 0,$$

$$\vec{F}_2 = \vec{F}_2(F_{2x}, F_{2y}, F_{2z}) = \vec{F}_2(t) = \left(\frac{\dot{m}_2}{m_2}\vec{V}_2 - \frac{\dot{m}_0}{m_0}\vec{V}_0\right) - \nu_1\left(\frac{\dot{m}_1}{m_1}\vec{V}_1 - \frac{\dot{m}_0}{m_0}\vec{V}_0\right) \neq 0,$$

are considered known and given.

To apply the perturbation theory it is convenient to rewrite the equations of motion in terms of the analogues of the second system of the Poincare elements. The first step in such transformation is to write the equations of motion in terms of the osculating elements of the aperiodic motion on the quasi-conic section using the Delaunay coordinates (see Refs. [4]). Then investigation of the secular perturbations of the Delanay elements is reduced to solving the following system of non-autonomous differential equations

$$\begin{aligned} \dot{\xi}_i &= \frac{\partial R^*_{isec}}{\partial \eta_i}, & \dot{p}_i &= \frac{\partial R^*_{isec}}{\partial q_i}, \\ \dot{\eta}_i &= -\frac{\partial R^*_{isec}}{\partial \xi_i}, & \dot{q}_i &= -\frac{\partial R^*_{isec}}{\partial p_i}. \end{aligned} \tag{4}$$

where R^*_{isec} are perturbation functions (Refs. [4, 5, 6]), ξ_i, η_i, p_i, q_i are analogues of the second system of the Poincaré elements (Refs. [1]).

In the present paper we consider the expansion of the perturbing function in terms of small quantities $m_1, m_2, e_1, e_2, i_1, i_2$ up to the second order inclusively (Refs. [4, 5, 6]). Then the secular expressions for R^*_{1sec}, R^*_{2sec} in the analogues of the second system of the Poincaré elements take the form (Refs. [5, 6])

$$
\begin{aligned}
R^*_{1sec} &= \frac{1}{\gamma_1^2}\cdot\frac{\tilde{\beta}_1^4}{2\mu_{10}\Lambda_1^2} + F_{01} + F_{12sec1} + F_{\rho 1sec} + \Phi_{1sec},\\
R^*_{2sec} &= \frac{1}{\gamma_2^2}\cdot\frac{\tilde{\beta}_2^4}{2\mu_{20}\Lambda_2^2} + F_{02} + F_{12sec2} + F_{\rho 2sec} + V_{sec} + \Phi_{2sec},
\end{aligned}
\tag{5}
$$

$$
F_{01} = -\frac{b_1\gamma_1^2 a_1^2}{2\psi_1} - f\frac{m_1 m_2}{\gamma_2\psi_1 a_2}, \quad F_{12sec1} = \frac{f}{\psi_1}\left[\frac{m_1 m_2}{r_{12}}\right]_{sec}, \quad F_{\rho 1sec} = -\frac{3b_1\gamma_1^2 a_1^2}{4\Lambda_1\psi_1}(\xi_1^2+\eta_1^2),
$$

$$
F_{02} = -\frac{b_2\gamma_2^2 a_2^2}{2\psi_2} - f\frac{m_1 m_2}{\gamma_2\psi_2 a_2}, \quad F_{12sec2} = \frac{f}{\psi_2}\left[\frac{m_1 m_2}{r_{12}}\right]_{sec}, \quad F_{\rho 2sec} = -\frac{3b_2\gamma_2^2 a_2^2}{4\Lambda_2\psi_2}(\xi_2^2+\eta_2^2),
$$

$$
V_{sec} = -\frac{9a_1 a_2\mu_2\gamma_2(2\dot{\gamma}_1\dot{\nu}_1 + \gamma_1\ddot{\nu}_1)}{14\sqrt{\Lambda_1}\sqrt{\Lambda_2}\psi_2}(\xi_1\xi_2 + \eta_1\eta_2),
$$

$$
\Phi_{1sec} = \frac{3a_1\gamma_1(t)}{2\psi_1\sqrt{\Lambda_1}}\left\{-F_{1x}(t)\xi_1 + F_{1y}(t)\eta_1 + \frac{F_{1z}(t)}{\sqrt{\Lambda_1}}[(-\xi_1 q_1 + \eta_1 p_1)]\right\},
$$

$$
\Phi_{2sec} = \frac{3a_2\gamma_2(t)}{2\psi_2\sqrt{\Lambda_2}}\left\{-F_{2x}(t)\xi_2 + F_{2y}(t)\eta_2 + \frac{F_{2z}(t)}{\sqrt{\Lambda_2}}[(-\xi_2 q_2 + \eta_2 p_2)]\right\}.
$$

Analysis of the expressions obtained (see Eqs. (4, 5)) shows that the equations of the secular perturbations in the presence of reactive forces (in the case when masses changing anisotropically) are not splitted into two systems with respect to the elements ξ_i, η_i and p_i, q_i.

The main purpose of this paper is to identify the explicit form of the equations (see Eqs. (4)) and to find their approximate analytical solutions using by Picard method. On the basis of these solutions we can obtain an explicit form of the equations of the analogues of the orbital elements.

3 Approximate Analytical Solutions

Using expressions (5), we can write the equations of motion in explicit form (see Eqs. (4))

$$
\begin{aligned}
\dot{\xi}_1 &= K_5 + K_6 p_1 + 2K_1\eta_1 + K_3\eta_2, \quad & \dot{\eta}_1 &= K_4 - K_6 q_1 + 2K_1\xi_1 + K_3\xi_2,\\
\dot{\xi}_2 &= K'_5 + K'_6 p_2 + 2K'_2\eta_2 + K'_3\eta_1, \quad & \dot{\eta}_2 &= K'_4 - K'_6 q_2 + 2K'_2\xi_2 + K'_3\xi_1,
\end{aligned}
\tag{6}
$$

$$\dot{p}_1 = -K_6\xi_1 + 2\psi_1^*(t)\left(\frac{q_1}{\Lambda_1} - \frac{q_2}{\sqrt{\Lambda_1\Lambda_2}}\right), \quad \dot{q}_1 = K_6\eta_1 + 2\psi_1^*(t)\left(\frac{p_1}{\Lambda_1} - \frac{p_2}{\sqrt{\Lambda_1\Lambda_2}}\right),$$
$$\dot{p}_2 = -K_6'\xi_2 + 2\psi_2^*(t)\left(\frac{q_2}{\Lambda_2} - \frac{q_1}{\sqrt{\Lambda_1\Lambda_2}}\right), \quad \dot{q}_2 = K_6'\eta_2 + 2\psi_2^*(t)\left(\frac{p_2}{\Lambda_2} - \frac{p_1}{\sqrt{\Lambda_1\Lambda_2}}\right). \tag{7}$$

In this formulation due to the anisotropical change of the masses and therefore adding of the reactive force new terms appear which have the form

$$K_4 = -\frac{3a_1F_{1x}(t)\gamma_1(t)}{2\psi_1\sqrt{\Lambda_1}}, \quad K_5 = \frac{3a_1F_{1y}(t)\gamma_1(t)}{2\psi_1\sqrt{\Lambda_1}} \quad K_6 = \frac{3a_1F_{1z}(t)\gamma_1(t)}{2\psi_1\Lambda_1},$$
$$K_4' = -\frac{3a_2F_{2x}(t)\gamma_2(t)}{2\psi_2\sqrt{\Lambda_2}}, \quad K_5' = \frac{3a_2F_{2y}(t)\gamma_2(t)}{2\psi_2\sqrt{\Lambda_2}} \quad K_6' = \frac{3a_2F_{2z}(t)\gamma_2(t)}{2\psi_2\Lambda_2},$$

and the values $K_1, K_2, K_3, K_1', K_2', K_3'$ were obtained in (Refs. [6]).

Using the method of Picard we can write the solutions of the equations (see Eqs. (6, 7)) as follows

$$\ni_k(t) = \ni_k(t_0) + \int_{t_0}^{t} \Pi_i^{**}(t, \ni_k(t_0))dt, \tag{8}$$

where $\Pi_i^{**}(t, \ni_k)$ are the right-hand sides of the equations (see Eqs. (6, 7)), $\ni_k$ are the elements ξ_i, η_i, p_i, q_i, and $\ni_{k0} = \ni_k(t_0)$ are their values at the initial time.

The solutions (see Eqs. (8)) allow to analyze the evolution of the analogues of eccentricities e_i, inclinations i_i, argument of pericenters ω_i and motions of the longitude of the ascending nodes Ω_i, the longitude of pericenters π_i given by

$$e_i^2 = \frac{\ni_{\xi_i}^2 + \ni_{\eta_i}^2}{\Lambda_i}, \quad \sin^2 i_i = \frac{\ni_{p_i}^2 + \ni_{q_i}^2}{\Lambda_i},$$
$$\Omega_i = -arctg\frac{\ni_{q_i}}{\ni_{p_i}}, \quad \pi_i = -arctg\frac{\ni_{\eta_i}}{\ni_{\xi_i}}, \quad \omega_i = \pi_i - \Omega_i, \quad i = 1, 2.$$

It should be noted that all of the analytical calculations have been done with the use of the system Wolfram Mathematica (Refs. [7]).

4 Conclusion

In the paper a general problem of three mutually attracting spherical celestial bodies with variable masses changing anisotropically in different rates is considered. A system of eight differential equations of the first order describing the secular perturbations of the orbital elements is obtained in terms of the analogues of the second system of the Poincaré elements in the presence of reactive forces. Approximate analytical solutions of these equations are found by the method of Picard. On the basis of these solutions it is possible to analyze the evolution of the analogues of orbital elements of the bodies that will be done if the next paper.

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