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Vladimir V. Mityushev
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Editors

Current Trends in Analysis and Its Applications

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Contents

Nonlinear PDE as Immersions

Zhanat Zhunussova

Abstract Investigating of the nonlinear PDE including their geometric nature is one of the topical problems. With geometric point of view the nonlinear PDE are considered as immersions. We consider some aspects of the simplest soliton immersions in multidimensional space in Fokas–Gelfand's sense (Ceyhan et al. in J. Math. Phys. 41:2551–2270, 2000). In $(1+1)$-dimensional case nonlinear PDE are given in compatibility condition some system of linear equations (Lakshmanan and Myrzakulov in J. Math. Phys. 39:3765–3771, 1998). In this case there is a surface with immersion function. We find the second quadratic form in Fokas–Gelfand's sense associated to one soliton solution of nonlinear Schrödinger equation.

Keywords Immersion · Soliton · Surface · Evolution equation

1 Introduction

Over the last twenty years in the field of mathematical physics a large number of researches is devoted to the study of nonlinear equations. Some nonlinear wave equations can occur in problems of the different physical nature [1, 2]. For example, such equations are the well-known Korteweg de Vries equation, the nonlinear Schrödinger equation, sin-Gordon equation.

Soliton theory is a powerful apparatus for studying nonlinear equations including their geometrical nature. With a geometrical point of view soliton systems are considered as immersion of infinite-dimensional spaces. In other words, the hierarchy of soliton equations considered as a system of defining immersion of a manifold V^n in space V^m, where $n < m$. Connection between theory of solitons and theory of surfaces is set by introducing evolution equations that associated with algebra. The relation $(1+1)$-dimensional soliton equations with the theory of surfaces are given by the Gauss–Codazzi–Mainardi equation. In this case, the soliton equations are considered as some integrable reductions of the Gauss–Codazzi–Mainardi equation.

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In this work, we would like to review the simplest aspects of soliton investments in multi-dimensional space, in Fokas–Gelfand sense [1].

In $(1+1)$-dimensional case nonlinear partial differential equations are given as a condition of zero curvature $U_t - V_x + [U,V] = 0$, where $[U,V] = UV - VU$, matrix U is given, and the matrix V is expressed in terms of elements of the matrix U. Also the nonlinear partial differential equation is the compatibility condition the system of linear equations $\phi_x = U\phi$, $\phi_t = V\phi$. In this case there is a surface with immersion function $P(x,t)$ defined by the formulas $\frac{\partial P}{\partial x} = \phi^{-1}X\phi$, $\frac{\partial P}{\partial t} = \phi^{-1}Y\phi$. Surface defined by $P(x,t)$ identified a surface in three-dimensional space defined by the coordinates [1] $x_j = P_j(x,t)$, $j = 1,2,3$. Frame on the surface is given by a triple [1] $\frac{\partial P}{\partial x} = \phi^{-1}X\phi$, $\frac{\partial P}{\partial t} = \phi^{-1}Y\phi$, $N = \phi^{-1}J\phi$, where $J = \frac{[X,Y]}{\|[X,Y]\|}$, $|X| = \sqrt{\langle X,X\rangle}$. Here, by definition, $\langle X,Y\rangle = -\frac{1}{2}\operatorname{tr}(XY)$, where X, Y are some matrixes. And the first and second quadratic forms of the surface are given by

$$I = \langle X,X\rangle dx^2 + 2\langle X,Y\rangle dx dt + \langle Y,Y\rangle dt^2, \tag{1.1}$$

$$II = \left\langle \frac{\partial X}{\partial x} + [X,U], J\right\rangle dx^2 + 2\left\langle \frac{\partial X}{\partial t} + [X,V], J\right\rangle dx dt + \left\langle \frac{\partial Y}{\partial t} + [Y,V], J\right\rangle dt^2. \tag{1.2}$$

As shown in [1] immersion function P can be defined as $P = \gamma_0\phi^{-1}\phi_\lambda + \phi^{-1}M_1\phi = \sum_{j=1}^{3} P_j f_j$, where M_1 is a matrix function, which depends on λ, x, t. Here $f_j = -\frac{i}{2}\sigma_j$ is basis of the corresponding algebra, σ_j are Pauli matrices and $[f_i, f_j] = f_k$. In this case, X, Y can be written as $X = \gamma_0 U_\lambda + M_{1x} + [M_1, U]$, $Y = \gamma_0 V_\lambda + M_{1t} + [M_1, V]$.

2 Soliton Immersions in $(1+1)$-Dimension

Let the matrixes X, Y, J have the form

$$X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad Y = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad J = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \tag{2.1}$$

In this case, the elements of the matrix J are expressed through the elements of the matrix X and Y in accordance with the formulas

$$c_{11} = \frac{a_{12}b_{21} - b_{12}a_{21}}{\|[X,Y]\|}, \qquad c_{21} = \frac{a_{21}(b_{11} - b_{22}) + b_{21}(a_{22} - a_{11})}{\|[X,Y]\|}, \tag{2.2}$$

$$c_{12} = \frac{b_{12}(a_{11} - a_{22}) + a_{12}(b_{22} - b_{11})}{\|[X,Y]\|}, \qquad c_{22} = \frac{a_{21}b_{12} - b_{21}a_{12}}{\|[X,Y]\|}. \tag{2.3}$$

Then the first fundamental form (1.1) of two-dimensional surface becomes $I = Edx^2 + 2Fdxdt + Gdt^2$, where

$$\begin{aligned} E &= -\frac{1}{2}(a_{11}^2 + 2a_{12}a_{21} + a_{22}^2), \\ F &= -\frac{1}{2}(a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}), \end{aligned} \tag{2.4}$$

$$G = -\frac{1}{2}(b_{11}^2 + 2b_{12}b_{21} + b_{22}^2). \tag{2.5}$$

As an example of a soliton equation that yields such immersion we consider the nonlinear Schrödinger equation $i\psi_t + \psi_{xx} + 2\beta|\psi|^2\psi = 0$, where $\beta = +1$, ψ is complex function. In this case, the matrix U, V have the form [3]

$$\begin{aligned} U &= \frac{\lambda\sigma_3}{2i} + U_0, \quad U_0 = i\begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix}, \\ V &= \frac{i\lambda^2}{2}\sigma_3 + i|q|^2\sigma_3 - i\lambda\begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{q}_x \\ -q_x & 0 \end{pmatrix}. \end{aligned} \tag{2.6}$$

The theorem is held.

Theorem 2.1 *Second quadratic form in the sense of Fokas–Gelfand corresponding to soliton solution q of nonlinear Schrödinger equation has the form*

$$II = Ldx^2 + 2Mdxdt + Ndt^2, \tag{2.7}$$

where

$$\begin{aligned} L = -\frac{1}{2}\{&a_{11x}c_{11} + a_{12x}c_{21} + a_{21x}c_{12} + a_{22x}c_{22} \\ &- \lambda i(a_{21}c_{12} - a_{12}c_{21}) \\ &+ iq(a_{12}c_{11} + a_{22}c_{12} - a_{11}c_{12} - a_{12}c_{22}) \\ &+ i\bar{q}(a_{21}c_{22} + a_{11}c_{21} - a_{22}c_{21} - a_{21}c_{11})\}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} M = -\frac{1}{2}\{&a_{11t}c_{11} + a_{12t}c_{21} + a_{21t}c_{12} + a_{22t}c_{22} \\ &+ i(\lambda^2 + 2|q|^2)(a_{21}c_{12} - a_{12}c_{21}) \\ &+ (q_x + \lambda iq)(a_{11}c_{12} + a_{12}c_{22} - a_{12}c_{11} - a_{22}c_{12}) \\ &+ (\bar{q}_x - \lambda i\bar{q})(a_{11}c_{21} + a_{21}c_{22} - a_{21}c_{11} - a_{22}c_{21})\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} N = -\frac{1}{2}\{&b_{11t}c_{11} + b_{12t}c_{21} + b_{21t}c_{12} + b_{22t}c_{22} \\ &+ i(\lambda^2 + 2|q|^2)(b_{21}c_{12} - b_{12}c_{21}) \end{aligned}$$

$$+ (q_x + \lambda i q)(b_{11}c_{12} + b_{12}c_{22} - b_{12}c_{11} - b_{22}c_{12})$$
$$+ (\bar{q}_x - \lambda i \bar{q})(b_{11}c_{21} + b_{21}c_{22} - b_{21}c_{11} - b_{22}c_{21})\big], \quad (2.10)$$

Proof By direct substitution of the matrix (2.1), (2.6) to (1.2) we obtain (2.7), (2.8)–(2.10). Theorem is proved. □

3 One-Soliton Solution of the Nonlinear Schrödinger Equation Corresponding to the Surface

We consider the partial case of immersion at $\gamma_0 = 1$, $M_1 = 0$. For this case we have

$$X = U_\lambda = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = V_\lambda = -i\begin{pmatrix} -\lambda & \bar{q} \\ q & \lambda \end{pmatrix},$$
$$J = \begin{pmatrix} 0 & -\frac{\bar{q}}{\sqrt{q\bar{q}}} \\ \frac{q}{\sqrt{q\bar{q}}} & 0 \end{pmatrix}, \quad (3.1)$$

and $P = \phi^{-1}\phi_\lambda$. To calculate the explicit expressions for the functions of immersion P we consider the one-soliton solution of the nonlinear Schrödinger equation, which has the form

$$q(x,t) = Q\frac{\exp\{i(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)t}{4} - \frac{\pi}{2})\}}{\mathrm{ch}\{\frac{u}{2}(x - vt - x_0)\}}, \quad (3.2)$$

where we put $\lambda = \frac{u+iv}{2}$, Q is constant.

Theorem 3.1 (Main Theorem) *One-soliton solution of the nonlinear Schrödinger equation corresponds to the surface in the sense of Fokas–Gelfand with the corresponding coefficients of the first and second quadratic form*

$$E = \frac{Q^2(u^2+v^2)}{(\lambda - \bar{\lambda}_1)^4\,\mathrm{ch}^2\{\frac{u}{2}(x - vt - x_0)\}},$$
$$F = -\frac{v(u^2+v^2)Q^2}{2(\lambda - \bar{\lambda}_1)^4\,\mathrm{ch}^2\{\frac{u}{2}(x - vt - x_0)\}}, \quad (3.3)$$

$$G = \frac{(u^2+v^2)^2Q^2}{4(\lambda - \bar{\lambda}_1)^4\,\mathrm{ch}^2\{\frac{u}{2}(x - vt - x_0)\}},$$
$$L = -\frac{u(u^2+v^2)}{4(\lambda - \bar{\lambda}_1)^2\,\mathrm{ch}^2\{\frac{u}{2}(x - vt - x_0)\}}, \quad (3.4)$$

$$M = \frac{uv(u^2+v^2)}{8(\lambda-\bar{\lambda}_1)^2 \operatorname{ch}^2\{\frac{u}{2}(x-vt-x_0)\}},$$
$$N = -\frac{u(u^2+v^2)}{16(\lambda-\bar{\lambda}_1)^2 \operatorname{ch}^2\{\frac{u}{2}(x-vt-x_0)\}}, \tag{3.5}$$

where λ_1 is constant.

Proof The solution of the linear system we find in the form

$$\psi = \phi e^{-(\frac{\lambda\sigma_3}{2i}x + \frac{\lambda^2}{2}\sigma_3 t)}. \tag{3.6}$$

Taking into account (3.6), apply (2.6) we have

$$\psi_x = \left(\frac{\lambda\sigma_3}{2i} + U_0\right)\psi - \psi\frac{\lambda\sigma_3}{2i} = \frac{\lambda\sigma_3}{2i}\psi - \psi\frac{\lambda\sigma_3}{2i} + U_0\psi$$
$$= \left[\frac{\lambda\sigma_3}{2i}, \psi\right] + U_0\psi. \tag{3.7}$$

We take

$$\psi = I - \frac{\tilde{A}}{\lambda-\lambda_1^*}, \quad \text{where } \tilde{A} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}, \; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \lambda_1^*\text{-const.} \tag{3.8}$$

We substitute (3.8) to (3.7)

$$\psi_x = U_0 - \frac{U_0\tilde{A}}{\lambda-\lambda_1^*} - \frac{1}{2i}[\sigma_3, \tilde{A}] - \frac{\lambda_1^*}{2i(\lambda-\lambda_1^*)}[\sigma_3, \tilde{A}]. \tag{3.9}$$

On the other side of (3.8) follows

$$\psi_x = -\frac{\tilde{A}_x}{\lambda-\lambda_1^*}. \tag{3.10}$$

From (3.9) and (3.10) we have

$$-\frac{\tilde{A}_x}{\lambda-\lambda_1^*} = U_0 - \frac{U_0\tilde{A}}{\lambda-\lambda_1^*} - \frac{1}{2i}[\sigma_3, \tilde{A}] - \frac{\lambda_1^*}{2i(\lambda-\lambda_1^*)}[\sigma_3, \tilde{A}]. \tag{3.11}$$

Thus

$$\tilde{A}_x = U_0\tilde{A} + \frac{\lambda_1^*}{2i}[\sigma_3, \tilde{A}], \quad U_0 = \frac{1}{2i}[\sigma_3, A]. \tag{3.12}$$

Note that

$$[\sigma_3, \tilde{A}] = \sigma_3\tilde{A} - \tilde{A}\sigma_3 = 2\begin{pmatrix} 0 & \tilde{b} \\ -\tilde{c} & 0 \end{pmatrix}. \tag{3.13}$$

Then substituting (3.13) into (3.28), we have

$$U_0 = \frac{1}{i}\begin{pmatrix} 0 & \tilde{b} \\ -\tilde{c} & 0 \end{pmatrix}. \tag{3.14}$$

Substituting (3.13) to (3.12), we have

$$\begin{pmatrix} \tilde{a}_x & \tilde{b}_x \\ \tilde{c}_x & \tilde{d}_x \end{pmatrix} = \frac{1}{i}\begin{pmatrix} \tilde{b}\tilde{c} & \tilde{b}\tilde{d} \\ -\tilde{c}\tilde{a} & -\tilde{c}\tilde{b} \end{pmatrix} + \frac{\lambda_1^*}{i}\begin{pmatrix} 0 & \tilde{b} \\ -\tilde{c} & 0 \end{pmatrix}. \tag{3.15}$$

From (2.6) and (3.14) we have

$$i\begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix} = \frac{1}{i}\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \Rightarrow \begin{cases} i\bar{q} = \frac{1}{i}b \\ iq = -\frac{1}{i}c \end{cases} \Rightarrow \begin{cases} b = -\bar{q} \\ c = q. \end{cases} \tag{3.16}$$

Thus we have found a matrix $\tilde{A}$ implicitly, with components (3.15). From (3.15), (3.16) follows $\tilde{a} = -\frac{i\tilde{c}_x}{c} - \lambda_1^* \Rightarrow \tilde{a} = -\frac{iq_x}{q} - \lambda_1^*$. Using (3.2) we obtain

$$\tilde{a} = \frac{iu}{2}\,\mathrm{th}\left\{\frac{u}{2}(x - vt - x_0)\right\} + \frac{v}{2} - \lambda_1^*. \tag{3.17}$$

From (3.15) follows $\tilde{a}_x = \frac{1}{i}\tilde{b}\tilde{c} \Rightarrow \tilde{a}_x = \frac{1}{i}(-\bar{q})q, \Rightarrow \tilde{a} = -\frac{1}{i}\int \bar{q}q\,dx$. Using (3.2) we obtain

$$\tilde{a} = -\frac{2|Q|^2}{iu}\,\mathrm{th}\left\{\frac{u}{2}(x - vt - x_0)\right\} - \frac{2|Q|^2 c_1}{iu}. \tag{3.18}$$

From (3.17), (3.18) follows

$$\begin{cases} -\frac{2|Q|^2}{iu} = \frac{iu}{2}, \Rightarrow \frac{2|Q|^2}{iu} = -\frac{iu}{2}, \Rightarrow 4|Q|^2 = u^2, \Rightarrow |Q|^2 = \frac{u^2}{4}, \\ \left(\frac{v}{2} - \lambda_1^*\right) = -c_1\frac{2|Q|^2}{iu}, \Rightarrow c_1 = -\frac{iu}{2|Q|^2}\left(\frac{v}{2} - \lambda_1^*\right). \end{cases} \tag{3.19}$$

From (3.15), (3.16) follows

$$\tilde{d} = \frac{i\tilde{b}_x}{\tilde{b}} - \lambda_1^* \Rightarrow \tilde{d} = \frac{i(-\bar{q})_x}{(-\bar{q})} - \lambda_1^* \Rightarrow \tilde{d} = \frac{i\bar{q}_x}{\bar{q}} - \lambda_1^*. \tag{3.20}$$

Using (3.2) we have

$$\tilde{d} = -\frac{iu}{2}\,\mathrm{th}\left\{\frac{u}{2}(x - vt - x_0)\right\} + \left(\frac{v}{2} - \lambda_1^*\right). \tag{3.21}$$

From (3.15), (3.16) follows

$$\tilde{d}_x = -\frac{1}{i}\tilde{c}\tilde{b} \Rightarrow \tilde{d} = \frac{1}{i}\int q\bar{q}\,dx \Rightarrow \tilde{d} = -\tilde{a}. \tag{3.22}$$

We denote $c = c_1 \frac{2|Q|^2}{iu}$. From (3.21), (3.22) follows

$$\begin{cases} \left(\frac{v}{2} - \lambda_1^*\right) = c_1 \frac{2|Q|^2}{iu}, & \Rightarrow \quad c_1 = \frac{iu}{2|Q|^2}\left(\frac{v}{2} - \lambda_1^*\right), \\ \frac{2|Q|^2}{iu} = -\frac{iu}{2}, & \Rightarrow \quad 4|Q|^2 = u^2, \quad \Rightarrow \quad |Q|^2 = \frac{u^2}{4} \end{cases} \tag{3.23}$$

Taking into account c, (3.23), we obtain (3.18) in the form

$$\bar{a} = -\frac{u}{2i}\,\mathrm{th}\left\{\frac{u}{2}(x - vt - x_0)\right\} - c. \tag{3.24}$$

Thus, the matrix $\tilde{A}$ for one-soliton solution (3.2) of the nonlinear Schrödinger equation takes the form

$$\tilde{A} = \begin{pmatrix} -\frac{u}{2i}\,\mathrm{th}\{\frac{u}{2}(x - vt - x_0)\} - c & -Q\frac{\exp(-i(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{\mathrm{ch}\{\frac{u}{2}(x - vt - x_0)\}} \\ Q\frac{\exp\{i(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{\mathrm{ch}\{\frac{u}{2}(x - vt - x_0)\}} & \frac{u}{2i}\,\mathrm{th}\{\frac{u}{2}(x - vt - x_0)\} + c \end{pmatrix}. \tag{3.25}$$

Now we take $\phi = I - \frac{A}{(\lambda - \lambda_1)^2}$, where λ_1 is constants, then from (3.1) we have

$$P = \phi^{-1}\phi_\lambda = \left(I + \frac{\tilde{A}}{\lambda - \lambda_1}\right)\frac{\tilde{A}}{(\lambda - \bar{\lambda}_1)^2} \tag{3.26}$$

On the other hand, we obtain

$$P = \sum_{j=1}^{3} P_j f_j = -\frac{i}{2}\sum_{j=1}^{3} P_j \sigma_j = \begin{pmatrix} -\frac{i}{2}P_3 & -\frac{i}{2}P_1 - \frac{1}{2}P_2 \\ -\frac{i}{2}P_1 + \frac{1}{2}P_2 & \frac{i}{2}P_3 \end{pmatrix}. \tag{3.27}$$

From (3.26), (3.27) by (3.22) we have $P_3 = \frac{2i\bar{a}}{(\lambda - \bar{\lambda}_1)^2}$. Now with the help of (3.24) we find P_3 explicitly for solution of the nonlinear Schrödinger equation

$$P_3 = -\frac{4|Q|^2 c_1}{u(\lambda - \bar{\lambda}_1)^2} - \frac{u\,\mathrm{th}\{\frac{u}{2}(x - vt - x_0)\}}{(\lambda - \bar{\lambda}_1)^2}. \tag{3.28}$$

From (3.26), (3.27) we have $P_2 = \frac{\bar{c} - \bar{b}}{(\lambda - \bar{\lambda}_1)^2}$. Thus $P_1 = \frac{i(\bar{c} + \bar{b})}{(\lambda - \bar{\lambda}_1)^2}$, $P_2 = \frac{(\bar{c} - \bar{b})}{(\lambda - \bar{\lambda}_1)^2}$, $P_3 = \frac{2i\bar{a}}{(\lambda - \bar{\lambda}_1)^2}$. From (3.26), (3.2) using the known formulas

$$\begin{aligned} \mathrm{sh}\,\zeta &= \frac{e^{\zeta} - e^{-\zeta}}{2}; & \mathrm{ch}\,\zeta &= \frac{e^{\zeta} + e^{-\zeta}}{2}; \\ \cos\zeta &= \frac{e^{i\zeta} + e^{-i\zeta}}{2}; & \sin\zeta &= \frac{e^{i\zeta} - e^{-i\zeta}}{2i}. \end{aligned} \tag{3.29}$$

where $\zeta = (\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})$ we obtain the explicit values of P_1, P_2, P_3 matrix P

$$P_1 = -\frac{2Q\sin(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{(\lambda - \bar{\lambda}_1)^2 \operatorname{ch}\{\frac{u}{2}(x - vt - x_0)\}},$$
$$P_2 = \frac{2Q\cos(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{(\lambda - \bar{\lambda}_1)^2 \operatorname{ch}\{\frac{u}{2}(x - vt - x_0)\}}, \tag{3.30}$$

$$P_3 = -\frac{4|Q|^2 c_1}{u(\lambda - \bar{\lambda}_1)^2} - \frac{u \operatorname{th}\{\frac{u}{2}(x - vt - x_0)\}}{(\lambda - \bar{\lambda}_1)^2}. \tag{3.31}$$

Now we can calculate the coefficients on the first quadratic form i.e.

$$E = P_{1x}^2 + P_{2x}^2 + P_{3x}^2. \tag{3.32}$$

For this, we compute P_{1x}, P_{2x}, P_{3x}. Now the first derivatives are raised separately to the 2nd power and substitute into (3.32), then

$$E = \frac{Q^2(u^2 + v^2)}{(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^2\{\frac{u}{2}(x - vt - x_0)\}}.$$

Similarly, according to the formulae $F = P_{1x}P_{1t} + P_{2x}P_{2t} + P_{3x}P_{3t}$, $G = P_{1t}^2 + P_{2t}^2 + P_{3t}^2$ we obtain the values

$$F = -\frac{v(u^2 + v^2)Q^2}{2(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^2\{\frac{u}{2}(x - vt - x_0)\}},$$
$$G = \frac{Q^2(u^2 + v^2)^2}{4(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^2\{\frac{u}{2}(x - vt - x_0)\}}. \tag{3.33}$$

Now, using (3.30), (3.31) we calculate coefficients of the second form L, M, N. For this, we have to calculate

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_t}{\sqrt{\Lambda}}, \quad \sqrt{\Lambda} = \sqrt{EG - F^2}. \tag{3.34}$$

Directly substituting the values of (3.30)–(3.31) to (3.34) we calculate the components vector $\mathbf{n}$. Here we present the calculation

$$n_1 = -\frac{u^2(u^2 + v^2)Q\sin(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{4\sqrt{\Lambda}(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^3\{\frac{u}{2}(x - vt - x_0)\}}. \tag{3.35}$$

$$n_2 = \frac{u^2(u^2 + v^2)Q\cos(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{4\sqrt{\Lambda}(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^3\{\frac{u}{2}(x - vt - x_0)\}}, \tag{3.36}$$

$$n_3 = -\frac{Q^2 u(u^2+v^2)\,\mathrm{sh}[\frac{u}{2}(x-vt-x_0)]}{2\sqrt{\Lambda}(\lambda-\bar{\lambda}_1)^4\,\mathrm{ch}^3\{\frac{u}{2}(x-vt-x_0)\}}. \tag{3.37}$$

We calculate with the help of (3.33)

$$\sqrt{\Lambda} = \left(EG-F^2\right)^{\frac{1}{2}} = \left\{\frac{Q^4(u^2+v^2)^2u^2}{4(\lambda-\bar{\lambda}_1)^8\,\mathrm{ch}^4\{\frac{u}{2}(x-vt-x_0)\}}\right\}^{\frac{1}{2}}. \tag{3.38}$$

Now we find P_{1xx}, P_{2xx}, P_{3xx}. Then we can find L. By the similar way we calculate M, N. Now, using (3.38), (3.34) Gaussian and mean curvature K and H can be calculated

$$K = \frac{1}{4u^2}(1-v^2)(\lambda-\bar{\lambda}_1)^4, \qquad H = \frac{1}{2u^3}(v^2-u^2-1)(\lambda-\bar{\lambda}_1)^2. \tag{3.39}$$

Now, from (2.4), (2.5) using (3.1) for the case γ_0, $M_1 = 0$ we have coefficients of the first fundamental form corresponding to (3.2) as $E = \frac{1}{4}$, $F = -\frac{\lambda}{2}$, $G = \lambda^2 + \bar{q}q$. Respectively, from (2.8)–(2.10) using (3.1), we have coefficients of the second quadratic form. Now we can calculate $\Lambda = EG - F^2 = \frac{1}{4}\bar{q}q$. Theorem is proved. □

4 Conclusion

Thus, we have examined the soliton immersion in $(1+1)$-dimension and obtained the corresponding formulae. As an example of such immersion we consider $(1+1)$-dimensional nonlinear Schrodinger equation. It is found integrable surface corresponding to the one-soliton solution of the nonlinear Schrödinger equation given by the first and second quadratic forms with coefficients (3.3)–(3.5). We have calculated the Gaussian and mean curvature of found surface. We see, that the geometric equation describing the n-curvilinear coordinate systems in flat Euclidean and pseudo-Euclidean space allow some integrable reductions. In addition, we have assumed that immersion 3- and 4-dimensional manifolds arbitrarily embedded in R^n admit integrable cases.

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