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ε-approximation of the equations of heat convection for the Kelvin-Voight fluids

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Abstract. We study one an ε - approximation for the initial-boundary value problem with free surface condition for the heat convection for Kelvin-Voight fluids in bounded domain $\Omega \subset R^m$, $m = 2,3$ with a smooth boundary.The theorems of existence and uniqueness of smooth solutions of $\varepsilon-$ regularization initial value problem in Sobolev spaces are proved. The estimate for rate of convergence of solution for $\varepsilon \to 0$ is obtained.

Keywords: ε-approximation, Kelvin-Voight fluids, Heat convection
PACS: 02.30.Mv

INTRODUCTION. STATEMENT OF THE PROBLEM

In the work [1], the unique solvability of the following initial-boundary value problem for the system of the nonlinear partial differential equations describing motion of the linear viscoelastic incompressible Kelvin-Voight fluids has been investigated:

$$\vec{v}_t - \nu\Delta\vec{v} + v_k\vec{v}_{x_k} + grad\, p - \chi\Delta\vec{v}_t = \vec{f}\,(x,t) + g\vec{\gamma}\theta,\ \vec{\gamma} = (0,0,1)\,, \tag{1}$$

$$\mathrm{div}\vec{v} = 0, \tag{2}$$

$$\theta_t - \lambda\Delta\theta + (\vec{v}\cdot\nabla)\,\theta = q(x,t), \tag{3}$$

$$\vec{v}|_{t=0} = \vec{v}_0\,(x)\,,\ \theta|_{t=0} = \theta_0\,(x)\,, \tag{4}$$

$$\vec{v}_n|_{\partial\Omega} = 0, (\,rot\vec{v}\times n)|_{\partial\Omega} = 0,\ \theta|_{\partial\Omega} = 0, \tag{5}$$

where v_n is normal component of the vector-function (velocity of a fluid) $\vec{v}(x,t)$ on $\partial\Omega$, $p(x,t)$ is pressure, $\theta(x,t)$ is temperature, $\vec{f}(x,t)$ is denoted the external forces, $q(x,t)$ is density of the external heat flow, ν,λ and χ are some positive physical coefficients.

Thus, the system (1)-(5) is not evolutionary, so that the direct application of method of fractional steps is difficult [2]. To overcome these difficulties due to the incompressibility condition (2), in the works [3–5] some $\varepsilon-$ approximations for system of Navier-Stokes equations have been proposed, at which the incompressibility condition (2) is approximated by some equations with a small parameters $\varepsilon > 0$. Thus, the system of the Cauchy-Kowalewskii type is obtained as a result.

By arguing [6, 7], we approximate equations (1) and (3) by following equations:

$$\vec{v}^{\varepsilon}_t - \nu\Delta\vec{v}^{\varepsilon} + v^{\varepsilon}_k\vec{v}^{\varepsilon}_{x_k} - \chi\Delta\vec{v}^{\varepsilon}_t + \frac{1}{2}\vec{v}^{\varepsilon}\mathrm{div}\vec{v}^{\varepsilon} - \nabla p = \vec{f}\,(x,t) + g\vec{\gamma}\theta^{\varepsilon},\ \vec{\gamma} = (0,0,1)\,, \tag{6}$$

$$\theta^{\varepsilon}_t - \lambda\Delta\theta^{\varepsilon} + (\vec{v}^{\varepsilon}\cdot\nabla)\,\theta^{\varepsilon} + \frac{1}{2}\theta^{\varepsilon}\mathrm{div}\vec{v}^{\varepsilon} = q(x,t), \tag{7}$$

and equation (2) is approximated by the equation

$$\varepsilon {p_t}^{\varepsilon} + \mathrm{div}\vec{v}^{\varepsilon} = 0,\ p^{\varepsilon}(x,0) = p_0(x). \tag{8}$$

The system of equations (6)-(8) after the transformations

$$p^{\varepsilon} = p_0\,(x) - \frac{1}{\varepsilon}\int_0^t \mathrm{div}\vec{v}^{\varepsilon}d\tau,\ \vec{\omega}^{\varepsilon} \equiv \int_0^t \mathrm{div}\vec{v}^{\varepsilon}d\tau$$

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reduces to the system

$$L_1(\vec{v}^\varepsilon,\theta^\varepsilon)\equiv\vec{v}_t^\varepsilon-\nu\Delta\vec{v}^\varepsilon+v_k^\varepsilon\vec{v}_{x_k}^\varepsilon-\chi\Delta\vec{v}_t^\varepsilon+\frac{1}{2}\vec{v}^\varepsilon\mathrm{div}\vec{v}^\varepsilon-\frac{1}{\varepsilon}\mathrm{graddiv}\vec{\omega}^\varepsilon=\vec{f}(x,t)+g\vec{\gamma}\theta^\varepsilon,\quad \vec{\omega}_t^\varepsilon=\vec{v}^\varepsilon, \tag{9}$$

$$L_2(\theta^\varepsilon,\vec{v}^\varepsilon)\equiv\theta_t^\varepsilon-\lambda\Delta\theta^\varepsilon+(\vec{v}^\varepsilon\cdot\nabla)\theta^\varepsilon+\frac{1}{2}\theta^\varepsilon\mathrm{div}\vec{v}^\varepsilon=q(x,t), \tag{10}$$

where we denoted $\nabla p_0+f(x,t)$ again by $f(x,t)$ for simplicity.

We study the system of equations (9)-(10) in Q_T with initial conditions

$$\vec{v}^\varepsilon|_{t=0}=\vec{v}_0(x),\ \vec{\omega}^\varepsilon|_{t=0}=0,\ \theta^\varepsilon|_{t=0}=\theta_0(x), \tag{11}$$

and free surface conditions [8]

$$\vec{v}_n^\varepsilon\equiv\vec{v}^\varepsilon\cdot n|_{\partial\Omega}=0,\ (\mathrm{rot}\vec{v}^\varepsilon\times n)|_{\partial\Omega}=0,\ \vec{\omega}_n^\varepsilon|_{\partial\Omega}=0,\ (\mathrm{rot}\vec{\omega}^\varepsilon\times n)|_{\partial\Omega}=0,\ \theta^\varepsilon|_{\partial\Omega}=0. \tag{12}$$

An $\varepsilon-$ approximation for the system (1)-(2) were investigated in [9] where the equation (2) has been approximated by $\varepsilon p^\varepsilon+\mathrm{div}\vec{v}^\varepsilon=0$.

We use the following notation of functional spaces and their norms studied in [7]:

$$H^k(\Omega)\equiv W_2^k(\Omega),\ k=1,2,...,$$
$$H_n^1(\Omega)\equiv\left\{u\in H^1(\Omega):\ u_n|_{\partial\Omega}=0\right\},$$
$$H_n^2(\Omega)\equiv\left\{u(x)\in H^2(\Omega)\cap H_n^1(\Omega):\ (rot\vec{u}\times\vec{n})|_{\partial\Omega}=0\right\},$$
$$J_n^2(\Omega)\equiv\left\{u(x)\in H_n^2(\Omega):\ \mathrm{div}\vec{u}(x)=0,\ x\in\Omega\right\},$$

where $W_2^k(\Omega)$ and $L_2(\Omega)$ are classical Sobolev spaces.

We also apply (see [6]) the Poincare's inequality

$$\|\vec{v}\|_{2,\Omega}\leq C_p(\Omega)\|\nabla\vec{v}\|_{2,\Omega},\qquad\forall\vec{v}\in H_0^1(\Omega),\qquad(\text{or } H_n^1(\Omega)), \tag{13}$$

Ladyzhenskaya's inequality

$$\|\vec{v}\|_{4,\Omega}\leq\sqrt[4]{4}\|\vec{v}\|_{2,\Omega}^{\frac{1}{4}}\cdot\|\vec{v}_x\|_{2,\Omega}^{\frac{3}{4}},\qquad\Omega\subset R^3, \tag{14}$$

and the following inequalities

$$c(\Omega)\|v\|_{H^1(\Omega)}\leq\left(\|\mathrm{rot}v\|^2+\|\mathrm{div}v\|^2\right)^{\frac{1}{2}}\leq c'(\Omega)\|v\|_{H^1(\Omega)},\quad\forall v\in H_n^1(\Omega), \tag{15}$$

$$C(\Omega)\|\vec{v}\|_{H^2(\Omega)}\leq\|\Delta\vec{v}\|\leq C'(\Omega)\|\vec{v}\|_{H^2\Omega)},\quad\forall\vec{v}\in H_n^2(\Omega). \tag{16}$$

UNIQUE EXISTENCE AND CONVERGENCE OF THE SOLUTION OF (9)-(12)

The following theorem is the main theorem of the work.

Theorem 1. *Let be $\vec{v}_0(x)\in J_n^2(\Omega),\ \theta_0(x)\in\mathring{W}_2^1(\Omega),\ \vec{f}(x,t),\ \vec{f}_t(x,t)\in L_2(Q_T)$.*

Then, the initial-boundary value problem (9)-(12) for $\forall\varepsilon>0$ has a unique solution $(\vec{v}^\varepsilon,\ \vec{\omega}^\varepsilon,\ \theta^\varepsilon)$ such that

$$\vec{v}^\varepsilon,\ \vec{\omega}^\varepsilon\in W_\infty^1\left(0,T;H_n^2\right),\ \theta^\varepsilon\in W_2^1\left(0,T;W_2^2\right)\cap L_\infty\left(0,T;\mathring{W}_2^1\right)$$

and the following estimate holds:

$$\|\vec{v}^\varepsilon(x,t)\|^2_{W_\infty^1(0,T;H^2(\Omega))}+\|\theta^\varepsilon\|^2_{L_\infty(0,T;\mathring{W}_2^1(\Omega))}+\frac{1}{\varepsilon}\|qraddiv\vec{v}^\varepsilon\|^2_{L_\infty(0,T;L_2(\Omega))}+\|\theta_t^\varepsilon\|^2_{2,Q_T}$$
$$+\|\theta^\varepsilon\|^2_{L_2(0,T;W_2^2(\Omega)\cap\mathring{W}_2^1(\Omega))}+\frac{1}{\varepsilon^2}\|qraddiv\vec{\omega}^\varepsilon\|^2_{L_\infty(0,T;L_2(\Omega))}\leq C_0<\infty. \tag{17}$$

Moreover, the strong solution $(\vec{v}^{\varepsilon}, \vec{\omega}^{\varepsilon}, \theta^{\varepsilon})$ *of (9)-(12) converges for* $\varepsilon \to 0$ *to the smooth solution* $(\vec{v}(x,t), \nabla p(x,t), \theta(x,t))$ *of the initial-boundary value problem (1)-(5) such that*

$$\vec{v} \in W^1_\infty\left(0,T;J^2_n\right), \quad \nabla p \in L_\infty(0,T;L_2), \quad \theta \in W^1_2\left(0,T;W^2_2\right) \cap L_\infty\left(0,T;\overset{\circ}{W}{}^1_2\right),$$

and the following estimate holds

$$\|\vec{v}\|^2_{W^1_\infty(0,T;H^2(\Omega))} + \|\theta\|^2_{L_\infty(0,T;\overset{\circ}{W}{}^1_2(\Omega))} + \|\nabla p\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\theta_t\|^2_{Q_T} + \|\theta\|^2_{L_2(0,T;W^2_2(\Omega)\cap\overset{\circ}{W}{}^1_2(\Omega))} \le C.$$

Here C_i *denotes the constants depending only on initials of the problem and independent on the small parameter* ε.

It is well known [4, 5], to prove the Theorem 1 it suffices to prove a priori estimate (17), then the solution $(\vec{v}^{\varepsilon}, \vec{\omega}^{\varepsilon}, \theta^{\varepsilon})$ of the problem (9)-(12) will be constructed by the Faedo-Galerkin method and the convergence of the solution $(\vec{v}^{\varepsilon}, \vec{\omega}^{\varepsilon}, \theta^{\varepsilon})$ of the perturbed problem (9)-(12) for $\varepsilon \to 0$ to the smooth solution $(\vec{v}, \nabla p, \theta)$ of the initial-boundary value problem (1)-(5) follows from well known compactness theorems [4]-[5].

Proof of the estimate (17). In order to prove (17), at first we multiply the equation (10) by θ^{ε} and integrate over Ω. After integrating by parts and using Hölder's, Cauchy's inequalities and the Gronwall's lemma, we get the estimate

$$\|\theta^{\varepsilon}\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\theta^{\varepsilon}_x\|^2_{L_2(0,T;L_2(\Omega))} \le C_1\left(\lambda^{-1}, \|q\|^2_{2,Q_T}, \|\theta_0\|^2_{2,\Omega}\right). \tag{18}$$

We multiply the equation (9) by $\vec{v}^{\varepsilon}$, $\Delta\vec{v}^{\varepsilon}$, $\frac{1}{\varepsilon}\mathrm{graddiv}\vec{\omega}^{\varepsilon}$, the equation (10) by $\Delta\theta^{\varepsilon}$, θ^{ε}_t and a priori differentiated by t equation (9) by $\vec{v}^{\varepsilon}_t$, $\Delta\vec{v}^{\varepsilon}_t$, and integrate the obtained results over Ω. Then using the following Green's formulas (see [6]), which are valid for all functions $\vec{v}$, $\vec{\omega} \in H^k_n(\Omega)$, $k = 1,2$, satisfying the boundary condition (5)

$$\begin{aligned}
&(-\Delta\vec{v}, \vec{\omega})_{2,\Omega} = -(\mathrm{grad\ div}\vec{v}, \vec{\omega})_{2,\Omega} + \left(\mathrm{rot}^2\vec{v}, \omega\right)_{2,\Omega} \\
&= -\int_{\partial\Omega} \mathrm{div}\vec{v}\cdot\vec{\omega}_n dS + (\mathrm{div}\vec{v}, \mathrm{div}\vec{\omega})_{2,\Omega} + \int_{\partial\Omega} \vec{\omega}\,(\mathrm{rot}\vec{v}\times\vec{n})\,dS + (\mathrm{rot}\vec{v}, \mathrm{rot}\vec{\omega})_{2,\Omega} \\
&= (\mathrm{div}\vec{v}, \mathrm{div}\vec{\omega})_{2,\Omega} + (\mathrm{rot}\vec{v}, \mathrm{rot}\vec{\omega})_{2,\Omega},
\end{aligned} \tag{19}$$

$$\begin{aligned}
(\mathrm{grad\ div}\vec{v}, \Delta\vec{\omega})_{2,\Omega} &= (\mathrm{grad\ div}\vec{v},\ \mathrm{grad\ div}\vec{\omega})_{2,\Omega} - \int_{\partial\Omega} \mathrm{grad\ div}\vec{v}\,(\mathrm{rot}\vec{\omega}\times\vec{n})\,dS \\
&- (\mathrm{rot\ graddiv}\vec{v},\ \mathrm{rot}\vec{\omega})_{2,\Omega} = (\mathrm{grad\ div}\vec{v},\ \mathrm{grad\ div}\vec{\omega})_{2,\Omega},
\end{aligned} \tag{20}$$

we arrive at the following integral relations:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\vec{v}^{\varepsilon}\|^2_{2,\Omega} + \chi\left(\|\mathrm{div}\vec{v}^{\varepsilon}\|^2_{2,\Omega} + \|rot\vec{v}^{\varepsilon}\|^2_{2,\Omega}\right) + \frac{1}{\varepsilon}\|\mathrm{div}\vec{\omega}^{\varepsilon}\|^2_{2,\Omega}\right) \\
&+\nu\left(\|\mathrm{div}\vec{v}^{\varepsilon}\|^2_{2,\Omega} + \|rot\vec{v}^{\varepsilon}\|^2_{2,\Omega}\right) = \left(\vec{f} + \vec{\gamma}g\theta^{\varepsilon}, \vec{v}^{\varepsilon}\right)_{2,\Omega},
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathrm{div}\vec{v}^{\varepsilon}\|^2_{2,\Omega} + \|rot\vec{v}^{\varepsilon}\|^2_{2,\Omega} + \chi\|\Delta\vec{v}^{\varepsilon}\|^2_{2,\Omega} + \frac{1}{\varepsilon}\|\mathrm{graddiv}\vec{\omega}^{\varepsilon}\|^2_{2,\Omega}\right) + \nu\|\Delta\vec{v}^{\varepsilon}\|^2_{2,\Omega} \\
&= B\left((\vec{v}^{\varepsilon}, \vec{v}^{\varepsilon}), \Delta\vec{v}^{\varepsilon}\right)_{2,\Omega} - \left(\vec{f} + \vec{\gamma}g\theta^{\varepsilon}, \Delta\vec{v}^{\varepsilon}\right)_{2,\Omega},
\end{aligned} \tag{22}$$

$$\begin{aligned}
\frac{1}{\varepsilon^2}\|\mathrm{graddiv}\vec{\omega}^{\varepsilon}\|^2_{2,\Omega} &= \frac{1}{\varepsilon}\left(\vec{v}^{\varepsilon}_t - \varphi\Delta\vec{v}^{\varepsilon}_t - \nu\Delta\vec{v}^{\varepsilon} - \vec{f} - g\vec{\gamma}\theta^{\varepsilon}, \mathrm{graddiv}\vec{\omega}^{\varepsilon}\right) \\
&+\frac{1}{\varepsilon}\left(B(\vec{v}^{\varepsilon}, \vec{v}^{\varepsilon}), \mathrm{graddiv}\vec{\omega}^{\varepsilon}\right)_{2,\Omega},
\end{aligned} \tag{23}$$

$$\frac{1}{2}\frac{d}{dt}\|\theta^{\varepsilon}_x\|^2_{2,\Omega} + \gamma\|\Delta\theta^{\varepsilon}\|^2_{2,\Omega} = \left(B(\vec{v}^{\varepsilon}, \theta^{\varepsilon}) + q, \Delta\theta^{\varepsilon}\right)_{2,\Omega}, \quad \forall t \in (0,T), \tag{24}$$

$$\frac{\lambda}{2}\frac{d}{dt}\|\theta^{\varepsilon}_x\|^2_{2,\Omega} + \|\theta^{\varepsilon}_t\|^2_{2,\Omega} = \left(B(\vec{v}^{\varepsilon}, \theta^{\varepsilon}) + q, \theta^{\varepsilon}_t\right)_{2,\Omega}, \quad \forall t \in (0,T), \tag{25}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\vec{v}_t^\varepsilon\|_{2,\Omega}^2+\chi\left(\|\mathrm{div}\vec{v}_t^\varepsilon\|_{2,\Omega}^2+\|\mathrm{rot}\vec{v}_t^\varepsilon\|_{2,\Omega}^2\right)+\frac{1}{\varepsilon}\|\mathrm{div}\vec{v}^\varepsilon\|_{2,\Omega}^2\right)$$
$$+\nu\left(\|\mathrm{div}\vec{v}_t^\varepsilon\|_{2,\Omega}^2+\|\,\mathrm{rot}\vec{v}_t^\varepsilon\|_{2,\Omega}^2\right)=\left(\vec{f}_t+g\vec{\gamma}\theta_t^\varepsilon,\,\vec{v}_t^\varepsilon\right)-\left(\vec{v}_t^\varepsilon\nabla\vec{v}^\varepsilon+\frac{1}{2}\vec{v}_t^\varepsilon\mathrm{div}\vec{v}^\varepsilon,\,\vec{v}_t^\varepsilon\right), \tag{26}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathrm{div}\vec{v}_t^\varepsilon\|_{2,\Omega}^2+\|\mathrm{rot}\vec{v}_t^\varepsilon\|_{2,\Omega}^2+\chi\|\Delta\vec{v}_t^\varepsilon\|_{2,\Omega}^2+\frac{1}{\varepsilon}\|\mathrm{graddiv}\vec{v}^\varepsilon\|_{2,\Omega}^2\right)+\nu\|\Delta\vec{v}_t^\varepsilon\|_{2,\Omega}^2$$
$$=-\left(\vec{f}_t+g\vec{\gamma}\theta_t^\varepsilon,\Delta\vec{v}_t^\varepsilon\right)-\left(\frac{\partial}{\partial t}B(\vec{v}^\varepsilon,\,\vec{v}^\varepsilon),\,\Delta\vec{v}_t^\varepsilon\right), \tag{27}$$

where

$$\left(B\left(\vec{v}^\varepsilon,\,\theta^\varepsilon\right),\,\omega\right)=\int\limits_\Omega\left(\left(\vec{v}^\varepsilon\cdot\nabla\right)\theta^\varepsilon+\frac{1}{2}\theta^\varepsilon\mathrm{div}\vec{v}^\varepsilon\right)\omega dx$$

and we denote by $(\cdot,\cdot)$ the inner product in $L_2(\Omega)$.

Now, we estimate the right-hand side of (21) by Hölder's inequality, then using (15) and (18), we get the estimate

$$\|\vec{v}_x^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2+\|\vec{v}_x^\varepsilon\|_{2,Q_T}^2+\frac{1}{\varepsilon}\|\mathrm{div}\vec{\omega}^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2\le C_2\left(\nu^{-1},\Omega,\|f\|_{2,Q_T}^2,C_1,\|v_0\|^{(1)}\right). \tag{28}$$

The terms on the right-hand side of (22) can be estimated by Hölder's inequality, Poincare's inequality and the inequality (16). In consequence, using the estimates (18), (28), we obtain

$$\|\vec{v}_x^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2+\|\vec{v}_x^\varepsilon\|_{2,Q_T}^2+\frac{1}{\varepsilon}\|\mathrm{graddiv}\vec{\omega}^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2\le C_3\left(\Omega,\|f\|_{2,Q_T}^2,C_2^3,\|v_0\|^{(2)}\right). \tag{29}$$

Applying the same method to (24), we can easily get the following estimate

$$\|\theta_x^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2+\|\Delta\theta^\varepsilon\|_{2,Q_T}^2\le C_4<\infty. \tag{30}$$

Next, we estimate the integrals on the right-hand side in (25) by Hölder's, Young's, Poincare's inequalities and (18)-(12). Then using the Granwoll's lemma, we have

$$\|\theta_x^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2+\|\theta_t^\varepsilon\|_{2,Q_T}^2\le C_5<\infty. \tag{31}$$

Applying the Hölder's inequality, Ladyzhenskaya inequality (14), and the estimates (18)-(31) to right-hand side of (26), we get the estimate

$$\|\vec{v}_t^\varepsilon,\nabla\vec{v}_t^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2+\|\nabla\vec{v}_t^\varepsilon\|_{2,Q_T}^2+\frac{1}{\varepsilon}\|\mathrm{div}\vec{v}^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2\le C_6<\infty. \tag{32}$$

Analogical way as above, we get from (27) the estimate

$$\|\nabla\vec{v}_t^\varepsilon,\vec{v}_{xxt}^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2+\|\vec{v}_{xxt}^\varepsilon\|_{2,Q_T}^2+\frac{1}{\varepsilon}\|graddiv\vec{v}^\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}^2\le C_7\left(\|f,f_t\|_{2,Q_T}^2,\|v_0\|^{(2)}\right), \tag{33}$$

where we used the inequality

$$\|\,\vec{v}_t|_{t=0}\|^{(2)}\le C_8\left(\nu^{-1},\chi^{-1},\|\vec{v}_0\|^{(2)},\|f(x,0)\|\right).$$

Finally, estimating the terms on right-hand side of (24) by Hölder's, Young's inequalities, and the already obtained estimates, we obtain

$$\frac{1}{\varepsilon^2}\|graddiv\vec{\omega}^\varepsilon\|_{2,\Omega}^2\le C_9\left(\chi^{-1},\nu^{-1},\Omega,\|\vec{v}_0\|^{(2)},\|\theta_0\|,\|f,\,f_t\|_{2,Q_T}^2\right). \tag{34}$$

Estimates (18), (28)-(34) imply the estimate (17).

In numerical analysis an estimate of convergence rate is very important. For the rate of convergence the following theorem holds.

Theorem 2. *Let conditions of Theorem 1 are fulfilled. Then for the rate of convergence the following estimate holds*

$$\|\vec{v}(x,t)-\vec{v}^\varepsilon(x,t)\|_{L_\infty\left(0,T;H^1(\Omega)\right)}+\|\theta(x,t)-\theta^\varepsilon(x,t)\|_{L_\infty(0,T;L_2(\Omega))}$$
$$+\|\vec{v}(x,t)-\vec{v}^\varepsilon(x,t)\|_{L_2\left(0,T;H^1(\Omega)\right)}+\|\theta(x,t)-\theta^\varepsilon(x,t)\|_{L_2\left(0,T;W_2^1(\Omega)\right)}\le C_{10}\varepsilon^{\frac{1}{2}}.$$

Analogical way as in [9], one can prove the Theorem 2.

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