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Reconstruction of source function for parabolic equations with variable exponents

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Abstract. We study the inverse problem of reconstructing the right side for nonlinear parabolic equation with integral overdetermination. We prove the unique solvability of this problem. By using the method of successive approximations, the existence and uniqueness theorem for the solution of the inverse problem is proved. The proof of the uniqueness of a generalized solution is based on obtained a priori estimate.

Keywords: Inverse problem, Nonlinear parabolic equations, Nonlocal condition overdetermination
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INTRODUCTION

The questions of existence and uniqueness of solutions of degenerate parabolic equations with constant exponents of nonlinearity are studied in the works [1–5]. Parabolic and elliptic equations with variable exponents of nonlinearity arise in the mathematical description of the electrorheological fluids [6, 7]. The study of degenerate elliptic and parabolic equations with variable exponents of nonlinearity is considered in [8–11].

Boundary problem of determining the coefficients or source function for parabolic equation under the assumption of independence of the unknown coefficients or source function on the time variable or on the spatial variable is considered in [12–19].

STATEMENT OF THE PROBLEM

We consider the inverse problem for nonlinear parabolic equation in the cylinder $Q_T = \Omega \times (0,T)$, $\Omega \subset R^2$. We need to determine functions $u(x,t)$ and $f(t)$ that satisfy the following equation

$$\frac{\partial u}{\partial t} = \mu \Delta u - a(x,t,u)|u|^{\sigma(x,t)-2}u + f(t)\lambda(x,t), \tag{1}$$

boundary conditions

$$u|_{\Gamma_T} = 0, \quad \Gamma_T = \partial\Omega \times [0,T], \tag{2}$$

initial conditions

$$u(x,0) = \varphi(x), \tag{3}$$

and integral overdetermination conditions

$$\int_\Omega u(x,t)K(x,t)dx = e(t), \quad 0 \le t \le T, \tag{4}$$

where $a = a(x,t,u)$ is Caratheodory function defined for $(x,t,r) \in \overline{Q_T} \times R$ (and is measurable with respect to (x,t) for all $r \in R$ and continuous in r for almost all $(x,t) \in Q_T$) as,

$$\begin{cases} \forall (x,t,r) \in \overline{Q_T} \times R, \\ 0 < a_0 \le a(x,t,r) \le a_1 < \infty, \\ \left| \dfrac{\partial a(x,t,u)}{\partial u} \right| \le a_2 < \infty. \end{cases} \tag{5}$$

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We assume that $\sigma = \sigma(x,t)$ is a measurable function defined in Q_T and satisfies the inequality

$$\forall P = (x,t) \in \overline{Q}_T, \quad 2 \leq \sigma^- \leq \sigma(P) \leq \sigma^+ < \infty \tag{6}$$

with given constants σ^- and σ^+. Here $\lambda(x,t),\ \varphi(x),\ K(x,t),\ e(t)$ are given functions.

The inverse problem (1)-(4) can be interpreted as the problem of finding the exact control $f(t)$ such the desired or expected energy $e(t)$ is achieved.

A generalized solution of the inverse problem (1)-(4) is understood as follows.

Definition 1. *Functions $u(x,t)$ and $f(t)$ are said to be a generalized solution of the inverse problem (1)-(4), if $u(x,t) \in L_\infty(Q_T)\bigcap L_2(0,T;\overset{0}{W}{}_2^1(\Omega))$, $u_t \in L_2(0,T;\overset{0}{W}{}_2^{-1}(\Omega))$ and $f(t) \in L_\infty(0,T)$ such that the following integral identities are satisfied*

$$\int_{Q_T}\left[-u\cdot\xi_t + \mu\nabla u\cdot\nabla\xi + a(x,t,u)|u|^{\sigma(x,t)-2}u\cdot\xi\right]dxdt = \int_{Q_T} f(t)\lambda\cdot\xi\, dxdt + \int_\Omega \varphi(x)\xi(x,0)dx \tag{7}$$

for all $\xi(x,t) \in L_\infty(Q_T),\ \xi(x,t) \in L_{\sigma(x,t)}(Q_T),\ \xi(x,t) \in W_2^{1,1}(Q_T)\bigcap \overset{0}{W}{}_2^1(Q_T),\ \xi(x,T) = 0,$

$$e'(t) = \int_\Omega K_t u\, dx + \mu\int_\Omega u\cdot\Delta K dx - \int_\Omega a(x,t,u)|u|^{\sigma(x,t)-2}u\cdot K dx + f(t)\int_0^l \lambda\cdot K dx, \tag{8}$$

where

$$\begin{aligned} &K(x,t) \in C^1(0,T;\overset{0}{W}{}_2^1(\Omega)) \cap (0,T;W_2^2(\Omega)),\ e(t) \in W_2^1(0,T),\ \lambda(x,t) \in C(\bar{Q}_T),\\ &\varphi(x) \in L_\infty(\Omega),\ \int_\Omega K\cdot\lambda dx \neq 0,\ t \in [0,T]. \end{aligned} \tag{9}$$

We introduce the following notation $V(Q_T) := L_\infty(Q_T) \cap L_2(0,T;\overset{0}{W}{}_2^1(\Omega))$.

MAIN RESULT

Theorem 1. *Let conditions (9) hold. Then there exists a unique generalized solution $u(x,t) \in V(Q_T)$, $f(t) \in L_\infty(0,T)$ of the inverse problem (1)-(4).*

Proof. We apply the method of successive approximations. We take $u^0 = \varphi(x)$ (or $u^0 = 0$) as the zero approximation. We define $(u^m,\ f^m)$ by the following recursive relations

$$\begin{aligned} f^m(t) = \left(\int_0^l \lambda\cdot K dx\right)^{-1}\Bigg(e'(t) - \int_\Omega u^{m-1}K_t\, dx - \mu\int_\Omega u^{m-1}\cdot\Delta K dx \\ + \int_\Omega a(x,t,u^{m-1})|u^{m-1}|^{\sigma(x,t)-2}u^{m-1}\cdot K dx\Bigg), \end{aligned} \tag{10}$$

$$\frac{\partial u^m}{\partial t} = \mu\Delta u^m - a(x,t,u^m)|u^m|^{\sigma(x,t)-2}u^m + f^m(t)\lambda(x,t), \tag{11}$$

$$u^m = 0, \quad \Gamma_T = \partial\Omega\times[0,T], \tag{12}$$

$$u^m|_{t=0} = \varphi(x) \tag{13}$$

for $m = 1,2,...$

Definition 2. *A function $u(x,t) \in V(Q_T)$ is said to be a generalized solution of the problem (11)-(13), if it satisfies the following integral identity*

$$\int_{Q_T}\left[-u\cdot\xi_t + \mu\nabla u\cdot\nabla\xi + a(x,t,u)|u|^{\sigma(x,t)-2}u\cdot\xi\right]dxdt = \int_{Q_T} f(t)\lambda\cdot\xi\, dxdt + \int_\Omega \varphi(x)\xi(x,0)dx \tag{14}$$

for all $\xi(x,t) \in L_\infty(Q_T),\ \xi(x,t) \in L_{\sigma(x,t)}(Q_T),\ \xi(x,t) \in W_2^{1,1}(Q_T) \bigcap \overset{0}{W}{}_2^1(Q_T),\ \xi(x,T) = 0$, *where*

$$\lambda \in C(\bar{Q}_T),\ K(x,t) \in C^1(0,T;\overset{0}{W}{}_2^1(\Omega)) \cap (0,T;W_2^2(\Omega)),\ \varphi(x) \in L_\infty(\Omega),\ f^m \in L_\infty(0,T).$$

We assume that the functions $f^m(t)$ are known, since they can be expressed by $u^{m-1}(x,t)$ according to (11). The unique solvability in $V(Q_T) := L_\infty(Q_T) \bigcap L_2(0,T;\overset{0}{W}{}_2^1(\Omega))$ of the problem (11)-(13) follows from [8]-[9], where such problem has been investigated in more general form.

Thus, the sequence $\{(u^m,\ f^m)\}$ is correctly defined. We will prove that the sequence $\{(u^m,\ f^m)\}$ is a Cauchy sequence. Then, the completeness of the space $V(Q_T) \times L_2(0,T)$ implies that the pair of functions $(u,\ f)$ is the limit of the sequence $\{(u^m,\ f^m)\}$, i.e. $(u^m,\ f^m) \to (u,\ f)$ as $m \to \infty$. So, $(u,\ f)$ is the weak solution of the inverse problem (1)-(4).

We introduce the notations $\Phi^{m+1}(t) = f^{m+1}(t) - f^m(t)$, $D^{m+1} = u^{m+1} - u^m$. Then (10) and the problem (11)-(13) can be rewritten in the following form

$$\begin{aligned} \Phi^{m+1} = \left(\int_\Omega \lambda \cdot K dx\right)^{-1} \Big(-\mu \int_\Omega D^m \cdot \Delta K dx - \int_\Omega D^m \cdot K_t dx \\ + \int_\Omega \left(a(x,t,u^m)|u^m|^{\sigma-2}u^m - a(x,t,u^{m-1})|u^{m-1}|^{\sigma-2}u^{m-1}\right) K dx\Big), \end{aligned} \tag{15}$$

$$\frac{\partial D^{m+1}}{\partial t} = \mu \Delta D^{m+1} - \left(a(x,t,u^{m+1})|u^{m+1}|^{\sigma-2}u^{m+1} - a(x,t,u^m)|u^m|^{\sigma-2}u^m\right) + \Phi^{m+1}\lambda, \tag{16}$$

$$D^{m+1}\big|_{\Gamma_T} = 0, \quad D^{m+1}\big|_{t=0} = 0. \tag{17}$$

For differences $\Phi^{m+1}(t)$ and D^{m+1} the following inequalities (see [18]-[19])

$$\left\|\Phi^{m+1}(t)\right\|_{L_\infty(0,T)} \le c_2 \frac{e^{c_3 T} - 1}{c_3} \|\lambda\|_{\infty,Q_T} \|\Phi^m\|_{L_\infty(0,T)}, \tag{18}$$

$$\left\|D^{m+1}\right\|_{V(Q_T)} \le c_2 \frac{e^{c_3 T} - 1}{c_3} \|\lambda\|_{\infty,Q_T} \left(1 + \frac{\sqrt{c_3 + \|\lambda\|_{\infty,Q_T}}}{\sqrt{\mu}}\right) \|D^m\|_{V(Q_T)} \tag{19}$$

hold, where c_2 and c_3 do not depend on $m,\ u^m,\ f^m,\ \Phi^m,\ D^m$.

Suppose that

$$\begin{aligned} & c_2 \frac{e^{c_3 T} - 1}{c_3} \|\lambda\|_{\infty,Q_T} \le q_1 < 1, \\ & c_2 \frac{e^{c_3 T} - 1}{c_3} \|\lambda\|_{\infty,Q_T} \left(1 + \frac{\sqrt{c_3 + \|\lambda\|_{\infty,Q_T}}}{\sqrt{\mu}}\right) \le q_2 < 1. \end{aligned} \tag{20}$$

Then, the inequalities (18) and (19) can be rewritten in the following form

$$\left\|\Phi^{m+1}(t)\right\|_{L_\infty(0,T)} \le q_1 \|\Phi^m\|_{L_\infty(0,T)}, \tag{21}$$

$$\left\|D^{m+1}\right\|_{V(Q_T)} \le q_2 \|D^m\|_{V(Q_T)}. \tag{22}$$

Consequently, if inequalities (20) hold, then inequalities (21) and (22) imply that $(u^m,\ f^m)$ is a Cauchy sequence in the space $V(Q_T) \times L_\infty(0,T)$. By virtue of the above arguments, there exists a unique pair of functions $(u,\ f)$ from $V(Q_T) \times L_\infty(0,T)$, respectively, such that

$$u^m \to u \quad \text{in} \quad V(Q_T),$$

$$f^m \to f \quad \text{in} \quad L_\infty(0,T).$$

Taking the limit in (10) and (14) as $m \to \infty$ and by virtue of convergence of the sequence $(u^m,\ f^m)$, we obtain the generalized solution $(u,\ f)$ for the inverse problem (1)-(4) in $Q_T = \Omega \times (0,T)$.

From the fact that $\{(u^m,\ f^m)\}$ is a Cauchy sequence it follows that the solution of the inverse problem is unique. It can be proved by using inequalities (21) and (22). Suppose that there exist two solutions $(u_k,\ f_k)$, $k = 1,2$ of the inverse problem (1)-(4) in Q_T. Then by (21) and (22), we have

$$(1-q_1)\,\|f_1 - f_2\|_{L_\infty(0,T)} \leq 0,$$

$$(1-q_2)\,\|u_1 - u_2\|_{V(Q_T)} \leq 0.$$

Hence, $u_1 \equiv u_2$ and $f_1 \equiv f_2$. The proof of the theorem is complete. □

CONCLUSION

The global existence and uniqueness of a generalized solution of the inverse problem with an integral overdetermination condition for nonlinear parabolic equation is proved by using the method of successive approximations. The search of new methods to solve the problem of the existence of a global (local) solution of inverse problems for nonlinear equations is relevant. In this regard, the method proposed in this paper, of course, can be applied to the study of many other inverse problems, including the coefficient inverse problems.

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