Metadata of the chapter that will be visualized online

Chapter Title	Optimization Control Problems for Systems Described by Elliptic Variational Inequalities with State Constraints	
Copyright Year	2016	
Copyright Holder	Springer International Publishing Switzerland	
Author	Family Name	**Serovajsky**
	Particle	
	Given Name	**Simon**
	Suffix	
	Email	serovajskys@mail.ru
Abstract	The control system described by variational inequality is considered. It is approximated by the system described by a nonlinear equation with using the penalty method. The convergence of the approximate method is proved. The necessary conditions of optimality for approximate optimization control problem are obtained. The optimal control for the approximate optimization problem is chosen as an approximate solution of the initial problem.	
Keywords (separated by "-")	Necessary conditions of optimality - Optimization - Penalty method - Variational inequality	

Optimization Control Problems for Systems Described by Elliptic Variational Inequalities with State Constraints

Simon Serovajsky

Abstract The control system described by variational inequality is considered. It is approximated by the system described by a nonlinear equation with using the penalty method. The convergence of the approximate method is proved. The necessary conditions of optimality for approximate optimization control problem are obtained. The optimal control for the approximate optimization problem is chosen as an approximate solution of the initial problem.

Keywords Necessary conditions of optimality • Optimization • Penalty method • Variational inequality

Mathematics Subject Classification (2000). Primary 49K20, Secondary 35J85

1 Introduction

Many mathematical physics problems are described by variational inequalities (see, for example, [1–5]). The mathematical theory of these problems is well known (see [2–6]). So optimization control problems for these systems are interesting enough. A lot of results for optimization control problems of systems described by variational inequalities are known (see, for example, [7–15] for elliptic case, [7, 9, 16–18] for parabolic case, and [9] for hyperbolic case). The control systems for variational inequalities with state constraints are analyzed in [8, 10, 11, 15].

We consider the control system with state constraint in the form of the general inclusion. The analysis is based on the Warga's concept of the search of minimizing sequences, but not optimal controls [19] (see also [20, 21]). Besides we will use a double regularization of the optimization control problem. At first the variational inequality, which defines the state of the system, is approximated by a nonlinear

---

S. Serovajsky (✉)
71 al Farabi avenue, 050078 Almaty, Kazakhstan
e-mail: serovajskys@mail.ru

M. Ruzhansky, S. Tikhonov (eds.), *Methods of Fourier Analysis and Approximation Theory*, Applied and Numerical Harmonic Analysis,
DOI 10.1007/978-3-319-27466-9_14

equation with using the penalty method. The analogical technique was used for the classical theory of variational inequalities (see [6]) and optimization control theory (see [7, 9, 11]). Hence we obtain the optimization control problem for a nonlinear elliptic equation with state constraint. It is approximated by the minimization problem for a penalty functional on the set of admissible pairs "state-control". This method was used in [22] for the analysis of the distributed singular systems without state constraints. However our system is regular and we have state constraints. Besides the penalty method was used in [22] for obtaining necessary conditions of optimality for the initial optimization problem (see also [7, 9, 11]). We apply it for finding minimizing sequences with using the idea of Warga [19] (see also [20, 21]). But this means was used for the extension of optimization controls problems in the case of its insolvability there. However we prove the solvability of our problem, and this method is applied for finding an optimal control.

Thus an approximate solution of the initial optimization control problem is chosen as the optimal control for approximate problem for large enough step of the algorithm. The necessary conditions of optimality for the approximate optimization control problem are obtained in the standard form.

2 Problem Statement

Let Ω be an open bounded n-dimensional set, where $n \leq 3$. Define the space $H_0^1(\Omega)$ and its subset Z that consists all functions with non-negative values. We consider the control system described by the variational inequality

$$\int_\Omega (\Delta y + v)(z - y)dx \leq 0 \ \forall z \in Z, \tag{1}$$

where v is the control, and y is the state function.

For any control v from the space $L_2(\Omega)$ the problem (1) is solvable on the set Z (see [6], Sect. 3, Example 5.1). The inequality (1) was approximated in [6] by the homogeneous Dirichlet problem for the nonlinear elliptic equation

$$-\Delta y + \frac{1}{\varepsilon_k} a(y) = v, \tag{2}$$

where $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$, $a(y) = 0$ for $y \geq 0$ and $a(y) = y^3$ if $y < 0$. By monotony method (see [6], Sect. 2, Theorem 2.1) for any $v \in L_2(\Omega)$ the Eq. (2) has a unique solution $y = y_k[v]$ from the space $H_0^1(\Omega) \cap H^2(\Omega)$, and the mapping

$$y_k[\cdot] : L_2(\Omega) \to H_0^1(\Omega)$$

is weakly continuous. Besides $y_k[v] \to y$ weakly in $H_0^1(\Omega)$ after extracting a subsequence by Theorem 5.2 (see [6], Chap. 3), where y is a solution of the variational inequality (1) for this control. Note that the norm of the solution of Eq. (2) is estimated by the norm of the absolute term by this theorem. Then the mentioned convergence is uniform with respect to v from any bounded subset of $L_2(\Omega)$.

Consider convex closed bounded subsets V of $L_2(\Omega)$ and Y of $H_0^1(\Omega)$. The pair (v, y) from the set $V \times Y$ is called admissible if it satisfies the inequality (1) (see [22]). By U denote the set of all admissible pairs. Suppose this set is non-empty for nontriviality of the problem. Consider the functional

$$I(v,y) = \frac{1}{2}\int_\Omega \left[(y - y_\partial)^2 + \chi v^2\right]dx,$$

where y_∂ is a given function from $H_0^1(\Omega)$, $\chi > 0$. We have the following optimization control problem.

Problem P1 *Minimize the functional I on the set U.*

Prove the weak continuity of the solution of the variational inequality (1) with respect to the control. By $y[v]$ denote its solution for the control v.

Lemma 2.1 *If* $\{v_s\} \subset V$ *and* $v_s \to v$ *weakly in* $L_2(\Omega)$, *then* $y[v_s] \to y[v]$ *weakly in* $H_0^1(\Omega)$ *after extracting a subsequence.*

Proof We have

$$y[v_s] - y[v] = \big(y[v_s] - y_k[v_s]\big) + \big(y_k[v_s] - y_k[v]\big) + \big(y_k[v] - y[v]\big).$$

Then $y_k[w] \to y[w]$ weakly in $H_0^1(\Omega)$ uniformly with respect to $w \in V$ after extracting a subsequence as $k \to \infty$. So $y_k[v] \to y[v]$ and $\big(y[v_s] - y_k[v_s]\big) \to 0$ weakly in $H_0^1(\Omega)$. Besides $y_k[v_s] \to y_k[v]$ weakly in $H_0^1(\Omega)$ for all k as $s \to \infty$. Hence the assertions of the lemma follow from the last equality. □

Theorem 2.2 *Problem P1 is solvable.*

Proof Let the sequence of pairs $\big\{(v_s, y_s)\big\}$ be minimizing. So we have the inclusions $v_s \in V$, $y_s \in Y$, the variational inequality

$$\int_\Omega (\Delta y_s + v_s)(z - y_s)dx \le 0 \ \forall z \in Z,$$

and the convergence $I(v_s) \to \inf I(U)$. The sequence $\{v_s\}$ is bounded in $L_2(\Omega)$ by the boundedness of V. Then $v_s \to v$ weakly in $L_2(\Omega)$ after extracting a subsequence. Using Lemma 2.1, we get $y_s \to y[v]$ weakly in $H_0^1(\Omega)$ after extracting a subsequence. So we obtain the inclusions $v \in V$ and $y[v] \in Y$ by the convexity and the closeness of the sets V and Y. Then

$$\big(v, y[v]\big) \in U.$$

Using the lower semicontinuity of the square of the norm for Hilbert space, we have

$$I\big(v, y[v]\big) \leq \varliminf_{s\to\infty} I(v_s, y_s).$$

Thus

$$I\big(v, y[v]\big) \leq I(U).$$

Therefore the pair $\big(v, y[v]\big)$ is a solution of our problem. This completes the proof of the Theorem 2.2. □

Hence the Problem P1 has a solution. Our aim is the development and the substantiation of the method of its resolution.

3 Approximation of the Optimization Control Problem

The optimization control problems for systems described by equations are easier than for systems described by variational inequalities. So we will use the known approximation of the system (1) by the nonlinear elliptic equation (2) for the analysis of Problem P1. Consider the set

$$V_k = \big\{v \in V \mid \ y_k[v] \in Y\big\}$$

and the functional

$$I_k(v) = \frac{1}{2}\int_{\Omega} \Big[(y_k[v] - y_\partial)^2 + \chi v^2\Big] dx,$$

Problem P2 *Minimize the functional I_k on the set V_k.*

Prove the non-triviality of the set V_k at first. We supposed that the set U is non-empty. Use now the more strong assumption. Suppose the existence of a point $v \in V$ such that the state $y[v]$ belongs to the interior of the set Y with respect to the weak topology of $H_0^1(\Omega)$.

Lemma 3.1 *Under this supposition the set V_k is non-empty for large enough value k.*

Proof By our assumption the state $y[v]$ belongs to the interior of the set Y for some control $v \in V$. Then there exists a neighborhood O of $y[v]$ such that it is the subset of this set. By convergence $y_k[v] \to y[v]$ weakly in $H_0^1(\Omega)$ the point $y_k[v]$ belongs to O for a large enough k. Then $y_k[v] \in Y$. So the set V_k is non-empty. □

Using the weakly continuity of the map

$$y_k[\cdot] : L_2(\Omega) \to H_0^1(\Omega),$$

we obtain the following result.

Lemma 3.2 *Problem P2 is solvable.*

By v_k denote a solution of Problem P2. Prove the convergence of the approximation method.

Theorem 3.3 *We have the convergence $I\big(v_k, y[v_k]\big) \to \inf I(U)$ as $k \to \infty$ and $v_k \to v_*$ in $L_2(\Omega)$ after extracting a subsequence, where v_* is a solution of Problem P1.*

Proof We have

$$I_k(v_k) = \min I_k(V_k) \le I_k(v_*).$$

Using the definition of the approximate functional, we get

$$I_k(v_*) = \frac{1}{2}\int_\Omega \Big\{(y_k[v_*] - y_\partial)^2 + \chi v^2\Big\}dx$$

$$= I(v_*) + \frac{1}{2}\int_\Omega \Big\{\big[(y_k[v_*] - y_\partial)\big]^2 - \big[(y[v_*] - y_\partial)\big]^2\Big\}dx.$$

Then

$$I_k(v_k) \le \inf I(U) + \frac{1}{2}\Big\|y_k[v_*] + y[v_*] - 2y_\partial\Big\|_2 \Big\|y_k[v_*] - y[v_*]\Big\|_2,$$

where $\|\cdot\|_p$ is the norm of the space $L_p(\Omega)$. The sequence $\{y_k[v_*]\}$ is bounded in the space $H_0^1(\Omega)$. Besides $y_k[v_*] \to y[v_*]$ weakly in $H_0^1(\Omega)$ and strongly in $L_2(\Omega)$ by Rellich–Kondrashov Theorem. Then we obtain

$$\overline{\lim_{k\to\infty}}\, I_k(v_k) \le \inf I(U). \tag{3}$$

The sequences $\{v_k\}$ and $\{y_k\}$, where $y_k = y_k[v_k]$, are bounded in the spaces $L_2(\Omega)$ and $H_0^1(\Omega)$ because of the boundedness of the set V and Y. Then we get $v_k \to v$ weakly in $L_2(\Omega)$ and $y_k \to y$ weakly in $H_0^1(\Omega)$ after extracting subsequences. Using convexity and closeness of the set V and Y, we get $v \in V$ and $y \in Y$. We have the equality

$$-\Delta y_k + \frac{1}{\varepsilon_k} a(y_k) = v_k. \qquad (4)$$

Then

$$a(y_k) = \varepsilon_k (v_k + \Delta y_k).$$

By boundedness of the sequence $\{y_k\}$ in $H_0^1(\Omega)$ the sequence $\{\Delta y_k\}$ is bounded in $H^{-1}(\Omega)$. Using the convergence $y_k \to y$ weakly in $H_0^1(\Omega)$, we have $\Delta y_k \to \Delta y$ weakly in $H^{-1}(\Omega)$. After passing to the limit in the last equality, we get $a(y_k) \to 0$ weakly in $H^{-1}(\Omega)$.

By Sobolev Theorem we have the continuous embedding $H_0^1(\Omega) \subset L_4(\Omega)$ and $L_{4/3}(\Omega) \subset H^{-1}(\Omega)$. Then the sequence $\{y_k\}$ is bounded in the space $L_4(\Omega)$. Using the definition of the function , we obtain

$$\|a(y_k)\|_{4/3}^{4/3} = \int_\Omega |a(y_k)|^{4/3} dx = \int_{\Omega_k} |y_k|^4 dx \le \|y_k\|_4^4,$$

where

$$\Omega_k = \{x \in \Omega |\, y_k(x) \le 0\}.$$

Then the sequence $\{a(y_k)\}$ is bounded in the space $L_{4/3}(\Omega)$. By Rellich–Kondrashov Theorem we have the convergence $y_k \to y$ strongly in $L_2(\Omega)$ and a.e. in Ω after extracting a subsequence. So $a(y_k) \to a(y)$ a.e. in Ω . Using Lemma 1.3 [6, Chap. 1], we have $a(y_k) \to a(y)$ weakly in $L_{4/3}(\Omega)$ and in $H^{-1}(\Omega)$ too. Then $a(y) = 0$, so $y \ge 0$ on Ω. Hence the inclusion $y \in Z$ is true.

Using the equality (1), we have

$$\int_\Omega (\Delta y_k + v_k)(z - y_k) dx = \frac{1}{\varepsilon_k} \int_\Omega a(y_k)(z - y_k) dx = -\frac{1}{\varepsilon_k} \int_\Omega [a(z) - a(y_k)](z - y_k) dx$$

$$= -\frac{1}{\varepsilon_k} \int_{\Omega_k} [z^3 - (y_k)^3](z - y_k) dx \ \forall z \in Z. \qquad (5)$$

Besides we get

$$\int_\Omega \Delta y_k(y_k - y)dx = -\int_\Omega v_k(y_k - y)dx + \frac{1}{\varepsilon_k}\int_\Omega a(y_k)(y_k - y)dx = -\int_\Omega a(y_k)(y_k - y)dx$$

$$= -\int_\Omega v_k(y_k - y)dx + \frac{1}{\varepsilon_k}\int_\Omega \big[a(y_k) - a(y)\big](y_k - y)dx \ge -\int_\Omega v_k(y_k - y)dx.$$

Then

$$\overline{\lim_{k\to\infty}} \int_\Omega \Delta y_k(y_k - y)dx \ge -\overline{\lim_{k\to\infty}} \int_\Omega v_k(y_k - y)dx = 0. \tag{6}$$

By inequalities (5) and (1) we have

$$\int_\Omega (\Delta y + v)(z - y)dx = \lim_{k\to\infty} \int_\Omega \Big[\Delta y_k(z - y) + v_k(z - y)\Big]dx$$

$$= \lim_{k\to\infty} \int_\Omega \big[\Delta y_k(z - y_k) + v_k(z - y_k) + \Delta y_k(y_k - y)\big]dx$$

$$\le \overline{\lim_{k\to\infty}} \int_\Omega (\Delta y_k + v_k)(z - y_k)dx + \overline{\lim_{k\to\infty}} \int_\Omega \Delta y_k(y_k - y)dx \le 0 \ \forall z \in Z.$$

So $y = y[v]$, then $(v, y) \in U$.

Using the convergence $v_k \to v$ weakly in $L_2(\Omega)$ and $y_k \to y$ weakly in $H_0^1(\Omega)$, we get

$$\|v\|_2 \le \inf \lim_{k\to\infty} \|v_k\|_2, \quad \|y - y_\partial\|_2 \le \inf \lim_{k\to\infty} \|y_k - y_\partial\|_2.$$

Then $I_k(v_k) \to \inf I(U)$.

We have the inequality

$$\Big|I_k(v_k) - I(v_k, y_k)\Big| \le \frac{1}{2}\int_\Omega \Big|\big(y_k[v_k] - y_\partial\big)^2 - \big(y[v_k] - y_\partial\big)^2\Big|dx$$

$$\le \frac{1}{2}\big\|y_k[v_k] - y[v_k]\big\|_2 \big\|y_k[v_k] + y[v_k] - 2y_\partial\big\|_2$$

$$\le \frac{1}{2}\Big\{\big\|y_k] - y\big\|_2 + \big\|y[v_k] - y[v]\big\|_2\Big\}\big\|y_k[v_k] + y[v_k] - 2y_\partial\big\|_2.$$

By the convergence $v_k \to v$ weakly in $L_2(\Omega)$ we get $y[v_k] \to y[v]$ weakly in $H_0^1(\Omega)$. Using the convergence $y_k \to y$ weakly in $H_0^1(\Omega)$, we obtain $y[v_k] \to y[v]$ and $y_k \to y$ strongly in $L_2(\Omega)$. Using the last inequality, we have

$$\lim_{k\to\infty} \big| I_k(v_k) - I(v_k) \big| \to 0,$$

so $I(v_k, y_k) \to \inf I(U)$.

We proved that a subsequence of solutions of Problem P2 is minimizing for the Problem P1. Suppose the existence of a subsequence of $\{I(v_k, y_k)\}$ such that it does not have $\inf I(U)$ as a limit point. Using considered technique, extract its subsequence that convergences to $\inf I(U)$. So the whole sequence $\{I(v_k, y_k)\}$ converges to $\inf I(U)$.

By the convergence $v_k \to v$ weakly in $L_2(\Omega)$ and $y_k \to y$ strongly in $L_2(\Omega)$ we have

$$\|v\|_2 \le \underline{\lim}_{k\to\infty} \|v_k\|_2, \quad \|y[v] - y_\partial\|_2 = \lim_{k\to\infty} \|y_k[v] - y_\partial\|_2.$$

Then

$$I(v, y) \le \underline{\lim}_{k\to\infty} I(v_k, y_k) = \inf I(U).$$

Using the inclusion $(v, y) \in U$, we prove that v is a solution of Problem P1.

By $\{v_k\}$ denote the subsequence, which correspond the lower limit of last inequalities. Suppose the strong inequality

$$\|v\|_2 < \inf \lim_{k\to\infty} \|v_k\|_2.$$

Then we obtain the strong inequality

$$I(v, y) < \inf I(U).$$

This contradiction prove the convergence $\|v_k\|_2 \to \|v\|_2$. Using the convergence $v_k \to v$ weakly in $L_2(\Omega)$, we prove that $v_k \to v$ strongly in $L_2(\Omega)$. This completes the proof of Theorem 3.3. □

Remark 3.4 Problem P1 can have many solutions. In this case different subsequences of $\{v_k\}$ can converge to different solutions of this problem. However our conclusions are true for all its convergent subsequence. Therefore the set of limit points of $\{v_k\}$ consists of solutions of Problem P1. However it is possible that some solution does not belong to this set.

The known results the optimization control problems for systems described by variational inequalities include as a rule the justification of the necessary conditions of optimality (see, for example, [7–14]). However we solve optimization control problems only approximately. The known necessary conditions of optimality are difficult enough. So it is more naturally to find the approximate solution of the problem, rather than necessary conditions of optimality. This idea was used in [19–21] in the case of insolvability of extremum problems. By Theorem 2 we can choose the optimal control for Problem P2 for large enough value of k as an approximate solution of Problem P1. So we will solve the solution of Problem P2. It is easier than Problem P1 because the system is described by equation, rather than variational inequality.

4 Second Approximation of the Problem

The general difficulty of Problem P2 is the state constraint. We cannot to use the standard variational method for this case because we do not know how we can change the control for saving the state constraint. We could apply results of the general extremum theory (Lagrange principle and some other methods, see, for example, [23–25]). But it uses very difficult properties of the linearized operator and the state constraint. However some results for optimization control problems for nonlinear elliptic equations with state constraints are known (see, for example, [26–32]). Our aim is the search of minimizing sequences in contrast to these results. Then we transform our problem to an easier one. Using the penalty method [22], we change our optimization problem by the minimization problem for the penalty functional on the set of admissible "control-state" pairs. Note that this technique was used in [22] for the case of the absence of the state constraint. The unique solvability of the state equation was not guarantee there. However our boundary problem is well-posed, but we have the state constraint.

Define the functional

$$I_k^m(v, y) = \frac{1}{2} \int_\Omega \Big\{ (y - y_\partial)^2 + \chi v^2 + \frac{1}{\delta_m} \big[\Delta y + \varepsilon_k^{-1} a(y) + v \big]^2 \Big\} dx,$$

where $\delta_m > 0$ and $\delta_m \to 0$ as $m \to \infty$. Define the space

$$W = L_2(\Omega) \times H_0^1(\Omega)$$

and the set $U_\partial = V \times Y$. We have the following problem.

Problem P3 *Minimize the functional I_k^m on the set U_∂.*

Lemma 4.1 *Problem P3 is solvable.*

Proof Let $\{u_s\} = \{v_s, y_s\}$ be a minimizing sequence for the Problem P3, so $u_s \in U_\partial$ and $I_k^m \to \inf I_k^m(U_\partial)$ as $s \to \infty$. Using the boundedness of the set U_∂, we prove that the sequence $\{u_s\}$ is bounded in the space W. By definition of the functional we have the equality

$$-\Delta y_s = \varepsilon_k^{-1} a(y_s) + v_s + f_s,$$

where the sequence $\{f_s\}$ is bounded in the space $L_2(\Omega)$. Using the boundedness of the sequence $\{y_s\}$ in $H_0^1(\Omega)$ and in $L_6(\Omega)$ too because of Sobolev Embedding Theorem, we prove the boundedness of the sequence $\{a(y_s)\}$ in the space $L_2(\Omega)$. Then the term in the right side of the last equality is bounded in the space $L_2(\Omega)$. So $\{\Delta y_s\}$ is bounded in $L_2(\Omega)$. Hence we get $v_s \to v$ weakly in $L_2(\Omega)$, $y_s \to y$ weakly in $H_0^1(\Omega)$, $a(y_s) \to \varphi$ weakly in $L_2(\Omega)$, $\Delta y_s \to \Delta y$ weakly in $L_2(\Omega)$ after extracting subsequences. Using the convexity and the closeness of the sets V and Y, we have the inclusions $v \in V$ and $y \in Y$, then $u \in U_\partial$, where $u = (v, y)$. By Rellich–Kondrashov Theorem we get $y_s \to y$ strongly in $L_2(\Omega)$ and a.e. on Ω, then $a(y_s) \to a(y)$ a.e. on Ω. Using Lemma 1.3 (see [6], Chap. 1), we obtain $a(y_s) \to a(y)$ weakly in $L_2(\Omega)$, so $\varphi = a(y)$. By the weak lower semicontinuous of the norm in Hilbert spaces we get

$$I_k^m(u) \le \inf I_k^m(U_\partial),$$

so u is a solution of Problem P3. This completes the proof of Lemma 4.1. □

Let $u_k^m = \left(v_k^m, y_k^m\right)$ be a solution of Problem P3.

Theorem 4.2 *For any k $I_k(v_k^m) \to \inf I_k(V_k)$ as $m \to \infty$, besides $v_k^m \to v_k$ in $L_2(\Omega)$ after extracting a subsequence.*

Proof We have the inequality

$$I_k^m(u_k^m) = \min I_k^m(U_\partial) \le I_k^m\left(v_k, y_k[v_k]\right) = I_k(v_k). \tag{7}$$

By boundedness of the set U_∂ the sequence $\{u_k^m\}$ is bounded in the space W. Using the inequality (7) and the definition of the functional I_k^m, we get

$$-\Delta y_k^m = \varepsilon_k^{-1} a(y_k^m) + v_k^m + \sqrt{\delta_m} f_k^m, \tag{8}$$

where the sequence $\{f_k^m\}$ is bounded in $L_2(\Omega)$. Then (see the proof of Lemma 4.1), the sequence $\{\Delta y_k^m\}$ is bounded in $L_2(\Omega)$. Then $v_k^m \to v_k^*$ weakly in $L_2(\Omega)$, $y_k^m \to y_k^*$ weakly in $H_0^1(\Omega)$, $f_k^m \to f_k$ weakly in $L_2(\Omega)$, and $\Delta y_k^m \to \Delta y_k^*$ weakly in $L_2(\Omega)$ as $m \to \infty$ after extracting subsequences. Using the technique from the proof of Lemma 4.1, we obtain $v_k^* \in V$, $y_k^* \in Y$, and $a(y_k^m) \to a(y_k^*)$ weakly in $L_2(\Omega)$. After passing to the limit in the equality (8) we get $y_k^* = y_k[v_k^*]$.

By definition of the functional I_k^m we have

$$I_k^m(u_k^m) \ge \frac{1}{2} \int\limits_\Omega \left[\left(y_k^m - y_\partial\right)^2 + \chi\left(v_k^m\right)^2\right] dx.$$

Hence

$$\min I_k(V_k) = I_k(v_k) \le I_k(v_k^*) = \frac{1}{2}\int_\Omega \Big\{\big(y_k[v_k^*] - y_\partial\big)^2 + \chi\big(v_k^*\big)^2\Big\}dx$$

$$\le \frac{1}{2}\varliminf_{m\to\infty}\int_\Omega \Big[\big(y_k^m - y_\partial\big)^2 + \chi\big(v_k^m\big)^2\Big]dx \le \varliminf_{m\to\infty} I_k^m(u_k^m).$$

Using (7), we obtain $I_k^m(u_k^m) \to \min I_k(V_k)$. By inequalities

$$I_k(v_k) \le I_k(v_k^m) \le I_k^m(u_k^m)$$

we have $I_k(v_k^m) \to \inf I_k(V_k)$. The proof is ended with using the technique from Theorem 4.2 . □

Remark 4.3 All assertions of Remark 3.4 are true in this case.

By proved theorem a sequence of solutions of Problem P1 minimizes the functional I_k on the set V_k. So the value v_k^m for large enough m can be chosen as an approximate solution of Problem P2. Then the control v_k^m for large enough value m and k can be chosen as an approximate solution of Problem P1. Our next step is finding of this control. We will prove that the obtained result is sufficient for the analysis of the given optimization problem without any constraints.

5 Necessary Conditions of Optimality

We have the minimization problem for an integral functional on a convex set. The necessary condition of the minimum at the point u of Gateaux differentiable functional J on a convex set W is the variational inequality

$$\langle J'(u), w - u\rangle \ge 0 \ \forall w \in W, \tag{9}$$

where $\langle \varphi, \lambda\rangle$ is the value of a linear continuous functional φ at a point λ. We prove the differentiability of the functional I_k^m for using this result in our case.

Lemma 5.1 *The functional I_k^m has the partial derivatives*

$$I_{kv}^m(v,y) = \chi v + p_k^m(v,y),\ I_{ky}^m(v,y) = y - y_\partial + \Delta p_k^m(v,y) + \varepsilon_k^{-1}a'(y)p_k^m(v,y), \tag{10}$$

at the arbitrary point (v,y), where

$$p_k^m(v,y) = \frac{1}{\delta_m}\big[\Delta y + \varepsilon_k^{-1}a(y) + v\big]. \tag{11}$$

Proof For any function $h \in L_2(\Omega)$ and the value σ we have the equality

$$I_k^m(v+\sigma h, y) - I_k^m(v,y) = \frac{\chi}{2}\int_\Omega \Big[(v+\sigma h)^2 - v^2\Big]dx$$

$$+\frac{1}{2\delta_m}\int_\Omega \Big\{\big[\Delta y + \varepsilon_k^{-1}a(y) + v + \sigma h\big]^2 - \big[\Delta y + \varepsilon_k^{-1}a(y) + v\big]^2\Big\}dx$$

$$= \sigma\int_\Omega \big[\chi v + p_k^m(v,y)\big]h dx + \frac{\sigma}{2}\int_\Omega \Big\{\chi + \delta_m\big[p_k^m(v,y)\big]^2\Big\}h^2 dx.$$

So the first equality (10) is true. For any function $h \in H_0^1(\Omega)$ and the value σ we get

$$I_k^m(v, y+\sigma h) - I_k^m(v,y) = \frac{1}{2}\int_\Omega \big[(y - y_\partial + \sigma h)^2 - (y - y_\partial)^2\big]dx$$

$$+\frac{1}{2\delta_m}\int_\Omega \Big\{\big[\Delta(y+\sigma h) + \varepsilon_k^{-1}a(y+\sigma h) + v\big]^2 - \big[\Delta y + \varepsilon_k^{-1}a(y) + v\big]^2\Big\}dx$$

$$= \sigma\int_\Omega \Big\{(y - y_\partial)h + p_k^m(v,y)\big[\Delta h + \varepsilon_k^{-1}a'(y)h\big]\Big\}dx + \eta(\sigma)$$

$$+\sigma\int_\Omega \Big\{(y - y_\partial) + \big[\Delta p_k^m(v,y) + \varepsilon_k^{-1}a'(y)p_k^m(v,y)\big]\Big\}h dx + \eta(\sigma),$$

where $a'(y) = 0$ for $y \geq 0$, $a'(y) = 3y^2$ for $y < 0$ and $\eta(\sigma) \to 0$ as $\sigma \to 0$. So the first equality (10) is true. This completes the proof of Lemma 5.1. □

Thus by the inequality (9) we get a necessary condition of optimality.

Theorem 5.2 *The solution* $\big(v_k^m, y_k^m\big)$ *of Problem 3 satisfies the following system*

$$\int_\Omega \big(\chi v_k^m + p_k^m\big)\big(v - v_k^m\big)dx \geq 0 \ \forall v \in V, \tag{12}$$

$$\int_\Omega \Big[y_k^m - y_\partial + \Delta p_k^m + \varepsilon_k^{-1}a'(y_k^m)p_k^m\Big]\big(y - y_k^m\big)dx \geq 0 \ \forall y \in Y, \tag{13}$$

$$\Delta y_k^m + \varepsilon_k^{-1}a(y_k^m) + v_k^m = \delta_m p_k^m. \tag{14}$$

We obtain the standard necessary condition of optimality. It can be solved with using an iterative method (see, for example, [33–35]). Then the control v_k^m can be chosen as an approximate solution of the initial optimization problem for large enough values of k and m.

Remark 5.3 This system is simplified in the case of the absence of the state constraint. The variational inequality (13) can be transformed to the standard adjoint equation

$$\Delta p_k^m + \varepsilon_k^{-1} a'(y_k^m) p_k^m = y_\partial - y_k^m$$

in this case. Hence necessary conditions of optimality include the state equation (14), this adjoint equation and classical variational inequality (12). If we do not have any constraints, then we can find the control $v_k^m = -\chi p_k^m$ from (12). Then we obtain two elliptic equations

$$\begin{aligned} \Delta p_k^m + \varepsilon_k^{-1} a'(y_k^m) p_k^m &= y_\partial - y_k^m, \\ \Delta y_k^m + \varepsilon_k^{-1} a(y_k^m) + v_k^m &= \delta_m p_k^m. \end{aligned}$$

After solving this system we can find v_k^m by the obtained formula.

Analogical results could be obtained for controls systems described by parabolic and hyperbolic variational inequalities. Laplace operator can be substituted by general linear elliptic operators and some nonlinear elliptic operators. We could consider also a general integral functional with corresponding assumptions.

References

1. G. Fichera, Sul problema elastostatico di Signorini con ambigue condizioni al contorno. Rendiconti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche e Naturali **834**(2), 138–142 (1963)
2. G. Duvaut, J.-L. Lions, *Les inéquations en mécanique et en physique* (Dunod, Paris, 1972)
3. D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications* (Academic, New York, 1980)
4. R. Glowinski, J.-L. Lions, R. Trémolier, *Numerical Analysis of Variational Inequalities* (North Holland, Amsterdam, 1981)
5. C. Baiocchi, A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems* (Wiley, New York, 1984)
6. J.-L. Lions, *Quelques Méthods de resolution des problèmes aux limites Non linèaires* (Dunod, Gautier-Villars, Paris, 1969)
7. V. Barbu, *Optimal Control of Variational Inequalities*. Research Notes in Mathematics, vol. 100 (Pitman, Boston, 1984)
8. Z.X. He, State constrained control problems governed by variational inequalities. SIAM J. Control Optim. **25**, 1119–1145 (1987)
9. D. Tiba, *Optimal Control of Nonsmooth Distributed Parameter Systems*. Lecture Notes in Mathematics, vol. 1459 (Springer, Berlin, 1990)

10. J. Bonnans, E. Casas, An extension of Pontryagin's principle for state-constrained optimal control of semilinear elliptic equations and variational inequalities. SIAM J. Control Optim. **33**, 274–298 (1995)
11. G. Wang, Y. Zhao, W. Li, Some optimal control problems governed by elliptic variational inequalities with control and state constraint on the boundary. J. Optim. Theory Appl. **106**, 627–655 (2000)
12. Y. Ye, Q. Chen, Optimal control of the obstacle in a quasilinear elliptic variational inequality. J. Math. Anal. Appl. **294**, 258–272 (2004)
13. Q. Chen, D. Chu, R.C.E. Tan, Optimal control of obstacle for quasi-linear elliptic variational bilateral problems. SIAM J. Control Optim. **44**, 1067–1080 (2005)
14. Y.Y. Zhou, X.Q. Yang, K.L. Teo, The existence results for optimal control problems governed by a variational inequality. J. Math. Anal. Appl. **321**, 595–608 (2006)
15. S. Serovajsky, State-constrained optimal control of nonlinear elliptic variational inequalities. Optim. Lett. **8**(7), 1–13 (2013)
16. P. Neittaanmaki, D. Tiba, *Optimal Control of Nonlinear Parabolic Systems: Theory, Algorithms, and Applications* (Marcel Dekker, New York, 1994)
17. D.R. Adams, S. Lenhart, Optimal control of the obstacle for a parabolic variational inequality. J. Math. Anal. Appl. **268**, 602–614 (2002)
18. Q. Chen, Optimal obstacle control problem for semilinear evolutionary bilateral variational inequalities. J. Math. Anal. Appl. **307**, 677–690 (2005)
19. J. Warga, *Optimal Control of Differential and Functional Equations* (Academic, New York, 1972)
20. V. F. Krotov, *Global Methods in Optimal Control Theory* (Marcel Dekker, New York, 1996)
21. M.I. Sumin, Suboptimal control of a semilinear elliptic equation with state constraints. Izv. Vuzov **6**, 33–44 (2000)
22. J.-L. Lions, *Contrôle Optimal de systèmes Distributes Singuliers* (Gautier-Villars, Paris, 1983)
23. A.D. Ioffe, V.M. Tihomirov, *Extremal Problems Theory* (Nauka, Moscow, 1974)
24. A.S. Matveev, V.A. Yakubivich, *Abstract Theory of Optimal Control* (St. Petersburg University, St. Petersburg, 1994)
25. K. Madani, G. Mastroeni, A. Moldovan, Constrained extremum problems with infinite-dimensional image: selection and necessary conditions. J. Optim. Theory Appl. **135**, 37–53 (2007)
26. P. Michel, Necessary conditions for optimality of elliptic systems with positivity constraints on the state. SIAM J. Control Optim. **18**, 91–97 (1980)
27. E. Casas, Boundary control of semilinear elliptic equations with pointwise state constraints. SIAM J. Control Optim. **33**, 993–1006 (1993)
28. N. Arada, J.P. Raymond, State-constrained relaxed problems for semilinear elliptic equations J. Math. Anal. Appl. **223**, 248–271 (1998)
29. J.F. Bonnans, A. Hermant, Conditions d'optimalité du second ordre nécessaires ou suffisantes pour les problèmes de commande optimale avec une contrainte sur l'état et une commande scalaires. C. R. Acad. Sci. Paris Ser. I. **343**, 473–478 (2006)
30. A. Rösch, F. Tröltzsch. Sufficient second-order optimality conditions for an elliptic optimal control problem with pointwise control-state constraints. SIAM J. Optim. **17**, 776–794 (2006)
31. A. Rösch, F. Tröltzsch, On regularity of solutions and Lagrange multipliers of optimal control problems for semilinear equations with mixed pointwise control-state constraints. SIAM J. Control Optim. **46**, 1098–1115 (2007)
32. E. Casas, J.C. Reyes, F. Tröltzsch, Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints. SIAM J. Optim. **19**, 616–643 (2008)
33. R. Fletcher, *Practical Methods of Optimization* (Wiley, New York, 2000)
34. J.F. Bonnans, J.C. Gilbert, C. Lemaréchal, C. Sagastizábal, *Numerical Optimization: Theoretical and Practical Aspects*. Universitext (Springer, Berlin, 2006)
35. C. Floudas, P. Pardalos (eds.), *Encyclopedia of Optimization* (Springer, Berlin, 2008)

AUTHOR QUERIES

AQ1. Please provide organisation details for the author “Simon Serovajsky”.
AQ2. Please check whether the provided details for Ref. [15] is appropriate.
AQ3. Please check whether the provided information for Ref. [24] is appropriate.