Wiener-Hopf method in problems of plane wave diffraction by two opposite staggered perfectly conducting half-planes

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***Abstract*—Diffraction of a plane wave on two opposite staggered perfectly conducting half-planes is considered using the Wiener-Hopf method. The Neumann boundary value problem is reduced to solving a system of Fredholm integral equations of the second kind, the solution to which is expressed analytically through the integral operator in the form of the sum of the Neumann series.**

***Index Terms*—Wiener-Hopf method, factorization, diffraction, plane wave, Neumann problem, Fredholm integral equation**

I. Introduction

It is well known that the exact solution to the problem of diffraction on a perfectly conducting half-plane was obtained by Sommerfeld in his famous work [1]. The method and results of his paper were considered in details in the books of D. Jones, B. Noble and L. Weinstein [2]–[4] and K. Kobayashi [5]. The classical problem of diffraction of a plane wave at the open end of a waveguide consisting of two ideally conducting parallel half-planes was rigorously solved by Weinstein using the Wiener-Hopf method [4]. An explicit solution to the scattering of a plane wave by three parallel equidistant semi-infinite planes was derived by a matrix factorization method by Jones [6]. Recently, this method has been generalized as the so-called Fredholm factorization method, which belongs to the semianalytic methods and, in fact, reduces the problem of factorization of Wiener–Hopf matrix equations to Fredholm integral equations of the second kind, described by V. Daniele and G. Lombardi [7].

This paper proposes a rigorous analytical Wiener-Hopf method in the spirit of Weinstein's double equations [4] without the use of factorization of matrix equations. Therefore, to compare modifications of the W-H methods, the authors consider a boundary value problem on the diffraction of H-polarized plane wave by two opposite staggered perfectly electrically conducting half-planes, which coincides with the title of the article [8]. It should be noted that the new key problem is solved using the technique previously proposed by the authors for the problems of diffraction on a strip [9]–[11].

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II. Problem statement and its solution

Let a plane wave with a single magnetic field component $(H_x^0(y,z),0,0)$

$$H_x^0(y,z) = H_0 \exp\left\{ i\left(y\sqrt{k_0^2-h^2} + hz \right) \right\},\ h = k_0\cos\theta_0$$

fall normally on the edges of opposite staggered perfectly conducting half-planes, where its direction with respect to the z axis makes an angle θ_0 (see Figure 1). Then the electric field of the incident wave is defined as

$$\begin{cases} E_z^0 = \dfrac{\sqrt{k_0^2-h^2}}{k_0}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}H_x^0, \\ E_y^0 = -\dfrac{h}{k_0}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}H_x^0, \quad E_x^0 = 0. \end{cases} \tag{1}$$

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It is necessary to find the scattered field H_x of a plane wave of H-polarization, which is a solution to the Neumann problem

$$\frac{\partial^2}{\partial y^2}H_x + \frac{\partial^2}{\partial z^2}H_x + k_0^2 H_x = 0 \tag{2}$$

with boundary conditions

$$\frac{\partial}{\partial y}H_x = -ik_0\sqrt{\frac{\varepsilon_0}{\mu_0}}E_z^0 \qquad \text{for} \quad z<0,\, y=d; \quad z>0,\, y=-d. \tag{3}$$

To complete the formulation of the diffraction problem, and to ensure the uniqueness of its solution, the above wave equation and the boundary conditions are supplemented by the edge (Meixner condition) and Sommerfeld radiation condition for the scattered field. According to the Meixner condition, the component of current density normal to the edge (or H_x) vanishes at the edge as $\rho^{1/2}$ and the field components E_y, E_z vary as $\rho^{-1/2}$, where ρ is the distance from the edge [12].

The electric field of the scattered wave

$$\begin{cases} E_z = -\dfrac{i}{k_0}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{\partial}{\partial y}H_x, \\ E_y = \dfrac{i}{k_0}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{\partial}{\partial z}H_x, \quad E_x = 0 \end{cases}$$

can be expressed in terms of the magnetic field component H_x according to Maxwell's equation $-i\varepsilon_0\omega \boldsymbol{E} = \nabla \times \boldsymbol{H}$.

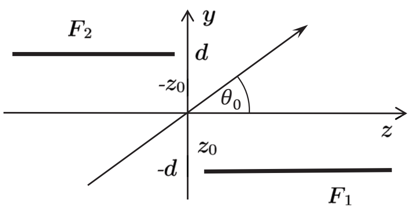

Fig. 1. Plane wave diffraction on half-planes.

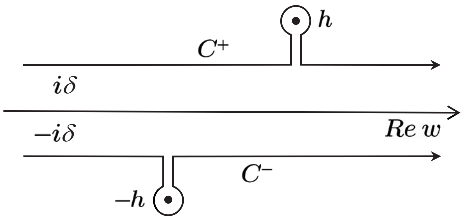

Fig. 2. Integration contours.

To solve the Neumann problem (2) we must have an integral representation of the magnetic field

$$\begin{aligned} H_x(y,z) = \operatorname{sgn}(y+d) \int\limits_{C^-} e^{i(wz+v|y+d|)} F_1(w)dw + \\ \operatorname{sgn}(y-d) \int\limits_{C^+} e^{i(wz+v|y-d|)} F_2(w)dw,\ v = \sqrt{k_0^2 - w^2}, \end{aligned} \tag{4}$$

created by surface currents (see (26), Appendix A), where F_1 and F_2 are the desired Fourier components of surface currents induced on the corresponding half-planes, C^- and C^+ are integration contours (IC), located parallel at a distance $i\delta \to 0$ ($\delta < |k_0|$) from the real axis and consisting of an infinitely narrow loop enclosing the point $u = h$ ($\mathrm{Im}(h) > 0$) below or above (Fig. 2).

In the figure, the values of h are fixed as $\pm h$ in the UHP and LHP w, respectively, as $h = k_0 \cos\theta_0$ is an algebraic quantity depending on θ_0. Therefore, it should be kept in mind that integration over the loop of the contour C^- actually corresponds to a residue at point h.

Taking into account the boundary conditions (3) and the integral representation (4), the boundary value problem can be reduced to a system of integral equations

$$\begin{cases} \int\limits_{C^-} e^{i(wz+2dv)} vF_1(w)dw + \int\limits_{C^+} e^{iwz} vF_2(w)dw + \\ \quad \sqrt{k_0^2 - h^2} H_x^0 e^{i\left(hz + d\sqrt{k_0^2-h^2}\right)} = 0,\ z<0,\ y=d; & \text{(5a)} \\ \int\limits_{C^-} e^{iwz} vF_1(w)dw + \int\limits_{C^+} e^{i(wz+2dv)} vF_2(w)dw + \\ \sqrt{k_0^2 - h^2} H_x^0 e^{i\left(hz - d\sqrt{k_0^2-h^2}\right)} = 0,\ z>0,\ y=-d; & \text{(5b)} \\ \int\limits_{C^-} e^{iwz} F_1(w)dw = 0, \quad z<0,\ y=-d; & \text{(5c)} \\ \int\limits_{C^+} e^{iwz} F_2(w)dw = 0, \quad z>0,\ y=d. & \text{(5d)} \end{cases}$$

The first pair of equations of the system provides fulfillment of the condition (3) about the absence of the tangential component of the electric field on the surfaces of an ideally conducting half-plane. The second pair corresponds to the absence of current density on the continuation of conducting half-planes or, in other words, the continuity of the magnetic field H_x.

Using the Wiener-Hopf technique for planar structures given in [9]–[11], we present the solution as

$$F_1(w) = \frac{e^{-iz_0 w}}{\sqrt{k_0 - w}} \big(A_1(w) + B^-(w)\big)\ (z_0 \to 0), \tag{6}$$

$$F_2(w) = \frac{e^{iz_0 w}}{\sqrt{k_0 + w}} \big(A_2(w) + B^+(w)\big). \tag{7}$$

From this follows, that the functions A_1 and A_2 have a simple pole at the point $w = h$, which corresponds to the amplitude of a plane wave incident on a half-plane edge. The function B^+ is regular in the upper half-plane (UHP), and B^- in the lower half-plane (LHP) of the complex variable w [13]. Their physical meaning corresponds to the amplitudes of secondary cylindrical waves reradiated from the edges of the half-planes.

The required functions are represented in the form of meromorphic functions

$$A_1(w) = \frac{a_1}{w-h},\ A_2(w) = \frac{a_2}{w-h}, \quad a_1, a_2 = \text{const}, \tag{8}$$

as well as Cauchy-type integrals [13]

$$B^+(w) = \frac{1}{2\pi i} \int\limits_{C^-} \frac{b_2(u)}{u-w} du,\ B^-(w) = \frac{i}{2\pi} \int\limits_{C^+} \frac{b_1(u)}{u-w} du, \tag{9}$$

which are regular in the UHP and the LHP u, respectively.

It should be noted that the equations (5c), (5d) are satisfied automatically, thanks to Jordan's lemma, since F_1 and F_2 are regular, respectively, in the LHP and the UHP of the complex variable w.

It's noted that the current densities satisfy the usal Meixner condition $F\ w^{-3/2}$ at $|w| \to \infty$.

The integral over the narrow loop of the contour C^+ in (5a) corresponds to the amplitude of a plane wave incident on the edge of the half-plane ($y = d$). Therefore, the tangential component of the electric field $E_z^0(d)$ on the conducting plane in (5a) must be canceled out by the integral over the contour loop C^+. Thus, by eliminating the pole $w = h$ corresponding to the plane wave in (5a), we find the constant

$$a_2 = H_x^0 \frac{\sqrt{k_0 + h}}{2\pi i} e^{i(d\sqrt{k_0^2 - w^2} - z_0 h)},$$

hence, the sought function in (8)

$$A_2(w) = \frac{H_x^0}{2\pi i} \frac{\sqrt{k_0 + h}}{(w - h)} e^{i(d\sqrt{k_0^2 - h^2} - z_0 h)}. \tag{10}$$

Having calculated the second integral in (5a), we use the residue at the pole $w = u$, with the closed integration contour in the lower half-plane (LHP) w, according to Jordan's lemma, and reduce the equation (5a) to the form

$$\int\limits_{C^-} e^{iuz} \Big[e^{i(2vd - z_0 u)} \sqrt{k_0 + u} \big(A_1(u) + B^-(u)\big) + e^{iz_0 u} \sqrt{k_0 - u}\, b_2(u) \Big] du = 0. \tag{11}$$

It is obvious that this equation does not contain the function A_2, which became possible due to the loop integral C^+.

In order to satisfy the above equation, it is sufficient to require that the integrand function be regular throughout the entire LHP u, i.e., all its remaining singular points in the LHP must be completely compensated by the function

$$b_2(u) = -e^{i2(vd - z_0 u)} \sqrt{\frac{k_0 + u}{k_0 - u}} \big(A_1(u) + B^-(u)\big). \tag{12}$$

Substituting (12) in (9), we find the sought function

$$B^+(w) = -\frac{1}{2\pi i} \int\limits_{C^-} \frac{e^{i2(d\sqrt{k_0^2 - u^2} - z_0 u)}}{u - w} \sqrt{\frac{k_0 + u}{k_0 - u}} \big(A_1(u) + B^-(u)\big) du. \tag{13}$$

Similarly, we calculate the residue at the pole $w = h$, substituting F_1 into the first integral of Eq. (5b), as a result of which the field of the incident plane wave will be compensated on the half-plane $z > 0$ ($y = -d$) due to the function

$$A_1(w) = -\frac{H_x^0}{2\pi i} \frac{\sqrt{k_0 - h}}{w - h} e^{-i(d\sqrt{k_0^2 - h^2} - z_0 h)}.$$

Further, in order to fully satisfy the equation (5b), first, it is necessary to calculate the integral using the residue at the pole $w = u$ from the integral representation of the function B^-, then compensate all the singularities of the integrant along the contour C^+ on the UHP u using the function

$$b_1(u) = -e^{i2(vd + z_0 u)} \sqrt{\frac{k_0 - u}{k_0 + u}} \big(A_2(u) + B^+(u)\big). \tag{14}$$

Thus, the system of singular integral equations (5) is reduced to a system of Fredholm integral equations of the second kind

$$\begin{cases} B^-(w) = \dfrac{1}{2\pi i} \displaystyle\int\limits_{C^+} \frac{e^{i2(d\sqrt{k_0^2 - u^2} + z_0 u)}}{u - w} \sqrt{\frac{k_0 - u}{k_0 + u}} \big(A_2(u) + B^+(u)\big) du, & \text{(15a)} \\ B^+(w) = \dfrac{1}{2\pi i} \displaystyle\int\limits_{C^+} \frac{e^{i2(d\sqrt{k_0^2 - u^2} + z_0 u)}}{u + w} \sqrt{\frac{k_0 - u}{k_0 + u}} \big(A_1(-u) + B^-(-u)\big) du. & \text{(15b)} \end{cases}$$

It should be noted that the technique of asymptotic solution of a system of this type with a given accuracy can be found in [9].

Introducing, for convenience, the integral operator

$$\mathbf{I}(w, u) = \frac{1}{2\pi i} \int\limits_{C^+} du \frac{e^{i2(d\sqrt{k_0^2 - u^2} + z_0 u)}}{u - w} \sqrt{\frac{k_0 - u}{k_0 + u}}, \tag{16}$$

we can write a system of functional equations (15)

$$\begin{cases} B^+(w) = \mathbf{I}(-w, u)\big(A_1(-u) + B^-(-u)\big), & \text{(17)} \\ B^-(w) = \mathbf{I}(w, u)\big(A_2(u) + B^+(u)\big). & \text{(18)} \end{cases}$$

Its solution can be obtained in the form

$$\begin{cases} B^+(w) = \displaystyle\sum_{n=1}^{\infty} \mathbf{I}^{2n-1}(-w) A_1(-w_0) + \mathbf{I}^{2n}(-w) A_2(w_0), & \text{(19)} \\ B^-(w) = \displaystyle\sum_{n=1}^{\infty} \mathbf{I}^{2n-1}(w) A_2(w_0) + \mathbf{I}^{2n}(w)\big(A_1(-w_0), & \text{(20)} \end{cases}$$

thanks to recursive equations where the following operator is used

$$\mathbf{I}^m(w) = \mathbf{I}(w, w_{m-1}) \prod_{i=m-1}^{1} \mathbf{I}(-w_i, w_{i-1}),\ m > 1, \quad \prod_{i=m}^{1} a_i = a_m a_{m-1} \cdots a_1,\ \mathbf{I}^1(w) \equiv \mathbf{I}(w, w_0). \tag{21}$$

III. Conclusions

The boundary value problem of the diffraction of a plane wave on two opposite staggered perfectly conducting half-planes is reduced to solving a system of Fredholm integral equations of the second kind by the W-H method. The solution is presented as a sum of an infinite series (19) where, as it is shown in [9], it can be found with any accuracy.

It should be noted that the resulting solution turned out to be identical in appearance to the solution to the problem of diffraction by a strip. This confirms the fact that the solutions to these problems actually describe the same wave diffraction processes occurring in similar planar structures in the form of multiple reflections of cylindrical waves from the edges of half-planes.

It is even easier to find an asymptotic solution to a system of Fredholm integral equations if we reduce it directly to a system of two algebraic equations by integrating it using the saddle point method [11]. The advantage of this approach is the presence of the Fredholm denominator, which is of interest for discrete spectrum problems.

The proposed method for solving the key problem can be useful in solving new boundary value problems of wave diffraction by finite planar structures

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APPENDIX A
INTEGRAL REPRESENTATION OF THE MAGNETIC FIELD OF SURFACE CURRENTS

The solution to the Helmholtz equation

$$(k_0^2 + \triangle)H_x = -\frac{\partial}{\partial y} j_z(y,z) \tag{22}$$

for an H-polarized wave is

$$H_x = -\frac{\partial}{\partial y} j_z(y,z) * \psi = -i\,\mathrm{F}^{-1}[k_y \tilde{j}_z \tilde{\psi}], \tag{23}$$

where $\mathrm{F}[\cdot]$ is the Fourier transform operator

$$\tilde{j}_z = \mathrm{F}[j_z],\ \tilde{\psi} = \mathrm{F}[\psi] = \frac{1}{k_0^2 - k^2},\ \psi = -\frac{1}{4\pi}\frac{e^{ik_0 r}}{r}. \tag{24}$$

Surface currents $J_z(z)$ induced on the plane $y = y_0$ correspond to the current density $j_z(z)$

$$j_z = -4\pi\delta(y-y_0)J_z(z),\ \tilde{j}_z = -8\pi^2\delta(k_x)e^{-ik_y y_0}F(k_z),$$

where $F(k_z) = \mathrm{F}_z[J_z(z)]$ is a Fourier component of the surface current density.

Taking into account the representation of the delta function and calculating the integral with respect to k_y

$$\mathrm{F}^{-1}[k_y \tilde{j}_z \tilde{\psi}] = -\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_y(y-y_0)+k_z z)} \frac{k_y F(k_z)}{k_0^2 - k_y^2 - k_z^2} dk_y dk_z \ \left(k_z = w,\ k_y = \sqrt{k_0^2 - w^2}\right) \tag{25}$$

using the theory of residues, from (23) we obtain the integral representation of the magnetic field

$$H_x = \mathrm{sgn}(y-y_0)\int_{-\infty}^{\infty} e^{iwz} e^{iv|y-y_0|} F(w)dw \tag{26}$$

of surface currents flowing parallel to the z axis on the plane surface $y = y_0$.