Diffraction by circular pin: Wiener-Hopf method

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***Abstract*—The paper considers the boundary value problem of the diffraction of a symmetric TE-wave by a semi-infinite pin located coaxially inside a circular waveguide. The problem is reduced to a system of singular integral equations, which is solved by the Wiener-Hopf method in the class of meromorphic functions. Based the factorized Bessel functions and the entire function, a technique for constructing a solution in the form of the Fourier component of the surface current density is developed. The Meixner condition is satisfied by shifting the zeros of the entire function. The correctness of the solution was verified using the stitching method. The solution also provides a limiting transition to a thin pin, where a semi-infinite pin and a hollow cylinder of the same radius are physically equivalent. In the case of the limiting transition, when the radius of the pin approaches the radius of the waveguide, the solution provides a complete reflection of the oncoming mode, which is the only mode of the waveguide propagating in the opposite direction.**

***Index Terms*—Factorization, Wiener-Hopf method, diffraction, circular waveguide, pin**

I. Introduction

The problem of diffraction of a symmetrical wave by a semi-infinite pin in a circular waveguide was previously considered in [1], where the solution of the linear initial problem was reduced to the solution of a system of nonlinear algebraic equations, which is very time-consuming for numerical calculations. A similar problem was considered in [2] with the use of the interpolation method of factorization [3], related to the Wiener-Hopf (W-H) method, where the systems of linear algebraic equations are solved using a numerical-analytical method of generalized stitching, the cumbersomeness and complexity of which increases significantly with the accuracy of the resulting solution, especially in the multi-wave mode.

The main obstacle in solving the problem analytically by the W-H method, in our opinion, was that it was difficult or even impossible to find such a combination of factorized functions in the class of meromorphic functions that the integrand did not contain either poles or branching points inside the integration contour.

In this work, based on the W-H method, an analytical solution to the key problem of a symmetrical TE-wave on a semi-infinite pin in a circular waveguide is proposed. The boundary value problem is reduced to solving a system of paired singular equations, which is solved by selecting combinations of factorized functions or by applying a special procedure to the entire function, so that inside the integration contour the integrand is regular according to the Jordan lemma and has no more than a power-law growth for $w \to \infty$ according to the Meixner condition.

The second section is devoted to the mathematical formulation of the boundary value problem, where the electrodynamic boundary conditions are formulated in the form of a system of integral equations. In the third section the methodology for solving a system of integral equations is considered. In the fourth section, using the theory of residues, the fields inside the waveguide are presented in the form of series of natural waves in each region. The fifth section provides factorization formulas for the kernels of integral equations. The fifth and sixth sections include discussion and conclusion.

II. Statement of the problem

Let a TE-wave with the longitudinal wave number h and the amplitude A_0 arrive at the left end of a semi-infinite circular pin of the radius a_1, coaxially located inside an infinite circular waveguide of radius a. This event is described by the formula:

$$E^i_\phi(r,z) = A_0 \mathrm{J}_1(Vr)e^{ih(z-z_0)},\ h=\sqrt{k^2-h^2}, \qquad (1)$$

where $z_0 < z_1$ is location of the emitter, z_1 is the coordinate of the end of the pin, V is one of the roots of the first-order Bessel function $\mathrm{J}_1(Va)$. The walls of the waveguide and pin will be considered ideally conductive.

The boundary value problem must satisfy the boundary conditions of the absence of a tangential component of the electric field on the surface of the pin and on the infinite circular waveguide of the radius a:

$$E_\phi(r,z) + E^i_\phi(r,z) = 0, \quad 0 \le r \le a_1,\ z_1 \le z; \qquad r=a,\ -\infty < z < \infty, \qquad (2)$$

where

$$E_\phi(r,z) = -\frac{kW}{2\pi i}\int_{-\infty}^{\infty} e^{iwz}L(r,w)F(w)dw \qquad (3)$$

is the azimuthal component of the electric field of secondary waves re-emitted by the semi-infinite pin, $W=(\mu_0/\varepsilon_0)^{1/2}$ is the wave impedance, $F(w)$ is the Fourier component of the

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density of surface currents induced on the side surface of the pin. The following notations are used here:

$$L(r,w)=\frac{1}{\mathrm{J}_1(va)}\begin{cases}\mathrm{J}_1(vr)(a_1',a'),\ r\le a_1,\\ \mathrm{J}_1(va_1)(r_1',a'),\ r\ge a_1,\end{cases}\tag{4}$$

for the combinations of Bessel functions

$$(r',a')=\mathrm{J}_1(vr)\mathrm{Y}_1(va)-\mathrm{J}_1(va)\mathrm{Y}_1(vr),\ v=\sqrt{k^2-w^2},$$

$\Im(v)>0$, Y_1 is the Neumann function of the first order.

Components of a magnetic field

$$H_r(r,z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{iwz}wL(r,w)F(w)dw,\tag{5}$$

$$H_z(r,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iwz}vL(r,w)F(w)dw.\tag{6}$$

are also expressed in terms of the desired function $F(w)$.

A. Integral equations

After rewriting the boundary conditions (2) using the integral representation of the field (3), and taking into account the continuity of the magnetic field on the extension of the lateral surface of the pin along z, we can reduce the boundary value problem to solving the paired W-H equations

$$\begin{cases}-\frac{kW}{2\pi i}\int_{-\infty}^{\infty}e^{iwz}L(r,w)F(w)dw+E_\phi^i(r,z)=0, & \text{(7a)}\\ \quad 0\le r\le a_1,\ z_1\le z;\ r=a,\ -\infty<z<\infty, & \\ \int_{-\infty}^{\infty}e^{iwz}F(w)dw=0,\quad z<z_1. & \text{(7b)}\end{cases}$$

III. Solving integral equations using the W-H method

The functional relations in (7) are fulfilled if F satisfies the following requirements:

I. The function $L(r,w)F(w)$ is regular for $\le r\le a_1$ and has no singular points in the upper half-plane (UHP) w, except for a simple pole $w=h$, which causes extinction of the incident wave to the right of the end of the pin, and has no more than a power-law growth $w^{-5/3}$ at infinity.

II. The function $F(w)$ is regular and has no singular points in the lower half-plane (LHP); at infinity it behaves as $w^{-2/3}$, according to the Meixner condition [4].

The function $F(w)$, determined by requirements I and II, can be represented as:

$$\begin{aligned}F(w)&=K\frac{L_-^{-1}(a_1,w)}{(w-h)}\Psi(w)e^{-iwz_1},\\ \Psi(w)&=\frac{\mathrm{J}_1(va_1)}{v(w+\tilde h)J(v)},\end{aligned}\tag{8}$$

where $K-$ const, $\tilde h=\sqrt{k^2-(Va/a_1)^2}$,

$$(a_1',a')_-=\frac{(a_1',a')}{(a_1',a')_+},\ \mathrm{J}_1(va)_-=\frac{\mathrm{J}_1(va)}{\mathrm{J}_1(va)_+}.$$

Here, the factorized functions with the subscript '$-$' are regular and have no zeros in the LHP w, whereas the functions with the subscript '$+$' have no zeros in the UHP.

The function $J(v)$ must have the asymptotic behavior $\sim\exp(-iva_1)$. Let us assume that it is an entire function, as it has an apparent advantage over a function with a branch point when calculating integrals. However, the entire function $J(v)$ should not contain zeros, at least inside the integration contours, since they are poles of the function $F(w)$. Therefore, we will need a procedure that removes all zeros to infinity, but does not change its asymptotic behavior.

Prior to performing this procedure, let us express the entire function $\tilde J(v,a_1)$

$$\begin{aligned}&\prod_{n=1}^{\infty}\left(1-\frac{\tilde v^2}{v_n^2}\right)=\frac{\Gamma^2(\triangle+1)}{\Gamma(\tilde v+\triangle+1)\Gamma(-\tilde v+\triangle+1)}=\\ &\frac{1}{\pi}\frac{\Gamma^2(\triangle+1)\Gamma(\tilde v-\triangle)}{\Gamma(\tilde v+\triangle+1)}\sin\pi(\tilde v-\triangle)\quad(\triangle<\tilde v)\sim\\ &\frac{1}{\pi}\Gamma^2(\triangle+1)\tilde v^{-1-2\triangle}\sin\pi(\tilde v-\triangle),\ v_n=\triangle+n\end{aligned}\tag{9}$$

in terms of sine, where $\tilde v=va_1/\pi$, $\triangle=2/3$ is a shift of zeros, which ensures that the Meixner condition is satisfied in the solution $F(w)$.

After partitioning the radius of the pin

$$0=b_0<b_1<b_2<\cdots<b_{N-1}<b_N=a_1\tag{10}$$

into N intervals, we can remove zeros of the entire function $\tilde J(v,a_1)$ at large distances using the transformation

$$J(v)=(-2i)^Ne^{-i\pi\triangle N}\prod_{m=1}^{N}\tilde J(v,b_m-b_{m-1}).\tag{11}$$

Finally, taking into account the transition of sine into (9)

$$\begin{aligned}&\sin\pi\Big(\frac{va_1}{\pi}-\triangle\Big)\to 2^N\exp\Big(-i\pi(\triangle+\tfrac{1}{2})N\Big)\\ &\left(\sin\pi\Big(\frac{va_1}{\pi N}-\triangle\Big)\right)^N\sim\exp(-iva_1),\ ka_1\ll N,\end{aligned}\tag{12}$$

we obtain the asymptotics of the entire function $J(v)$ in (11)

$$J(v)\sim(va_1)^{-1-2\triangle}\exp(-iva_1).\tag{13}$$

As the number $N\to\infty$ increases, all roots of the function $J(v)$ will be moved at large distances and have larger values. Since the zeros of $J(v)$ on the complex plane w are imaginary, the contributions of residues to the integral from the corresponding poles can be considered extremely small.

Having calculated the residue at the point $w=h$ in the UHP w, we find the constant

$$K=\frac{A_0}{kW}\frac{\mathrm{J}_1(Va_1)}{L_+(a_1,h)\Psi(h)}\exp\{ih(z_1-z_0)\}.\tag{14}$$

It should be noted that the solution $F(w)$ satisfies all the boundary conditions of the boundary value problem; the reliability of fulfillment of equations in (7) is directly verified using the theory of residues.

IV. Field calculation

Expressions for fields, for example, for the electric field, can be obtained from (3) using the theory of residues for each region of the waveguide

(a) $0 \le r \le a,\ z < 0.$

$$E_\phi = kW \sum_{n=1}^{\infty} e^{-iw_n^a z} w_n^a L^*(r, -w_n^a) F(-w_n^a) + A_0 \mathrm{J}_1(Vr) e^{ih(z-z_0)}, \tag{15}$$
$$L^*(r, -w_n^a) = \lim_{w \to -w_n^a} (w + w_n^a) L(r, w);$$

(c) $a_1 \le r \le a,\ 0 \le z.$

$$\begin{aligned} E_\phi &= -kW \sum_{n=1}^{\infty} e^{iw_n^c z} w_n^c L(r, w_n^c) F^*(w_n^c), \\ F^*(w_n^c) &= \lim_{w \to w_n^c} (w - w_n^c) F(w), \end{aligned} \tag{16}$$

where $w_n^a = \sqrt{k^2 - v_n^{a2}}$, v_n^a $(n = 1, 2, \dots)$ are the roots of the first order Bessel functions $\mathrm{J}_1(va)$, and $w_n^c = \sqrt{k^2 - v_n^{c2}}$, v_n^c correspond to zeros of the combination of Bessel functions (a_1', a').

The magnetic field components H_r and H_z are calculated similarly in the form of exponentially convergent series.

V. Discussion

The resulting solution $F(w)$ for the system of singular integral equations (7) satisfies the Meixner condition or the so-called sharp edge condition, which is achieved by shifting the zeros of the entire function $J(v)$ by 2/3.

The expression $F(w)$ is also a solution to the systems of infinite algebraic equations obtained by the stitching method with respect to the coefficients of the series of field expansion in eigenwaves A_n and B_m

$$\begin{aligned} &\sum_{n=1}^{\infty} A_n \frac{w_n^a \mathrm{J}_1(v_n^a a_1)}{w_n^{a2} - w_m^{c2}} - A \frac{h \mathrm{J}_1(V a_1)}{h^2 - w_m^{c2}} = \frac{B_m}{2} (a_1', a')_m^*, \\ &\sum_{n=1}^{\infty} A_n \frac{\mathrm{J}_1(v_n^a a_1)}{w_n^{a2} - w_m^{c2}} + A \frac{\mathrm{J}_1(V a_1)}{h^2 - w_m^{c2}} = \frac{-B_m}{2w_m^c} (a_1', a')_m^*, \\ &\sum_{n=1}^{\infty} A_n \frac{\mathrm{J}_1(v_n^a a_1)}{w_n^{a2} - w_n^{b2}} + A \frac{\mathrm{J}_1(V a_1)}{h^2 - w_m^{b2}} = 0. \end{aligned} \tag{17}$$

Indeed, these equations can be obtained using contour integrals over an infinite circumference through the Fourier component of the current density $F(w)$

$$\begin{aligned} &\oint w \frac{L(a_1, w) F(w)}{w^2 - w_m^{c2}} dw = 0, \\ &\oint \frac{L(a_1', w) F(w)}{w^2 - w_m^{c2}} dw = 0, \\ &\oint \frac{L(a_1', w) F(w)}{w^2 - w_m^{b2}} dw = 0, \end{aligned} \tag{18}$$

which are expressed as sums of residues

$$\begin{aligned} &\sum_{n=1}^{\infty} w_n^a \frac{L^*(a_1, -w_n^a) F(-w_n^a)}{w_n^{a2} - w_m^{c2}} - h \frac{L^*(a_1, h) F(h)}{h^2 - w_m^{c2}} = \frac{1}{2} L(a_1, w_m^c) F(w_m^c), \\ &\sum_{n=1}^{\infty} \frac{L^*(a_1, -w_n^a) F(-w_n^a)}{w_n^{a2} - w_m^{c2}} + \frac{L^*(a_1, h) F(h)}{h^2 - w_m^{c2}} = \frac{1}{2w_m^c} L(a_1, w_m^c) F(w_m^c), \\ &\sum_{n=1}^{\infty} \frac{L(a_1, -w_n^a) F(-w_n^a)}{w_n^{a2} - w_n^{b2}} + \frac{L^*(a_1, h) F(h)}{h^2 - w_m^{b2}} = 0. \end{aligned} \tag{19}$$

Thus, the unknown coefficients of the series can be expressed in terms of $F(w)$

$$\begin{aligned} B_m &= \frac{L(a_1, w_m^c) F(w_m^c)}{(a_1', a')_m^*}, \\ A_n &= \frac{L^*(a_1, -w_n^a) F(-w_n^a)}{\mathrm{J}_1(v_n^a a_1)},\ A = \frac{L^*(a_1, h) F(h)}{\mathrm{J}_1(V a_1)}. \end{aligned} \tag{20}$$

As a thin semi-infinite pin ($a_1 \to 0$) and a hollow cylinder of the same radius are equivalent from the physical point of view, this solution allows us to pass to the limit $a_1 \to 0$, $\triangle \!\to 0$, where the previously known expression for a thin semi-infinite cylinder obtained by the W-H method follows directly from (8), as the function $\Psi(w)$ in (8) tends to a constant value due to the limit

$$\lim_{a_1 \to 0} \Psi(w)/\Psi(h) = 1. \tag{21}$$

Finally, the obtained solution also enables us to make a transition to the limit when the waveguide is locked ($a_1 \to a$), and the incident spatial mode is completely reflected from the end of the pin. In this case, all reflected spatial modes with longitudinal wave numbers $-w_n^a$, $(n = 1, 2, \cdots)$, except for the only reflected harmonic $-\tilde{h} = -h$ at $a_1 \to a$ are completely canceled out by the Bessel function $\mathrm{J}_0(va_1)$ in $\Psi(w)$ contained in the analytical source $F(w)$ (8).

By calculating the residue at the single pole $w = -h$ in the LHP w of the integrand of the function $L(r, w)F(w)$ in (3), and passing to the limit $a_1 \to a$ we obtain the field of the reflected wave

$$E_\phi(r, z) = -A_0 \mathrm{J}_1(Vr) e^{ih(2z_1 - z_0 - z)}, \tag{22}$$

where it is seen that the imaginary source of the reflected wave and the radiator at z_0 are located symmetrically with respect to z_1.

VI. Conclusions

The boundary value problem of the diffraction of a TE wave by a semi-infinite pin located coaxially inside a circular waveguide is reduced to a system of singular integral equations. In the class of meromophic functions, the solution to paired integral equations (7) was constructed using the W-H method. The solution $F(w)$ in (8) was constructed so that the integrand had no poles other than those at infinity or the pole of the incident wave $w = h$, when the integration contour is closed along the real axis along the infinite semicircumference in UHP or LHP w, according to Jordan's lemma.

For this purpose, a sequence of the entire function $J(v)$ depending on the parameter N with a given behavior at infinity and all zeros removed to infinity was constructed. As on the complex plane w the zeros of $J(v)$ are imaginary and their absolute values tend to infinity, the sum of residues of the integrand at these points can be considered vanishingly small or all remote poles are beyond the integration contour.

The Meixner condition in solving the problem is satisfied due to the shift of the zeros of the function $J(v)$ in (9) by $\triangle = 2/3$.

The solution $F(w)$ is verified using the stitching method. It is shown that a system of infinite linear algebraic equations obtained by the stitching method can be derived by integrating the function $L(r, w)F(w)$ along an infinite circumference. All coefficients of the series of the stitching method are found through the function $L(r, w)F(w)$.

This solution allows us to pass to the limit when the radius of the pin is small $a_1 < 4/k$ or when the pin completely blocks the waveguide $a_1 = a$. In the first case, it is shown that the solutions for the pin and the hollow semi-infinite pipe are the same, as they are equivalent from the physical point of view. In the case of a locked waveguide $a_1 \to a$, the solution contains only a single pole in the LHP $w = -h$, corresponding to a mode completely reflected from the end of the waveguide. In this case, the locations of the sources for the incident and reflected modes turn out to be mirror-symmetrical with respect to the end.

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APPENDIX A
FACTORIZATION OF BESSEL FUNCTIONS

Here we present the expressions for the factorized Bessel functions $Q(w) = Q(w)_- Q(w)_+$

$$Q(w)_+ = (k+w)^\nu Q_0 e^{-\chi(w)} \prod_{m=1}^{\infty} \Big(1 + \frac{w}{w_m}\Big) e^{\frac{iw}{mp}},$$
$$Q(w)_+ \sim (k+w)^\nu \Big(\frac{w}{ip}\Big)^{-(\Delta+\frac{1}{2})} e^{\frac{iw}{p}\ln\frac{2w}{k}},\ w \to \infty, \tag{23}$$
$$\chi(w) = \frac{w}{ip}(1-\gamma) - i\frac{w}{p}\ln\frac{2ip}{k},\ Q(w_m) = 0,$$

where $\gamma = 0,577$ is Euler's constant, $Q_0 = \sqrt{Q(0)}$,

$$p = \left(\begin{array}{c} \pi/a \to \mathrm{J}_1(va) \\ \pi/(b-a) \to (a', b') \end{array} \right),\ w_m \simeq ip(m+\Delta),$$
$$\Delta = \left(\begin{array}{c} 1/4 \to \mathrm{J}_1(va), \\ 0 \to (a', b') \end{array} \right),\ \nu = \left(\begin{array}{c} 1/2 \to \mathrm{J}_1(va) \\ 0 \to (a', b') \end{array} \right). \tag{24}$$

The formula in (23) leads to the expression for factorization of the function in (8)

$$L_-^{-1}(a_1, w) = P_0 \prod_{m=1}^{\infty} \frac{\Big(1 - \dfrac{w}{w_m^a}\Big)\exp\Big\{i\dfrac{wa_1}{\pi}T\Big\}}{\Big(1 - \dfrac{w}{w_m^b}\Big)\Big(1 - \dfrac{w}{w_m^c}\Big)},$$
$$T = \tfrac{a}{a_1}\ln\tfrac{a}{a_1} - \Big(\tfrac{a}{a_1} - 1\Big)\ln\Big(\tfrac{a}{a_1} - 1\Big), \tag{25}$$

where $P_0 = \text{const}$, w_m^b are zeros of the function $\mathrm{J}_1(va_1)$.

It should be noted that when calculating factorized functions it is convenient to use the formula

$$L_-(a_1, w) = \sqrt{L(a_1, w)\frac{L_-(a_1, w)}{L_+(a_1, w)}}. \tag{26}$$