RESIDUAL INTERACTION AND NUCLEON PAIRING ENERGY*

F.M. Pen'kov, V.O. Kurmangalieva

Al-Farabi Kazakh National University, 050040 Almaty, Kazakhstan

T.K. Zholdybayev, P.M. Krassovitskiy

Institute of Nuclear Physics, 050032 Almaty, Kazakhstan

Received 14 November 2022, accepted 15 November 2022, published online 26 January 2023

The problem of describing the nucleon pairing energy in nuclei was considered using a realistic nucleon–nucleon potential that reproduces the parameters of nucleon–nucleon scattering. Satisfactory agreement between the calculated and experimental values of the isotriplet nucleon pairing energies for even–even nuclei was obtained.

DOI:10.5506/APhysPolBSupp.16.2-A15

1. Introduction

The pairing energies of nucleons in nuclei largely determine the properties of nuclei and nuclear matter (see, for example, reviews [1–3]). The authors conventionally consider superfluid states of nuclei as the main mechanism of pairing, in which the nucleon–nucleon interaction is not related to realistic nucleon–nucleon forces but represents some residual interaction from the nuclear forces spent on creating the potential of single-particle motion. How the pair interaction of nucleons in nuclei differs from that of free nucleons, at least for valence nucleons, remains an open question.

We define the experimental pairing energy E_{nn} (E_{pp}) as the difference between the bound energies of even $E_{\mathrm{b},N+2}$ ($E_{\mathrm{b},Z+2}$) and odd nucleons $E_{\mathrm{b},N+1}$ ($E_{\mathrm{b},Z+1}$) in the nuclei. Information can be obtained from [4] to analyze the pairing energies. The range of neutron pairing energies obtained from here can be estimated by comparing the maximum pairing energies of light and heavy elements. Thus, the maximum neutron pairing energy for carbon ($^{10}_{6}$C) comprises 7.02 MeV, while the maximum pairing energy for uranium

* Presented at the IV International Scientific Forum *Nuclear Science and Technologies*, Almaty, Kazakhstan, 26–30 September, 2022.

($^{224}_{92}$U) is 3.8 times less and is equal to 1.83 MeV. Such a large difference makes it difficult to analyze the original data. One of the reasons for this difference is the different scales of the nuclear energy units (nuclear quantum) for different nuclei. To exclude this reason, a standard definition of the energy of a nuclear quantum can be used $\hbar\omega = 41/A^{1/3}$ MeV with $A = N + Z$. In such units, the neutron pairing energy for the $^{10}_{6}$C nucleus was 0.371 and for $^{224}_{92}$U is 0.270; that is, the scaled energies differed by only 30%. This makes it possible to analyze the pairing energy from the average values of the neutron pairing energy for all isotopes of a given nucleus, that is, for a fixed Z. Similarly, the proton pairing energy can be analyzed by averaging all the proton pairing energies for a fixed N, that is, for the isotons.

Figure 1 presents the pairing values of neutrons and protons, calculated using the specified algorithm. To make the figures more suitable for analysis, only the pairing energies for the selected Z and N values are shown. The errors indicated in these figures are equal to the errors in determining the mean (one σ). To visualize the general trends, Fig. 1 shows the lines of the average values of $\bar{E}_{nn}$ ($\bar{E}_{pp}$) for the nuclei shown in the figure: dashed lines for even nuclei and dash-dotted lines for odd nuclei. It can be seen that the pairing energies of neutrons and protons for even nuclei are so close to each other $\bar{E}_{nn}/\hbar\omega = 0.248 \pm 0.005$ and $\bar{E}_{pp}/\hbar\omega = 0.232 \pm 0.004$ that a separate question arises about the smallness of the Coulomb repulsion of protons. A similar situation was observed for the pairing energies of neutrons and protons in odd nuclei: $\bar{E}_{nn}/\hbar\omega = 0.176 \pm 0.009$ and $\bar{E}_{pp}/\hbar\omega = 0.163 \pm 0.008$. In this case, the pairing energies for even and odd nuclei were significantly different. The different binding energies of nucleons in even and odd nuclei are reflected in the Weizsäcker formula. However, here, we deal with different nucleon pairing energies for even and odd nuclei.

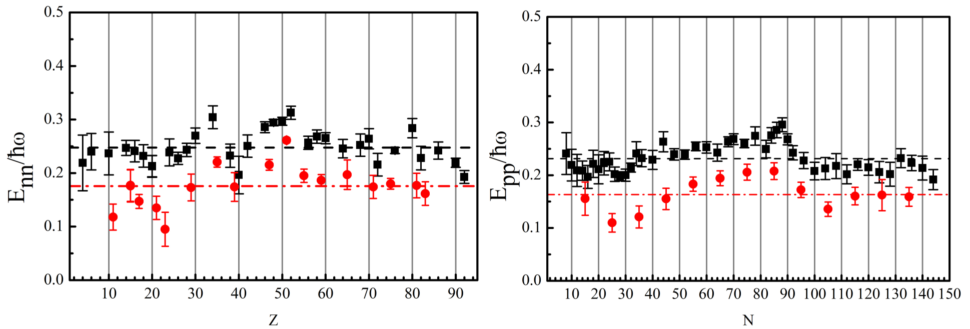

Fig. 1. Experimental energies of nn-pairing (left) and pp-pairing (right). Filled squares — even $Z(N)$, filled circles — odd $Z(N)$. Explanations are given in the text.

The proximity of the average pairing values of protons and neutrons indicates common pairing mechanisms for both neutrons and protons, which can be ordinary pair interactions in the isovector state ($T = 1$) for the singlet ($S = 0$) state of a pair of nucleons. Below, we consider the zero-range interaction model and the Yamaguchi-type pair interaction model. The model suitability criterion was the closeness of the calculated pairing energy to the experimental values shown in Fig. 1.

2. Model

For the sake of brevity, we will further assume $\hbar = 1$. A shell model with the possibility of separating the motion of the center of inertia of the pair is most suitable for pair interactions with realistic parameters. Therefore, we used a shell model with a single-particle motion potential in the form of a three-dimensional harmonic oscillator, whose Hamiltonian H_1 for a nucleon with mass m

$$H_1 = -\frac{1}{2m}\Delta + \frac{m\omega^2 r^2}{2} \tag{1}$$

generates a single-particle spectrum

$$E_1 = \omega\left(\frac{3}{2} + l + 2n\right), \qquad n = 0, 1, 2 \dots \tag{2}$$

The Fermi statistics of nucleons are considered only in the filling of the shells rule, considering the $\boldsymbol{ls}$-coupling, determining the orbital momentum l, and the radial quantum number n of the state. In addition, the shell-filling rule was used in the calculation model; the total momentum of a pair of odd and even nucleons was equal to zero. That is, the orbital moments of the nucleons of a pair must be the same, and the spectrum of the two non-interacting nucleons must have the form of

$$E_2 = \omega(3 + 2l + 4n_r), \qquad n_r = 0, 1, 2 \dots \tag{3}$$

The Hamiltonian H_2^0 of two non-interacting particles with coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in the oscillator well can be written in terms of the coordinates of the center of inertia of the pair $\boldsymbol{R} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$ and the coordinates of the relative motion $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ in the form of

$$H_2^0 = -\frac{1}{m}\Delta_r - \frac{1}{4m}\Delta_R + \frac{m\omega^2 r^2}{4} + m\omega^2 R^2 \tag{4}$$

allowing the variables to be separated, the energy of the two non-interacting nucleons can be written as the sum of the energies of the motion of the center of mass E_R and the relative motion E_r in forms similar to (3)

$$E_2 = \omega(3 + l_R + l_r + 2n_r + 2n_R), \qquad n_r = 0, 1, 2 \dots, \qquad n_R = 0, 1, 2 \dots \tag{5}$$

The orbital momentums of the relative motion l_r and the motion of the center of inertia l_R must be the same, making the net angular momentum equal to 0. If we assume that internucleon forces determine the pairing energy of nucleons, then determining the contribution of the S-wave pair interaction leads to the condition $l_r = 0$. Therefore, it follows from Eqs. (3) and (5) that

$$l + 2n = n_r + n_R \,, \tag{6}$$

that is, the sum of the radial quantum numbers n_r and n_R is determined by the number N of the quantum shell of the nucleus $N = l + 2n$. Considering the basic states of the nuclei, it is assumed that the center of inertia moves with minimum energy, that is, $n_R = 0$. Therefore, the energy of the relative motion without pair interaction can be written as

$$E_r^0 = \omega\left(\frac{3}{2} + 2N\right) , \qquad N = l + 2n \,. \tag{7}$$

The inclusion of the S-wave pair interaction allows us to write the radial Schrödinger equation for relative motion in the form of

$$-\frac{1}{m}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}}{\mathrm{d}r}\Psi + \frac{m\omega^2 r^2}{4}\Psi + V\Psi = E_r^{\mathrm{in}}\Psi \,. \tag{8}$$

In this case, the nucleon pairing energy E_{nn} (E_{pp}) is determined by the difference

$$E_{nn} = E_r^0 - E_r^{\mathrm{in}} \,. \tag{9}$$

Yamaguchi's separable potential [5] was chosen as a realistic nucleon–nucleon interaction model. In momentum representation, this potential takes the form of

$$V\left(p, p'\right) = v(p)v\left(p'\right) = -\frac{8\pi}{m}\frac{\beta(\beta+\kappa)^2}{\left(\beta^2 + p^2\right)\left(\beta^2 + p'^2\right)} \,, \tag{10}$$

where the parameters κ and β are determined by the pair length a_2 and the effective scattering radius r_{eff} [5]

$$a_2 = \frac{2(\beta+\kappa)^2}{\kappa(2\beta+\kappa)\beta} \,, \qquad r_{\mathrm{eff}} = \frac{(\beta+\kappa)^2 + 2\beta^2}{\beta(\beta+\kappa)^2} \,. \tag{11}$$

Note that at $\beta \gg \kappa$, parameters (11) are determined by the asymptotic form

$$a_2 \approx \frac{1}{\kappa} \,, \qquad r_{\mathrm{eff}} \approx \frac{3}{\beta} \,, \tag{12}$$

which allows for pair interaction to be set up via zero-range potentials. Therefore, for the asymptotic solution at $\beta \to \infty$, one can write the condition for pair interaction

$$\left.\frac{(r\Psi)'}{r\Psi}\right|_{r\to\infty} = \frac{1}{\kappa}\,, \tag{13}$$

which determines the scattering phase in such a formulation of the problem or the binding energy of a pair.

Since the Yamaguchi potential in the configuration space also has a separable form

$$V\left(r, r'\right) = v(r)v\left(r'\right) = -\frac{\beta(\beta+\kappa)^2}{2\pi m}\frac{\mathrm{e}^{-\beta r}}{r}\frac{\mathrm{e}^{-\beta r'}}{r'}\,, \tag{14}$$

then the solution to Eq. (8) is the solution of the equation for a harmonic oscillator with a non-homogeneous term of type (14). To write this solution, we introduce regular F and irregular G at the zero solution for the harmonic oscillator equation with energy E and oscillatory length $x_0^2 = 2/m\omega$

$$F = \mathrm{e}^{-\frac{r^2}{2x_0^2}} rM\left(\frac{2\omega - 2E}{4\omega}, \frac{3}{2}, \frac{r^2}{x_0^2}\right), \qquad G = \mathrm{e}^{-\frac{r^2}{2x_0^2}} rU\left(\frac{2\omega - 2E}{4\omega}, \frac{3}{2}, \frac{r^2}{x_0^2}\right), \tag{15}$$

which are expressed in terms of Kummer's functions $M(a, b, z)$ and function $U(a, b, z)$ (see, for example, [7]). The properties of these solutions near zero and at the infinity [7] allow to write down the solution of (7) in the form of

$$\phi(r; E) \equiv r\Psi \propto \frac{1}{W}\left(\int_0^r \mathrm{d}r' \mathrm{e}^{-\beta r'} G(r)F\left(r'\right) + \int_r^\infty \mathrm{d}r' \mathrm{e}^{-\beta r'} F(r)G\left(r'\right)\right). \tag{16}$$

At spectral points, at $E = E_r^{\mathrm{in}}$, defined by the equation

$$\begin{aligned} 1 \;=\; & \frac{2\beta(\beta+\kappa)^2}{W}\left(\int_0^\infty \mathrm{d}r\, \mathrm{e}^{-\beta r} G(r) \int_0^r \mathrm{d}r'\, \mathrm{e}^{-\beta r'} F\left(r'\right)\right. \\ & \left. + \int_0^\infty \mathrm{d}r\, \mathrm{e}^{-\beta r} F(r) \int_r^\infty \mathrm{d}r'\, \mathrm{e}^{-\beta r'} G\left(r'\right)\right), \end{aligned} \tag{17}$$

function (16) becomes square-integrable. The symbols W in (16) and (17) define the Wronskian of the solutions: $W = F'G - G'F$. Within the framework of the model described above, the solution to Eqs. (16) and (17) determines the nucleon-pairing energy.

Furthermore, the nucleon pairing energy was determined using the numerical solution of Eq. (17). Here, we present the results of the analysis for large values of the parameter β. Since the integrals in (17) depend on (15), the asymptotic parameter is βx_0. For the analysis, asymptotic methods can be used to estimate integrals in the simplest form by performing multiple integrations by parts. Consequently, the following simplified spectral equation was obtained

$$\frac{G'(0)}{G(0)} \approx -\frac{\kappa(2\beta+\kappa)\beta}{2(\beta+\kappa)^2} + \frac{3}{2\beta}K^2 + O\left((\beta x_0)^{-2}\right) , \tag{18}$$

where $K^2 = mE$. Considering (11) and (12), the resulting equation looks like an expansion of the effective radius only for discrete spectrum functions

$$\frac{G'(0)}{G(0)} \approx -\frac{1}{a_2} + \frac{r_{\rm eff}^2}{2}K^2 . \tag{19}$$

In particular, as $\beta \to \infty$, Eq. (18) or (19) tend toward the well-known equation for the spectrum of a pair of point bosons in harmonic traps [8]

$$\frac{\Gamma\left(\frac{3}{4} - \frac{E_r^{\rm in}}{2\omega}\right)}{\Gamma\left(\frac{1}{4} - \frac{E_r^{\rm in}}{2\omega}\right)}\frac{2}{x_0} = \frac{1}{a_2} , \tag{20}$$

which can be used to analyze the adequacy of the applicability of the point interaction with the experimental length of the pair interaction in describing the pair correlations.

3. Results and discussion

To solve Eq. (17), one should know the parameters β and κ of the Yamaguchi potential. These parameters can be obtained by comparing Eq. (11) with the experimental values of a_2 and $r_{\rm eff}$. Below, we use the accepted average values for isotriplet [9]

$$a_2 = -18.5 \pm 0.3 \text{ fm} , \qquad r_{\rm eff} = 2.75 \pm 0.11 \text{ fm} : \quad T = 1 , \quad S = 0 . \tag{21}$$

It can be seen that the effective radius for the isotriplet interactions is known with a quite large error (4%). Therefore, the error in the calculations below reflects the uncertainty of the experimental values in Eq. (21), which are displayed graphically as filled rectangles.

Let us first consider the applicability of the hypothesis of a possible description of the nucleon pairing energy using the point interaction (20) with experimental scattering lengths (21). The properties of Eq. (20) are quite obvious, and we can indicate that in the absence of the pair interaction ($a_2/x_0 \to 0$), the solution to this equation is determined by the features of the gamma function in numerator (20) when the argument of the gamma function is a negative integer or 0. That is, $E_r^{\rm in} = E_r^0$ from Eq. (9). In the case of large scattering lengths ($x_0/a_2 \to 0$), the spectrum (20) is determined by the singularities of the denominator and $E_r^{\rm in} = \omega(1/2+2N), N = l+2n$; that is, the energy in this limiting case is shifted by one nuclear quantum. By substituting the experimental scattering lengths from Eq. (21), we obtain almost the same values for the energy shifts. In particular, for the isotriplet interaction we obtain the solutions (20): $(E_r^{\rm in})/\omega = 0.5694, 2.5362, 4.5273, \ldots$, which in comparison with the spectrum of the harmonic oscillator $(E_r^0)/\omega = 1.5, 3.5, 5.5, \ldots$ give the shifts $E_{nn}/\omega = 0.931, 0.964, 0.973, \ldots$, which are very close to one nuclear quantum. The pairing energies obtained were approximately four times greater than the experimental values (Fig. 1) in even–even nuclei. The conclusion of this paragraph is that the pair interaction of nucleons in the form of point interaction with experimental scattering lengths does not describe the pairing energy of nucleons in the nuclei. Let us consider the solutions to Eq. (17) with a paired nucleon–nucleon potential of the Yamaguchi-type for isotriplet interactions. Figure 2 shows the experimental results and calculation from Eq. (17), the pairing energies E_{nn} and E_{pp} for even–even nuclei. The calculated values in the figure are displayed as filled rectangles, the heights of which were determined by 8% uncertainty of the experimental values (21). The influence of the Coulomb interaction $U_{pp} = e^2/r$ of protons was considered in the first-order perturbation theory.

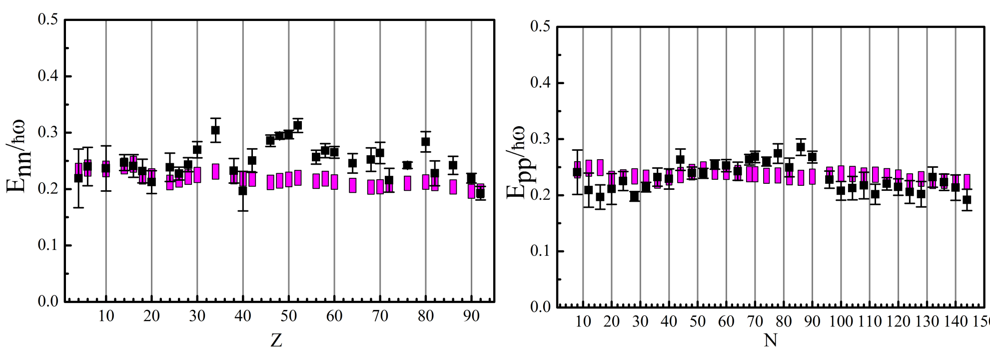

Fig. 2. Experimental and calculated energies of *nn*-pairing (left) and *pp*-pairing (right).

In addition to graphical information about pairing energies, Table 1 provides the average pairing energies of isotopes and isotones for even–even nuclei, as shown in Fig. 2. There is satisfactory agreement between the calculation and experiment; there are groups of nuclei where the experiment

Table 1. Average experimental and calculated nucleon pairing energies of even–even nuclei.

Nucleon pairing energy	Experiment	Calculation	Calculation without the Coulomb repulsion
$\bar{E}_{nn}$	0.248 ± 0.005	0.217 ± 0.013	
$\bar{E}_{pp}$	0.232 ± 0.004	0.195 ± 0.013	0.234 ± 0.013

and theory diverge by approximately 50%, providing a discrepancy between the average values of the binding energy. The ideal agreement between the experimental values of proton pairing and theoretical calculations without considering the Coulomb repulsion of protons is most likely accidental.

4. Conclusion

Calculations show that the Yamaguchi potential, which describes the interaction of two nucleons in vacuum, generally describes the experimental energy of the isotriplet pairing of nucleons in atomic nuclei for even–even nuclei.

This work was performed under grant AP09258757 from the Ministry of Education and Science of the Republic of Kazakhstan.

REFERENCES

[1] S. Frauendorf, A.O. Macchiavelli, *Prog. Part. Nucl. Phys.* **78**, 24 (2014).

[2] B.S. Ishkhanov, M.E. Stepanov, T.Yu. Tretyakova, *Moscow Univ. Phys. Bull.* **69**, 1 (2014).

[3] D.J. Dean, *Rev. Mod. Phys.* **75**, 607 (2003).

[4] Nuclear Wallet Cards database version of 5/16/2019, https://www.nndc.bnl.gov/nudat3/indx_sigma.jsp

[5] Y. Yamaguchi, *Phys. Rev.* **95**, 1628 (1954).

[6] J.R. Taylor, «Scattering Theory: The Quantum Theory on Nonrelativistic Collisions», *J. Wiley and Sons, Inc.*, New York, London, Sidney, Toronto 1972.

[7] M. Abramowitz, I. Stegun, «Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables», *National Bureau of Standards*, 1964.

[8] T. Busch, B.G. Englert, K. Rzażewski, M. Wilkens, *Found. Phys.* **28**, 549 (1998).

[9] I. Slaus, Y. Akaishi, H. Tanaka, *Phys. Rep.* **173**, 257 (1989).