Seminar on Computer Algebra
Faculty of Computational Mathematics and Cybernetics of MSU, and Computing Centre of RAS
September 29, 2021, Moscow, Russia

Symbolic Computation in Studying the Evolutionary Equations in the Problem of Many Bodies of Variable Masses

Alexander Prokopenya
Warsaw University of Life Sciences - SGGW, Poland
E-mail: **alexander_prokopenya@sggw.edu.pl**
M.Zh. Minglibayev, A.B. Kosherbayeva
al-Farabi Kazakh National University, Almaty, Kazakhstan
E-mail: **minglibayev@gmail.com; kosherbaevaayken@gmail.com**

Content

Physical problem
Equations of motion
Perturbing functions
Evolution equations
Conclusion

Physical Problem

Many-body problem is a well-known model in Celestial Mechanics. It is integrable in case of two interacting point masses or two finite-size bodies with spherically symmetric mass distribution, and its general solution is well-known. If at least one of the bodies has variable mass, the problem becomes much more complicated and its general solution may be written out in symbolic form only in some special cases. So the perturbation theory is usually applied for investigation of this problem.

Let us consider a system of $n+1$ mutually attracting bodies $P_0, P_1, P_2, \ldots, P_n$ with variable masses changing isotropically in different rates

$$m_0 = m_0(t),\quad m_j = m_j(t),\quad \frac{\dot m_0}{m_0} \ne \frac{\dot m_j}{m_j},\quad \frac{\dot m_i}{m_i} \ne \frac{\dot m_j}{m_j},\quad i, j = 1, 2, \ldots, n;$$

Using an inertial Cartesian coordinate system, one can write equations of motion of the system in the form

$$m_i \frac{d^2 \vec R_i}{dt^2} = \mathrm{grad}_{\vec R_i} U,\quad i = 0, 1, \ldots, n, \tag{1}$$

where f is a gravitational constant, and the force function U is given by

$$U = f \sum_{i<j}^{n} \frac{m_i m_j}{R_{ij}},\quad R_{ij} = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2}.$$

Equations of Motion in Relative Coordinates

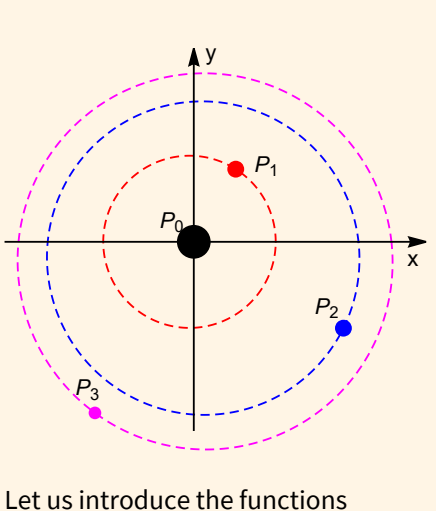
Using relative coordinates with the body P_0 situated in the origin ($\vec r_i \to \vec R_i - \vec R_0$, $i = 1, 2, \ldots, n$) we can rewrite equations of motion in the form

$$\frac{d^2 \vec r_i}{dt^2} + G(m_0 + m_i)\frac{\vec r_i}{r_i^3} = \mathrm{grad}_{\vec r_i} U_i,\quad U_i = f \sum_{j=1\,(\ne i)}^{n} m_j\left(\frac{1}{\Delta_{ij}} - \frac{\vec r_i \vec r_j}{r_j^3}\right), \tag{2}$$

where

$$\Delta_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} = r_{ij},\quad r_j = \sqrt{x_i^2 + y_i^2 + z_i^2},\quad i = 1, 2, \ldots, n.$$

We assume that the body P_0 is the most massive one and the rest bodies move around it and their trajectories do not intersect. This problem is known as the planetary problem of many bodies of variable masses.



Let us introduce the functions

$$\gamma_j = \gamma_j(t) = \frac{m_0(t_0) + m_j(t_0)}{m_0(t) + m_j(t)},$$

and rewrite equations (2) in the form

$$\frac{d^2 \vec r_j}{dt^2} + G(m_0 + m_j)\frac{\vec r_j}{r_j^3} - b_j \vec r_j = \mathrm{grad}_{\vec r_j} W_j, \tag{3}$$

where

$$W_j = -\frac{1}{2} b_j r_j^2 + U_j,\quad r_j^2 = x_j^2 + y_j^2 + z_j^2,$$

$$U_j = f \sum_{i=1\,(i\ne j)}^{n} m_i\left(\frac{1}{|\vec r_i - \vec r_j|} - \frac{\vec r_i \vec r_j}{r_i^3}\right),\quad b_j = b_j(t) = \frac{1}{\gamma_j}\frac{d^2\gamma_j}{dt^2},\ j = 1, 2, \ldots, n.$$

If $W_j = 0$ then we have n independent systems of differential equations describing motion of the bodies P_j about the body P_0.

Problem of Two Bodies of Variable Masses

Let us consider equations of motion in the two-body problem with variable masses

```
eq1 = {x''[t] == -K/γ[t] x[t]/(x[t]^2 + y[t]^2 + z[t]^2)^(3/2) + γ''[t] / γ[t] × x[t],
   y''[t] == -K/γ[t] y[t]/(x[t]^2 + y[t]^2 + z[t]^2)^(3/2) + γ''[t] / γ[t] × y[t],
   z''[t] == -K/γ[t] z[t]/(x[t]^2 + y[t]^2 + z[t]^2)^(3/2) + γ''[t] / γ[t] × z[t]};
```

where $K = K_j = f(m_{00} + m_{j0})$, $m_{j0} = m_j(t_0)$, and $\gamma(t) = \gamma_j(t)$, $j = 1, 2, \ldots, n$.

Let us rewrite the equations using quasi-spherical coordinates.

```
rul0 = {x → (γ[#] × ρ[#] Cos[φ[#]] Cos[θ[#]] &), y → (γ[#] × ρ[#] Cos[φ[#]] Sin[θ[#]] &), z → (γ[#] × ρ[#] Sin[φ[#]] &)};
```

```
eq2 = eq1 /. rul0;
eq3 = {ρ''[t] == (ρ''[t] /. Solve[eq2, {ρ''[t], φ''[t], θ''[t]}][[1]] // Simplify[#, γ[t] > 0 && ρ[t] > 0] &),
  φ''[t] == (φ''[t] /. Solve[eq2, {ρ''[t], φ''[t], θ''[t]}][[1]] // Simplify[#, γ[t] > 0 && ρ[t] > 0] &),
  θ''[t] == (θ''[t] /. Solve[eq2, {ρ''[t], φ''[t], θ''[t]}][[1]] // Simplify[#, γ[t] > 0 && ρ[t] > 0] &)}
```

$$\Big\{\rho''[t] = -\frac{K}{\gamma[t]^4\rho[t]^2} - \frac{2\gamma'[t]\rho'[t]}{\gamma[t]} + \rho[t]\left(\mathrm{Cos}[\varphi[t]]^2\theta'[t]^2 + \varphi'[t]^2\right),$$
$$\varphi''[t] = -\mathrm{Cos}[\varphi[t]]\,\mathrm{Sin}[\varphi[t]]\,\theta'[t]^2 - \frac{2(\rho[t]\gamma'[t] + \gamma[t]\rho'[t])\varphi'[t]}{\gamma[t]\times\rho[t]},$$
$$\theta''[t] = -\frac{2\theta'[t]\left(\gamma[t]\rho'[t] + \rho[t](\gamma'[t] - \mathrm{Tan}[\varphi[t]]\gamma[t]\varphi'[t])\right)}{\gamma[t]\times\rho[t]}\Big\}$$

To write the equations of motion in the Lagrange form we define the kinetic energy of the system.

```
K0 = 1/2 (x'[t]^2 + y'[t]^2 + z'[t]^2) /. rul0 // Collect[#, {ρ'[t], φ'[t], θ'[t], ρ[t]}, Simplify] &
```

$$\frac{1}{2}\rho[t]^2\gamma'[t]^2 + \frac{1}{2}\mathrm{Cos}[\varphi[t]]^2\gamma[t]^2\rho[t]^2\theta'[t]^2 + \gamma[t]\times\rho[t]\gamma'[t]\rho'[t] + \frac{1}{2}\gamma[t]^2\rho'[t]^2 + \frac{1}{2}\gamma[t]^2\rho[t]^2\varphi'[t]^2$$

Potential energy of the system is given by

```
U0 = (K/γ[t])/Sqrt[x[t]^2 + y[t]^2 + z[t]^2] + 1/2 γ''[t] / γ[t] (x[t]^2 + y[t]^2 + z[t]^2) /. rul0 // Simplify[#, γ[t] > 0 && ρ[t] > 0] &
```

$$\frac{K}{\gamma[t]^2\rho[t]} + \frac{1}{2}\gamma[t]\rho[t]^2\gamma''[t]$$

Then equations of motion in the Lagrange form are

```
eq4 = {D[D[K0, ρ'[t]], t]  == D[K0 + U0, ρ[t]],
    D[D[K0, φ'[t]], t]  == D[K0 + U0, φ[t]],
    D[D[K0, θ'[t]], t]  == D[K0 + U0, θ[t]]};
eq5 = {ρ''[t] == (ρ''[t] /. Solve[eq4, {ρ''[t], φ''[t], θ''[t]}][[1]] // Simplify),
   φ''[t] == (φ''[t] /. Solve[eq4, {ρ''[t], φ''[t], θ''[t]}][[1]] // Simplify),
   θ''[t] == (θ''[t] /. Solve[eq4, {ρ''[t], φ''[t], θ''[t]}][[1]] // Simplify)}
```

$$\Big\{\rho''[t] = -\frac{K}{\gamma[t]^4\rho[t]^2} - \frac{2\gamma'[t]\rho'[t]}{\gamma[t]} + \rho[t]\left(\mathrm{Cos}[\varphi[t]]^2\theta'[t]^2 + \varphi'[t]^2\right),$$
$$\varphi''[t] = -\mathrm{Cos}[\varphi[t]]\,\mathrm{Sin}[\varphi[t]]\,\theta'[t]^2 - \frac{2(\rho[t]\gamma'[t] + \gamma[t]\rho'[t])\varphi'[t]}{\gamma[t]\times\rho[t]},$$
$$\theta''[t] = -\frac{2\theta'[t]\left(\gamma[t]\rho'[t] + \rho[t](\gamma'[t] - \mathrm{Tan}[\varphi[t]]\gamma[t]\varphi'[t])\right)}{\gamma[t]\times\rho[t]}\Big\}$$

To write these equations in the Hamiltonian form we define the canonical momenta in the form

```
eq6 = {Pρ[t] == D[K0 + U0, ρ'[t]], Pφ[t] == D[K0 + U0, φ'[t]], Pθ[t] == D[K0 + U0, θ'[t]]}
```

$$\{P\rho[t] = \gamma[t]\times\rho[t]\gamma'[t] + \gamma[t]^2\rho'[t],\ P\varphi[t] = \gamma[t]^2\rho[t]^2\varphi'[t],\ P\theta[t] = \mathrm{Cos}[\varphi[t]]^2\gamma[t]^2\rho[t]^2\theta'[t]\}$$

```
sol6 = Solve[eq6, {ρ'[t], φ'[t], θ'[t]}][[1]]
```

$$\left\{\rho'[t] \to \frac{P\rho[t] - \gamma[t]\times\rho[t]\gamma'[t]}{\gamma[t]^2},\ \varphi'[t] \to \frac{P\varphi[t]}{\gamma[t]^2\rho[t]^2},\ \theta'[t] \to \frac{P\theta[t]\,\mathrm{Sec}[\varphi[t]]^2}{\gamma[t]^2\rho[t]^2}\right\}$$

Finally, the Hamiltonian takes the form

```
H0 = Pρ[t] × ρ'[t] + Pφ[t] × φ'[t] + Pθ[t] × θ'[t] - K0 - U0 /. sol6 // Collect[#, {Pρ[t], Pφ[t], Pθ[t], G}, Simplify] &;
```

$$H_0 = \frac{1}{2\gamma^2}(P_\rho - \rho\gamma\dot\gamma)^2 + \frac{1}{2\gamma^2\rho^2}P_\varphi^2 + \frac{1}{2\gamma^2\rho^2\cos^2\varphi}P_\theta^2 - \frac{K}{\gamma^2\rho} - \frac{1}{2}(\dot\gamma^2 + \gamma\ddot\gamma)\rho^2;\quad \gamma = \gamma(t) = \frac{m_0(t_0) + m_i(t_0)}{m_0(t) + m_i(t)}$$

Unperturbed motion

Equations of motion written in the Hamiltonian form may be integrated with the Hamilton-Jacobi method for any doubly continuously differentiable function $\gamma(t)$; the corresponding exact solution is given by

$$x_i = \gamma_i(t)\,\rho_i(t)\,(\cos(v_i + \omega_i)\cos\Omega_i - \sin(v_i + \omega_i)\sin\Omega_i\cos i_i),$$
$$y_i = \gamma_i(t)\,\rho_i(t)\,(\cos(v_i + \omega_i)\sin\Omega_i + \sin(v_i + \omega_i)\cos\Omega_i\cos i_i),$$
$$z_i = \gamma_i(t)\,\rho_i(t)\,\sin(v_i + \omega_i)\sin i_i,$$

where

$$\rho_i = \frac{a_i(1 - e_i^2)}{1 + e_i\cos v_i} = a_i(1 - e_i\cos w_i),\quad w_i - e_i\sin w_i = M_i = \frac{\sqrt{f(m_{00} + m_{i0})}}{a_i^{3/2}}(\Phi_i(t) - \Phi_i(\tau)),\quad \Phi_i(t) = \int_0^t \frac{dt}{\gamma_i^2(t)}.$$

where v_i is the true anomaly, w_i is the eccentric anomaly, $M_i = l_i$ is the mean anomaly, τ_i is the time of passage at pericenter.

If the perturbing function $W_i \ne 0$ is taken into account in the equations of motion the orbital parameters $a_j, e_j, i_j, \omega_j, \Omega_j$ become the functions of time. To derive the differential equations determining their evolution, it is convenient to use canonical variables known as Delaunay elements. The generating function for the corresponding canonical transformation is determined by the complete integral of the Hamilton–Jacobi equation. Finally, we determine three pairs of canonically conjugate coordinates and momenta $(L_j, l_j), (G_j, g_j), (H_j, h_j)$; they are related to the analogs of the Kepler orbital elements by

$$l_j = M_j,\ L_j = \sqrt{K_j a_j},\ g_j = \omega_j,\ G_j = \sqrt{K_j a_j(1 - e_j^2)},\ h_j = \Omega_j,\ H_j = G_j\cos i_j,\ K_j = f(m_{00} + m_{j0}).$$

The corresponding Hamiltonian is given by

$$H_{j0} = -\frac{K_j^2}{2\gamma_j^2 L_j^2} - W_j,\quad j = 1, 2, \ldots, n,$$

where the perturbing functions W_j must be expressed in terms of the canonical variables.

The Perturbing Functions

Taking into account the perturbing functions W_j, one can easily write out the equations of the perturbed motion but they are very complicated and their general solution cannot be found in symbolic form. So we'll assume that trajectories of the bodies are almost planar and their deviations from the equatorial plane are small or ***inclinations are small***: $i_j \ll 1$. Besides, we assume that ***eccentricities*** of the trajectories ***are also small***: $e_j \ll 1$. Then we have to choose such a system of canonical variables, for which some canonical variables are small for $i_j \ll 1$ and $e_j \ll 1$. Such a system is know in the literature as the ***second system of Poincare elements*** (according to C. Charlier definition). Three pairs of canonically conjugate coordinates and moments $(\lambda_j, \Lambda_j), (\eta_j, \xi_j), (q_j, p_j)$ are connected with the Delaunay elements by means of relationships

$$\lambda_j = l_j + g_j + h_j = l_j + \omega_j + \Omega_j,\quad \Lambda_j = L_j,$$
$$\xi_j = \sqrt{2\Gamma_j}\cos\delta_j,\quad \eta_j = -\sqrt{2\Gamma_j}\sin\delta_j,$$
$$p_j = \sqrt{2Z_j}\cos z_j,\quad q_j = -\sqrt{2Z_j}\sin z_j,$$

where

$$\Gamma_j = \Lambda_j\left(1 - \sqrt{1 - e_j^2}\right),\quad Z_j = \Lambda_j\sqrt{1 - e_j^2}\,(1 - \cos i_j),\quad \delta_j = \omega_j + \Omega_j,\quad z_j = \Omega_j.$$

If $e_j \ll 1$ and $i_j \ll 1$ then four new Poincare elements ξ_j, η_j, p_j, q_j are small quantities.

In the considered case, the perturbing functions W_j may be represented in the form of power series in terms of ξ_j, η_j, p_j, q_j. The first step is to construct a solution to the Kepler equation in the form of power series in terms of e_j.

```
rulw = w → Sum[e^k w_k, {k, 0, 4}];
eq21 = (w - e Sin[w] - M) /. rulw // Series[#, {e, 0, 4}] & // Normal // Collect[#, e, Simplify] &;
eq21a = Reduce[Table[Coefficient[eq21, e, k] == 0, {k, 0, 4}], Table[w_k, {k, 0, 4}]] // Simplify[#, M > 0] &
```

$$M = w_0\ \&\&\ \mathrm{Sin}[w_0] = w_1\ \&\&\ \mathrm{Cos}[w_0]w_1 = w_2\ \&\&\ \mathrm{Sin}[w_0]w_1^2 + 2w_3 = 2\mathrm{Cos}[w_0]w_2\ \&\&\ \frac{1}{6}\mathrm{Cos}[w_0]w_1^3 + \mathrm{Sin}[w_0]w_1w_2 + w_4 = \mathrm{Cos}[w_0]w_3$$

Then coefficients w_k in the expansion of the eccentric anomaly are given by

```
sol21 = {w_0 → M, w_1 → Sin[M], w_2 → Cos[M] Sin[M], w_3 → Cos[M]^2 Sin[M] - 1/2 Sin[M]^3, w_4 → 1/6 (-Sin[2 M] + 2 Sin[4 M])};
```

Finally, the eccentric anomaly is represented in the form of power series

```
w /. rulw /. sol21 // TrigReduce // Collect[#, e] &
```

$$M + e\,\mathrm{Sin}[M] + \frac{1}{2}e^2\mathrm{Sin}[2M] + \frac{1}{24}e^3(-3\mathrm{Sin}[M] + 9\mathrm{Sin}[3M]) + \frac{1}{24}e^4(-4\mathrm{Sin}[2M] + 8\mathrm{Sin}[4M])$$

The sine and cosine functions of the eccentric anomaly are

```
rul23 =
  {Cos[w], Sin[w]} -> Map[Collect[TrigReduce[Expand[Normal[Series[#, {e, 0, 4}]]]], e, Simplify] &,
     {Cos[w], Sin[w]} /. rulw /. sol21 // Evaluate] // Thread // Collect[#, e, Simplify] &
```

$$\Big\{\mathrm{Cos}[w] \to \mathrm{Cos}[M] + \frac{1}{3}e^3(-\mathrm{Cos}[2M] + \mathrm{Cos}[4M]) - e\,\mathrm{Sin}[M]^2 - \frac{3}{2}e^2\mathrm{Cos}[M]\mathrm{Sin}[M]^2 - \frac{5}{96}e^4(23\mathrm{Cos}[M] + 25\mathrm{Cos}[3M])\mathrm{Sin}[M]^2,\ \mathrm{Sin}[w] \to$$
$$\mathrm{Sin}[M] + e\,\mathrm{Cos}[M]\mathrm{Sin}[M] + \frac{1}{8}e^2(-\mathrm{Sin}[M] + 3\mathrm{Sin}[3M]) + \frac{1}{6}e^3(-\mathrm{Sin}[2M] + 2\mathrm{Sin}[4M]) + \frac{1}{384}e^4(2\mathrm{Sin}[M] - 81\mathrm{Sin}[3M] + 125\mathrm{Sin}[5M])\Big\}$$

```
Cos[w]^2 + Sin[w]^2 /. rul23 // Expand // Series[#, {e, 0, 4}] & // Normal // Simplify
```

$$1$$

We'll do the calculations accurate up to the ***second order in eccentricity***, then

```
rul23 =
  {Cos[w], Sin[w]} -> Map[Collect[TrigReduce[Expand[Normal[Series[#, {e, 0, 2}]]]], e, Simplify] &,
    {Cos[w], Sin[w]} /. rulw /. sol21 // Evaluate] // Thread
```

$$\left\{\mathrm{Cos}[w] \to \mathrm{Cos}[M] - e\,\mathrm{Sin}[M]^2 - \frac{3}{2}e^2\mathrm{Cos}[M]\mathrm{Sin}[M]^2,\ \mathrm{Sin}[w] \to \mathrm{Sin}[M] + e\,\mathrm{Cos}[M]\mathrm{Sin}[M] + \frac{1}{8}e^2(-\mathrm{Sin}[M] + 3\mathrm{Sin}[3M])\right\}$$

Next step: we express the small variables $e_j \ll 1$ and $i_j \ll 1$ in terms of the Poincare elements ξ_j, η_j, p_j, q_j which are small quantities.
Using an exact equation

$$\xi_j^2 + \eta_j^2 = 2\Lambda_j\left(1 - \sqrt{1 - e_j^2}\right),$$

we find an expression for e_j^2

```
sole2 = Solve[2 Λ (1 - Sqrt[1 - e2]) - (ξ^2 + η^2) == 0, e2][[1]] // Collect[#, Λ, Simplify] &
```

$$\left\{e2 \to \frac{\eta^2 + \xi^2}{\Lambda} - \frac{(\eta^2 + \xi^2)^2}{4\Lambda^2}\right\}$$

and define the following rules

```
rul25 = {e Cos[δ] → ξ/Sqrt[Λ], e Sin[δ] → -η/Sqrt[Λ]};
```

Thus, we have obtained for the expressions $e\cos\delta$ and $e\sin\delta$ the expansions in terms of small quantities ξ and η accurate to the second order.

Using exact equations

$$p = \sqrt{2\Lambda\sqrt{1 - e^2}\,(1 - \cos i)}\,\cos z,\quad q = -\sqrt{2\Lambda\sqrt{1 - e^2}\,(1 - \cos i)}\,\sin z,$$
$$p^2 + q^2 = 2\Lambda\sqrt{1 - e^2}\,(1 - \cos i) = 2\Lambda\sqrt{1 - e^2}\left(1 - \sqrt{1 - \sin i^2}\right)$$

and definition of the Poincare elements, in a similar way we obtain for small inclinations $i_j \ll 1$

```
rul26 = {Cos[i] → 1 - (p^2 + q^2)/(2 Λ)};
rul27 = {Cos[z] Sin[i] → p/Sqrt[Λ], Sin[z] Sin[i] → -q/Sqrt[Λ]};
rul28 = {e^2 Cos[δ]^2 → ξ^2/Λ, e^2 Sin[δ]^2 → η^2/Λ, e^2 Cos[δ] Sin[δ] → -ξ η/Λ};
```

These results are accurate up to the second order in small parameters.

The distances r_j can be found in the form

```
rulρ = ρ → a (1 - e Cos[w]);
rulξηpq = {ξ → ε ξ, η → ε η, p → ε p, q → ε q};
```

```
((Series[ρ/a /. rulρ /. rul23 /. M → (λ - δ), {e, 0, 2}] // Normal // Expand // TrigExpand) /. rul25 /. rul28 /. e^2 → e2 /. sole2 /.
   rulξηpq // Series[#, {ε, 0, 2}] & // Normal) /. ε → 1 // Collect[#, {ξ, η}, TrigReduce[Expand[Simplify[#]]] &] &
```

$$1 + \frac{\xi^2(1 - \mathrm{Cos}[2\lambda])}{2\Lambda} + \frac{\eta^2(1 + \mathrm{Cos}[2\lambda])}{2\Lambda} + \frac{\eta\,\mathrm{Sin}[\lambda]}{\sqrt\Lambda} + \xi\left(-\frac{\mathrm{Cos}[\lambda]}{\sqrt\Lambda} + \frac{\eta\,\mathrm{Sin}[2\lambda]}{\Lambda}\right)$$

where we substitute $\lambda - \delta$ instead of mean anomaly $l \equiv M$.

First, we rewrite Cartesian coordinates in new variables

```
x11 = a (Sin[ω] (-Sqrt[1 - e^2] Cos[Ω] Sin[w] + Cos[i] (e - Cos[w]) Sin[Ω]) + Cos[ω] ((-e + Cos[w]) Cos[Ω] - Sqrt[1 - e^2] Cos[i] Sin[w] Sin[Ω]));
```

```
(((x11 / a /. {Ω → z, ω → -z + δ} // Expand // TrigExpand) /. rul23 /. M → λ - δ // Series[#, {e, 0, 2}] & // Normal // Expand //
      TrigExpand) /. rul25 /. rul28 /. e^2 → e2 /. sole2 /. rul26 /. rulξηpq // Series[#, {ε, 0, 2}] & // Normal) /.
   ε → 1 // Collect[#, {ξ, η, p, q}, Simplify] & // Expand // TrigExpand;
```

```
x1exp =
  ((% /. {p → a Cos[z], q → -a Sin[z]} // Simplify) // Expand // TrigExpand) /.
    {a^2 Cos[z]^2 → p^2, a^2 Sin[z]^2 → q^2, a^2 Cos[z] Sin[z] → -p q} /. a^2 → (p^2 + q^2) // TrigReduce // Expand
```

$$-\frac{3\xi}{2\sqrt\Lambda} + \mathrm{Cos}[\lambda] - \frac{q^2\mathrm{Cos}[\lambda]}{2\Lambda} - \frac{5\eta^2\mathrm{Cos}[\lambda]}{8\Lambda} - \frac{3\xi^2\mathrm{Cos}[\lambda]}{8\Lambda} + \frac{\xi\,\mathrm{Cos}[2\lambda]}{2\sqrt\Lambda} - \frac{3\eta^2\mathrm{Cos}[3\lambda]}{8\Lambda} + \frac{3\xi^2\mathrm{Cos}[3\lambda]}{8\Lambda} - \frac{p\,q\,\mathrm{Sin}[\lambda]}{2\Lambda} - \frac{\eta\,\xi\,\mathrm{Sin}[\lambda]}{4\Lambda} - \frac{\eta\,\mathrm{Sin}[2\lambda]}{2\sqrt\Lambda} - \frac{3\eta\,\xi\,\mathrm{Sin}[3\lambda]}{4\Lambda}$$

```
y11 = -a (Cos[i] Cos[Ω] (-Sqrt[1 - e^2] Cos[ω] Sin[w] + (e - Cos[w]) Sin[ω]) + ((e - Cos[w]) Cos[ω] + Sqrt[1 - e^2] Sin[w] Sin[ω]) Sin[Ω]);
```

```
(((y11 / a /. {Ω → z, ω → -z + δ} // Expand // TrigExpand) /. rul23 /. M → λ - δ // Series[#, {e, 0, 2}] & // Normal // Expand //
      TrigExpand) /. rul25 /. rul28 /. e^2 → e2 /. sole2 /. rul26 /. rulξηpq // Series[#, {ε, 0, 2}] & // Normal) /.
   ε → 1 // Collect[#, {ξ, η, p, q}, Simplify] & // Expand // TrigExpand;
```

```
y1exp =
  ((% /. {p → a Cos[z], q → -a Sin[z]} // Simplify) // Expand // TrigExpand) /.
    {a^2 Cos[z]^2 → p^2, a^2 Sin[z]^2 → q^2, a^2 Cos[z] Sin[z] → -p q} /. a^2 → (p^2 + q^2) // TrigReduce // Expand
```

$$\frac{3\eta}{2\sqrt\Lambda} - \frac{p\,q\,\mathrm{Cos}[\lambda]}{2\Lambda} - \frac{\eta\,\xi\,\mathrm{Cos}[\lambda]}{4\Lambda} + \frac{\eta\,\mathrm{Cos}[2\lambda]}{2\sqrt\Lambda} + \frac{3\eta\,\xi\,\mathrm{Cos}[3\lambda]}{4\Lambda} + \mathrm{Sin}[\lambda] - \frac{p^2\mathrm{Sin}[\lambda]}{2\Lambda} - \frac{3\eta^2\mathrm{Sin}[\lambda]}{8\Lambda} - \frac{5\xi^2\mathrm{Sin}[\lambda]}{8\Lambda} + \frac{\xi\,\mathrm{Sin}[2\lambda]}{2\sqrt\Lambda} - \frac{3\eta^2\mathrm{Sin}[3\lambda]}{8\Lambda} + \frac{3\xi^2\mathrm{Sin}[3\lambda]}{8\Lambda}$$

```
zz1 = Sin[i] (Sqrt[1 - e^2] Cos[ω] Sin[w] + (-e + Cos[w]) Sin[ω]);
```

```
z1exp =
  ((((zz1 /. {Ω → z, ω → -z + δ} // Expand // TrigExpand) /. rul23 /. M → λ - δ // Series[#, {e, 0, 2}] & // Normal // Expand //
        TrigExpand) /. rul25 /. rul28 /. e^2 → e2 /. sole2 /. rul26 /. rul27 /. rulξηpq // Series[#, {ε, 0, 2}] & //
      Normal) /. ε → 1 // Collect[#, {ξ, η, p, q}, Simplify] & // Expand // TrigExpand)
```

$$\frac{3p\eta}{2\Lambda} - \frac{3q\xi}{2\Lambda} + \frac{q\,\mathrm{Cos}[\lambda]}{\sqrt\Lambda} + \frac{p\eta\,\mathrm{Cos}[\lambda]^2}{2\Lambda} + \frac{q\xi\,\mathrm{Cos}[\lambda]^2}{2\Lambda} + \frac{p\,\mathrm{Sin}[\lambda]}{\sqrt\Lambda} - \frac{q\eta\,\mathrm{Cos}[\lambda]\mathrm{Sin}[\lambda]}{\Lambda} + \frac{p\xi\,\mathrm{Cos}[\lambda]\mathrm{Sin}[\lambda]}{\Lambda} - \frac{p\eta\,\mathrm{Sin}[\lambda]^2}{2\Lambda} - \frac{q\xi\,\mathrm{Sin}[\lambda]^2}{2\Lambda}$$

Now we can define the Cartesian coordinates of the bodies P_j in the form of power series in terms of small parameters up to the second order. To simplify the calculations we'll consider the case of three bodies and define relative coordinates of the bodies P_1 and P_2.

```
q1 = a_1 γ_1 (x1exp  // Collect[#, {ξ, η, p, q}, Simplify] &) /. {ξ → ξ_1, η → η_1, p → p_1, q → q_1, Λ → Λ_1, λ → λ_1}
```

$$a_1\gamma_1\Big(\mathrm{Cos}[\lambda_1] - \frac{\mathrm{Sin}[\lambda_1]p_1q_1}{2\Lambda_1} - \frac{\mathrm{Cos}[\lambda_1]q_1^2}{2\Lambda_1} - \frac{(5\mathrm{Cos}[\lambda_1] + 3\mathrm{Cos}[3\lambda_1])\eta_1^2}{8\Lambda_1} - \frac{\mathrm{Cos}[\lambda_1]\mathrm{Sin}[\lambda_1]\eta_1}{\sqrt{\Lambda_1}} + \left(-\frac{(\mathrm{Sin}[\lambda_1] + 3\mathrm{Sin}[3\lambda_1])\eta_1}{4\Lambda_1} + \frac{-3 + \mathrm{Cos}[2\lambda_1]}{2\sqrt{\Lambda_1}}\right)\xi_1 - \frac{3\mathrm{Cos}[\lambda_1]\mathrm{Sin}[\lambda_1]^2\xi_1^2}{2\Lambda_1}\Big)$$

```
q1p = a_2 γ_2 (x1exp  // Collect[#, {ξ, η, p, q}, Simplify] &) /. {ξ → ξ_2, η → η_2, p → p_2, q → q_2, Λ → Λ_2, λ → λ_2}
```

$$a_2\gamma_2\Big(\mathrm{Cos}[\lambda_2] - \frac{\mathrm{Sin}[\lambda_2]p_2q_2}{2\Lambda_2} - \frac{\mathrm{Cos}[\lambda_2]q_2^2}{2\Lambda_2} - \frac{(5\mathrm{Cos}[\lambda_2] + 3\mathrm{Cos}[3\lambda_2])\eta_2^2}{8\Lambda_2} - \frac{\mathrm{Cos}[\lambda_2]\mathrm{Sin}[\lambda_2]\eta_2}{\sqrt{\Lambda_2}} + \left(-\frac{(\mathrm{Sin}[\lambda_2] + 3\mathrm{Sin}[3\lambda_2])\eta_2}{4\Lambda_2} + \frac{-3 + \mathrm{Cos}[2\lambda_2]}{2\sqrt{\Lambda_2}}\right)\xi_2 - \frac{3\mathrm{Cos}[\lambda_2]\mathrm{Sin}[\lambda_2]^2\xi_2^2}{2\Lambda_2}\Big)$$

```
q2 = a_1 γ_1 (y1exp  // Collect[#, {ξ, η, p, q}, Simplify] &) /. {ξ → ξ_1, η → η_1, p → p_1, q → q_1, Λ → Λ_1, λ → λ_1}
```

$$a_1\gamma_1\Big(\mathrm{Sin}[\lambda_1] - \frac{\mathrm{Sin}[\lambda_1]p_1^2}{2\Lambda_1} - \frac{\mathrm{Cos}[\lambda_1]p_1q_1}{2\Lambda_1} - \frac{3\mathrm{Cos}[\lambda_1]^2\mathrm{Sin}[\lambda_1]\eta_1^2}{2\Lambda_1} + \frac{(3 + \mathrm{Cos}[2\lambda_1])\eta_1}{2\sqrt{\Lambda_1}} + \left(-\frac{(\mathrm{Cos}[\lambda_1] - 3\mathrm{Cos}[3\lambda_1])\eta_1}{4\Lambda_1} + \frac{\mathrm{Cos}[\lambda_1]\mathrm{Sin}[\lambda_1]}{\sqrt{\Lambda_1}}\right)\xi_1 + \frac{(-5\mathrm{Sin}[\lambda_1] + 3\mathrm{Sin}[3\lambda_1])\xi_1^2}{8\Lambda_1}\Big)$$

```
q2p = a_2 γ_2 (y1exp  // Collect[#, {ξ, η, p, q}, Simplify] &) /. {ξ → ξ_2, η → η_2, p → p_2, q → q_2, Λ → Λ_2, λ → λ_2}
```

$$a_2\gamma_2\Big(\mathrm{Sin}[\lambda_2] - \frac{\mathrm{Sin}[\lambda_2]p_2^2}{2\Lambda_2} - \frac{\mathrm{Cos}[\lambda_2]p_2q_2}{2\Lambda_2} - \frac{3\mathrm{Cos}[\lambda_2]^2\mathrm{Sin}[\lambda_2]\eta_2^2}{2\Lambda_2} + \frac{(3 + \mathrm{Cos}[2\lambda_2])\eta_2}{2\sqrt{\Lambda_2}} + \left(-\frac{(\mathrm{Cos}[\lambda_2] - 3\mathrm{Cos}[3\lambda_2])\eta_2}{4\Lambda_2} + \frac{\mathrm{Cos}[\lambda_2]\mathrm{Sin}[\lambda_2]}{\sqrt{\Lambda_2}}\right)\xi_2 + \frac{(-5\mathrm{Sin}[\lambda_2] + 3\mathrm{Sin}[3\lambda_2])\xi_2^2}{8\Lambda_2}\Big)$$

```
q3 = a_1 γ_1 (z1exp  // Collect[#, {ξ, η, p, q}, Simplify] &) /. {ξ → ξ_1, η → η_1, p → p_1, q → q_1, Λ → Λ_1, λ → λ_1}
```

$$a_1\gamma_1\left(\eta_1\left(\frac{(3 + \mathrm{Cos}[2\lambda_1])p_1}{2\Lambda_1} - \frac{\mathrm{Cos}[\lambda_1]\mathrm{Sin}[\lambda_1]q_1}{\Lambda_1}\right) + \frac{\mathrm{Sin}[\lambda_1]p_1}{\sqrt{\Lambda_1}} + \frac{\mathrm{Cos}[\lambda_1]q_1}{\sqrt{\Lambda_1}} + \left(\frac{\mathrm{Cos}[\lambda_1]\mathrm{Sin}[\lambda_1]p_1}{\Lambda_1} + \frac{(-3 + \mathrm{Cos}[2\lambda_1])q_1}{2\Lambda_1}\right)\xi_1\right)$$

```
q3p = a_2 γ_2 (z1exp  // Collect[#, {ξ, η, p, q}, Simplify] &) /. {ξ → ξ_2, η → η_2, p → p_2, q → q_2, Λ → Λ_2, λ → λ_2}
```

$$a_2\gamma_2\left(\eta_2\left(\frac{(3 + \mathrm{Cos}[2\lambda_2])p_2}{2\Lambda_2} - \frac{\mathrm{Cos}[\lambda_2]\mathrm{Sin}[\lambda_2]q_2}{\Lambda_2}\right) + \frac{\mathrm{Sin}[\lambda_2]p_2}{\sqrt{\Lambda_2}} + \frac{\mathrm{Cos}[\lambda_2]q_2}{\sqrt{\Lambda_2}} + \left(\frac{\mathrm{Cos}[\lambda_2]\mathrm{Sin}[\lambda_2]p_2}{\Lambda_2} + \frac{(-3 + \mathrm{Cos}[2\lambda_2])q_2}{2\Lambda_2}\right)\xi_2\right)$$

When we have all the coordinates we can calculate all terms of the Hamiltonian in the form of power series in terms of small parameters up to the second order.
To be able to expand the Hamiltonian in power series we add small multiplier ε to each coordinate and define the rule **rul31**.

```
rul31 = {ξ_k_ → ε ξ_k, η_k_ → ε η_k, p_k_ → ε p_k, q_k_ → ε q_k};
```

Then r_1^2 is represented in the form of power series in terms of small parameters up to the second order as

```
r1sqε2 = (q1^2 + q2^2 + q3^2 /. rul31 // Series[#, {ε, 0, 2}] & // Normal // Collect[#, {a_1, γ_1, ε}, Simplify] &) /. ε → 1
```

$$a_1^2\gamma_1^2\left(1 + \frac{2(\mathrm{Sin}[\lambda_1]\eta_1 - \mathrm{Cos}[\lambda_1]\xi_1)}{\sqrt{\Lambda_1}} + \frac{(3 + \mathrm{Cos}[2\lambda_1])\eta_1^2 + 2\mathrm{Sin}[2\lambda_1]\eta_1\xi_1 - (-3 + \mathrm{Cos}[2\lambda_1])\xi_1^2}{2\Lambda_1}\right)$$

The corresponding expression r_1 is represented as

```
r1aγ =
  a_1 γ_1 ((r1sqε2 / (a_1^2 γ_1^2))^(1/2) /. rul31 // Series[#, {ε, 0, 2}] & // Normal // Expand // TrigReduce // Expand //
      Collect[#, {a_1, γ_1, ε}, Simplify] &) /. ε → 1
```

$$a_1\gamma_1\left(1 + \frac{\mathrm{Sin}[\lambda_1]\eta_1 - \mathrm{Cos}[\lambda_1]\xi_1}{\sqrt{\Lambda_1}} + \frac{(\mathrm{Cos}[\lambda_1]\eta_1 + \mathrm{Sin}[\lambda_1]\xi_1)^2}{\Lambda_1}\right)$$

Then r_2^2 is represented in the form of power series in terms of small parameters as

```
r2sqε2 = (q1p^2 + q2p^2 + q3p^2 /. rul31 // Series[#, {ε, 0, 2}] & // Normal // Collect[#, {a_2, γ_2, ε}, Simplify] &) /. ε → 1
```

$$a_2^2\gamma_2^2\left(1 + \frac{2(\mathrm{Sin}[\lambda_2]\eta_2 - \mathrm{Cos}[\lambda_2]\xi_2)}{\sqrt{\Lambda_2}} + \frac{(3 + \mathrm{Cos}[2\lambda_2])\eta_2^2 + 2\mathrm{Sin}[2\lambda_2]\eta_2\xi_2 - (-3 + \mathrm{Cos}[2\lambda_2])\xi_2^2}{2\Lambda_2}\right)$$

The corresponding expression r_2 is represented as

```
r2aγ =
  a_2 γ_2 ((r2sqε2 / (a_2^2 γ_2^2))^(1/2) /. rul31 // Series[#, {ε, 0, 2}] & // Normal // Expand // TrigReduce //
      Collect[#, {a_2, γ_2, ε}, Simplify] &) /. ε → 1
```

$$a_2\gamma_2\left(1 + \frac{\mathrm{Sin}[\lambda_2]\eta_2 - \mathrm{Cos}[\lambda_2]\xi_2}{\sqrt{\Lambda_2}} + \frac{(\mathrm{Cos}[\lambda_2]\eta_2 + \mathrm{Sin}[\lambda_2]\xi_2)^2}{\Lambda_2}\right)$$

Remind that the disturbing functions are

$$W_j = -\frac{1}{2}b_j r_j^2 + U_j,\quad r_j^2 = x_j^2 + y_j^2 + z_j^2,\quad j = 1, 2, \ldots, n;$$

$$U_j = f\sum_{i=1\,(i\ne j)}^{n} m_i\left(\frac{1}{|\vec r_j - \vec r_i|} - \frac{\vec r_i \vec r_j}{r_i^3}\right),\quad b_j = b_j(t) = \frac{1}{\gamma_j}\frac{d^2\gamma_j}{dt^2}.$$

The expression $\vec r_1 \cdot \vec r_2$ is represented as

```
r1r2ε2 =
  (q1 q1p + q2 q2p + q3 q3p /. rul31 // Series[#, {ε, 0, 2}] & // Normal //
     Collect[#, {a_1, γ_1, a_2, γ_2, ε, η_1, η_2, ξ_1, ξ_2, q_1, q_2, p_1, p_2}, Simplify] &) /. ε → 1
```

$$a_1a_2\gamma_1\gamma_2\Big(\mathrm{Cos}[\lambda_1 - \lambda_2] - \frac{\mathrm{Sin}[\lambda_1]\mathrm{Sin}[\lambda_2]p_1^2}{2\Lambda_1} - \frac{\mathrm{Cos}[\lambda_1]\mathrm{Cos}[\lambda_2]q_1^2}{2\Lambda_1} - \frac{\mathrm{Cos}[\lambda_1](3\mathrm{Cos}[2\lambda_1 - \lambda_2] + \mathrm{Cos}[\lambda_2])\eta_1^2}{4\Lambda_1} + \frac{(-\mathrm{Sin}[2\lambda_1 - \lambda_2] + 3\mathrm{Sin}[\lambda_2])\eta_1}{2\sqrt{\Lambda_1}} +$$
$$q_2\left(-\frac{\mathrm{Sin}[\lambda_1 + \lambda_2]p_2}{2\Lambda_2} + \frac{\mathrm{Cos}[\lambda_2]\mathrm{Sin}[\lambda_1]p_1}{\sqrt{\Lambda_1}\sqrt{\Lambda_2}}\right) + q_1\left(-\frac{\mathrm{Sin}[\lambda_1 + \lambda_2]p_1}{2\Lambda_1} + \frac{\mathrm{Cos}[\lambda_1]\mathrm{Sin}[\lambda_2]p_2}{\sqrt{\Lambda_1}\sqrt{\Lambda_2}} + \frac{\mathrm{Cos}[\lambda_1]\mathrm{Cos}[\lambda_2]q_2}{\sqrt{\Lambda_1}\sqrt{\Lambda_2}}\right) -$$
$$\frac{\mathrm{Sin}[\lambda_1]\mathrm{Sin}[\lambda_2]p_2^2}{2\Lambda_2} - \frac{\mathrm{Cos}[\lambda_1]\mathrm{Cos}[\lambda_2]q_2^2}{2\Lambda_2} - \frac{(\mathrm{Cos}[\lambda_1] + 3\mathrm{Cos}[\lambda_1 - 2\lambda_2])\mathrm{Cos}[\lambda_2]\eta_2^2}{4\Lambda_2} + \frac{(3\mathrm{Sin}[\lambda_1] + \mathrm{Sin}[\lambda_1 - 2\lambda_2])\eta_2}{2\sqrt{\Lambda_2}} +$$
$$\frac{\mathrm{Sin}[\lambda_1]\mathrm{Sin}[\lambda_2]p_1p_2}{\sqrt{\Lambda_1}\sqrt{\Lambda_2}} + \frac{(\mathrm{Cos}[2\lambda_1 - \lambda_2] - 3\mathrm{Cos}[\lambda_2])\xi_1}{2\sqrt{\Lambda_1}} - \frac{\mathrm{Sin}[\lambda_1](3\mathrm{Sin}[2\lambda_1 - \lambda_2] + \mathrm{Sin}[\lambda_2])\xi_1^2}{4\Lambda_1} +$$
$$\frac{(-3\mathrm{Cos}[\lambda_1] + \mathrm{Cos}[\lambda_1 - 2\lambda_2])\xi_2}{2\sqrt{\Lambda_2}} + \frac{(9 - 3\mathrm{Cos}[2\lambda_1] + \mathrm{Cos}[2(\lambda_1 - \lambda_2)] - 3\mathrm{Cos}[2\lambda_2])\xi_1\xi_2}{4\sqrt{\Lambda_1}\sqrt{\Lambda_2}} - \frac{(\mathrm{Sin}[\lambda_1] - 3\mathrm{Sin}[\lambda_1 - 2\lambda_2])\mathrm{Sin}[\lambda_2]\xi_2^2}{4\Lambda_2} +$$
$$\eta_2\left(\frac{(3\mathrm{Sin}[2\lambda_1] + \mathrm{Sin}[2(\lambda_1 - \lambda_2)] + 3\mathrm{Sin}[2\lambda_2])\xi_1}{4\sqrt{\Lambda_1}\sqrt{\Lambda_2}} - \frac{(-3\mathrm{Sin}[\lambda_1 - 3\lambda_2] + \mathrm{Sin}[\lambda_1 + \lambda_2])\xi_2}{4\Lambda_2}\right) +$$
$$\eta_1\left(\frac{(9 + 3\mathrm{Cos}[2\lambda_1] + \mathrm{Cos}[2(\lambda_1 - \lambda_2)] + 3\mathrm{Cos}[2\lambda_2])\eta_2}{4\sqrt{\Lambda_1}\sqrt{\Lambda_2}} - \frac{(3\mathrm{Sin}[3\lambda_1 - \lambda_2] + \mathrm{Sin}[\lambda_1 + \lambda_2])\xi_1}{4\Lambda_1} - \frac{(-3\mathrm{Sin}[2\lambda_1] + \mathrm{Sin}[2(\lambda_1 - \lambda_2)] - 3\mathrm{Sin}[2\lambda_2])\xi_2}{4\sqrt{\Lambda_1}\sqrt{\Lambda_2}}\right)\Big)$$

The term $r_{12} = \sqrt{|\vec r_2 - \vec r_1|^2} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2}$ is represented as

```
R12sq = (r1sqε2 + r2sqε2 - 2 r1r2ε2 /. rul31 // Series[#, {ε, 0, 2}] & // Normal // Collect[#, {ε, a_1, γ_1, a_2, γ_2}, Simplify] &)
```

$$a_1^2\gamma_1^2 - 2\mathrm{Cos}[\lambda_1 - \lambda_2]a_1a_2\gamma_1\gamma_2 + a_2^2\gamma_2^2 + \varepsilon\Big(\frac{2a_1^2\gamma_1^2(\mathrm{Sin}[\lambda_1]\eta_1 - \mathrm{Cos}[\lambda_1]\xi_1)}{\sqrt{\Lambda_1}} + \frac{2a_2^2\gamma_2^2(\mathrm{Sin}[\lambda_2]\eta_2 - \mathrm{Cos}[\lambda_2]\xi_2)}{\sqrt{\Lambda_2}} + a_1a_2\gamma_1\gamma_2$$
$$\left(\frac{(\mathrm{Sin}[2\lambda_1 - \lambda_2] - 3\mathrm{Sin}[\lambda_2])\eta_1}{\sqrt{\Lambda_1}} - \frac{(3\mathrm{Sin}[\lambda_1] + \mathrm{Sin}[\lambda_1 - 2\lambda_2])\eta_2}{\sqrt{\Lambda_2}} - \frac{(\mathrm{Cos}[2\lambda_1 - \lambda_2] - 3\mathrm{Cos}[\lambda_2])\xi_1}{\sqrt{\Lambda_1}} - \frac{(-3\mathrm{Cos}[\lambda_1] + \mathrm{Cos}[\lambda_1 - 2\lambda_2])\xi_2}{\sqrt{\Lambda_2}}\right)\Big) +$$
$$\varepsilon^2\Big(\frac{a_1^2\gamma_1^2((3 + \mathrm{Cos}[2\lambda_1])\eta_1^2 + 2\mathrm{Sin}[2\lambda_1]\eta_1\xi_1 - (-3 + \mathrm{Cos}[2\lambda_1])\xi_1^2)}{2\Lambda_1} + \frac{a_2^2\gamma_2^2((3 + \mathrm{Cos}[2\lambda_2])\eta_2^2 + 2\mathrm{Sin}[2\lambda_2]\eta_2\xi_2 - (-3 + \mathrm{Cos}[2\lambda_2])\xi_2^2)}{2\Lambda_2} +$$
$$\frac{1}{2\Lambda_1\Lambda_2}a_1a_2\gamma_1\gamma_2\big(2\mathrm{Sin}[\lambda_1]\mathrm{Sin}[\lambda_2]p_2^2\Lambda_1 + 2\mathrm{Sin}[\lambda_1 + \lambda_2]p_2q_2\Lambda_1 + 2\mathrm{Cos}[\lambda_1]\mathrm{Cos}[\lambda_2]q_2^2\Lambda_1 +$$
$$(\mathrm{Cos}[\lambda_1] + 3\mathrm{Cos}[\lambda_1 - 2\lambda_2])\mathrm{Cos}[\lambda_2]\eta_2^2\Lambda_1 - 4\mathrm{Sin}[\lambda_1]\mathrm{Sin}[\lambda_2]p_1p_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} - 4\mathrm{Cos}[\lambda_2]\mathrm{Sin}[\lambda_1]p_1q_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} +$$
$$2q_1(-2\mathrm{Cos}[\lambda_1]\mathrm{Sin}[\lambda_2]p_2\sqrt{\Lambda_1} - 2\mathrm{Cos}[\lambda_1]\mathrm{Cos}[\lambda_2]q_2\sqrt{\Lambda_1} + \mathrm{Sin}[\lambda_1 + \lambda_2]p_1\sqrt{\Lambda_2})\sqrt{\Lambda_2} + 2\mathrm{Sin}[\lambda_1]\mathrm{Sin}[\lambda_2]p_1^2\Lambda_2 +$$
$$2\mathrm{Cos}[\lambda_1]\mathrm{Cos}[\lambda_2]q_1^2\Lambda_2 + \mathrm{Cos}[\lambda_1](3\mathrm{Cos}[2\lambda_1 - \lambda_2] + \mathrm{Cos}[\lambda_2])\eta_1^2\Lambda_2 - (3\mathrm{Sin}[2\lambda_1] + \mathrm{Sin}[2(\lambda_1 - \lambda_2)] + 3\mathrm{Sin}[2\lambda_2])\eta_2\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1 +$$
$$\mathrm{Sin}[\lambda_1](3\mathrm{Sin}[2\lambda_1 - \lambda_2] + \mathrm{Sin}[\lambda_2])\Lambda_2\xi_1^2 + (-3\mathrm{Sin}[\lambda_1 - 3\lambda_2] + \mathrm{Sin}[\lambda_1 + \lambda_2])\eta_2\Lambda_1\xi_2 -$$
$$(9 - 3\mathrm{Cos}[2\lambda_1] + \mathrm{Cos}[2(\lambda_1 - \lambda_2)] - 3\mathrm{Cos}[2\lambda_2])\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1\xi_2 + (\mathrm{Sin}[\lambda_1] - 3\mathrm{Sin}[\lambda_1 - 2\lambda_2])\mathrm{Sin}[\lambda_2]\Lambda_1\xi_2^2 -$$
$$\eta_1\sqrt{\Lambda_2}\big((9 + 3\mathrm{Cos}[2\lambda_1] + \mathrm{Cos}[2(\lambda_1 - \lambda_2)] + 3\mathrm{Cos}[2\lambda_2])\eta_2\sqrt{\Lambda_1} -$$
$$(3\mathrm{Sin}[3\lambda_1 - \lambda_2] + \mathrm{Sin}[\lambda_1 + \lambda_2])\sqrt{\Lambda_2}\xi_1 + (3\mathrm{Sin}[2\lambda_1] - \mathrm{Sin}[2(\lambda_1 - \lambda_2)] + 3\mathrm{Sin}[2\lambda_2])\sqrt{\Lambda_1}\xi_2\big)\big)\Big)$$

```
Coefficient[R12sq, ε, 0]
```

$$a_1^2\gamma_1^2 - 2\mathrm{Cos}[\lambda_1 - \lambda_2]a_1a_2\gamma_1\gamma_2 + a_2^2\gamma_2^2$$

Now we can calculate $\frac{1}{r_{12}}$.

```
R12Ser = (Series[1/Sqrt[R12sq] /. Coefficient[R12sq, ε, 0] → Δ0^2, {ε, 0, 2}] // Collect[#, {ε, Δ0}, Simplify[#, Δ0 > 0] &] &) /. ε → 1
```

$$\frac{1}{\Delta0} + \frac{1}{2\Delta0^3\sqrt{\Lambda_1}\sqrt{\Lambda_2}}\Big(-2a_1^2\gamma_1^2\sqrt{\Lambda_2}(\mathrm{Sin}[\lambda_1]\eta_1 - \mathrm{Cos}[\lambda_1]\xi_1) + 2a_2^2\gamma_2^2\sqrt{\Lambda_1}(-\mathrm{Sin}[\lambda_2]\eta_2 + \mathrm{Cos}[\lambda_2]\xi_2) +$$
$$a_1a_2\gamma_1\gamma_2\big((3\mathrm{Sin}[\lambda_1] + \mathrm{Sin}[\lambda_1 - 2\lambda_2])\eta_2\sqrt{\Lambda_1} + (-\mathrm{Sin}[2\lambda_1 - \lambda_2] + 3\mathrm{Sin}[\lambda_2])\eta_1\sqrt{\Lambda_2} +$$
$$\mathrm{Cos}[2\lambda_1 - \lambda_2]\sqrt{\Lambda_2}\xi_1 - 3\mathrm{Cos}[\lambda_2]\sqrt{\Lambda_2}\xi_1 - 3\mathrm{Cos}[\lambda_1]\sqrt{\Lambda_1}\xi_2 + \mathrm{Cos}[\lambda_1 - 2\lambda_2]\sqrt{\Lambda_1}\xi_2\big)\Big) + \frac{1}{8\Delta0^5\Lambda_1\Lambda_2}$$
$$3\Big(-2a_1^2\gamma_1^2\sqrt{\Lambda_2}(\mathrm{Sin}[\lambda_1]\eta_1 - \mathrm{Cos}[\lambda_1]\xi_1) + 2a_2^2\gamma_2^2\sqrt{\Lambda_1}(-\mathrm{Sin}[\lambda_2]\eta_2 + \mathrm{Cos}[\lambda_2]\xi_2) + a_1a_2\gamma_1\gamma_2\big((3\mathrm{Sin}[\lambda_1] + \mathrm{Sin}[\lambda_1 - 2\lambda_2])\eta_2\sqrt{\Lambda_1} +$$
$$(-\mathrm{Sin}[2\lambda_1 - \lambda_2] + 3\mathrm{Sin}[\lambda_2])\eta_1\sqrt{\Lambda_2} + \mathrm{Cos}[2\lambda_1 - \lambda_2]\sqrt{\Lambda_2}\xi_1 - 3\mathrm{Cos}[\lambda_2]\sqrt{\Lambda_2}\xi_1 - 3\mathrm{Cos}[\lambda_1]\sqrt{\Lambda_1}\xi_2 + \mathrm{Cos}[\lambda_1 - 2\lambda_2]\sqrt{\Lambda_1}\xi_2\big)\Big)^2 -$$
$$\frac{1}{8\Delta0^3\Lambda_1\Lambda_2}\Big(2a_1^2\gamma_1^2\Lambda_2((3 + \mathrm{Cos}[2\lambda_1])\eta_1^2 + 2\mathrm{Sin}[2\lambda_1]\eta_1\xi_1 - (-3 + \mathrm{Cos}[2\lambda_1])\xi_1^2) +$$
$$2a_2^2\gamma_2^2\Lambda_1((3 + \mathrm{Cos}[2\lambda_2])\eta_2^2 + 2\mathrm{Sin}[2\lambda_2]\eta_2\xi_2 - (-3 + \mathrm{Cos}[2\lambda_2])\xi_2^2) +$$
$$a_1a_2\gamma_1\gamma_2\big(4\mathrm{Sin}[\lambda_1]\mathrm{Sin}[\lambda_2]p_2^2\Lambda_1 + 4\mathrm{Cos}[\lambda_1]\mathrm{Cos}[\lambda_2]q_2^2\Lambda_1 + 3\mathrm{Cos}[\lambda_1 - 3\lambda_2]\eta_2^2\Lambda_1 + 4\mathrm{Cos}[\lambda_1 - \lambda_2]\eta_2^2\Lambda_1 + \mathrm{Cos}[\lambda_1 + \lambda_2]\eta_2^2\Lambda_1 +$$
$$4p_2(\mathrm{Sin}[\lambda_1 + \lambda_2]q_2\Lambda_1 - 2\mathrm{Sin}[\lambda_2](\mathrm{Sin}[\lambda_1]p_1 + \mathrm{Cos}[\lambda_1]q_1)\sqrt{\Lambda_1}\sqrt{\Lambda_2}) - 8\mathrm{Cos}[\lambda_2](\mathrm{Sin}[\lambda_1]p_1 + \mathrm{Cos}[\lambda_1]q_1)q_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} -$$
$$18\eta_1\eta_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} - 6\mathrm{Cos}[2\lambda_1]\eta_1\eta_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} - 2\mathrm{Cos}[2\lambda_1 - 2\lambda_2]\eta_1\eta_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} - 6\mathrm{Cos}[2\lambda_2]\eta_1\eta_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} + 2\mathrm{Cos}[\lambda_1 - \lambda_2]p_1^2\Lambda_2 -$$
$$2\mathrm{Cos}[\lambda_1 + \lambda_2]p_1^2\Lambda_2 + 4\mathrm{Sin}[\lambda_1 + \lambda_2]p_1q_1\Lambda_2 + 2\mathrm{Cos}[\lambda_1 - \lambda_2]q_1^2\Lambda_2 + 2\mathrm{Cos}[\lambda_1 + \lambda_2]q_1^2\Lambda_2 + 4\mathrm{Cos}[\lambda_1 - \lambda_2]\eta_1^2\Lambda_2 + 3\mathrm{Cos}[3\lambda_1 - \lambda_2]\eta_1^2\Lambda_2 +$$
$$\mathrm{Cos}[\lambda_1 + \lambda_2]\eta_1^2\Lambda_2 - 6\mathrm{Sin}[2\lambda_1]\eta_2\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1 - 2\mathrm{Sin}[2\lambda_1 - 2\lambda_2]\eta_2\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1 - 6\mathrm{Sin}[2\lambda_2]\eta_2\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1 + 6\mathrm{Sin}[3\lambda_1 - \lambda_2]\eta_1\Lambda_2\xi_1 +$$
$$2\mathrm{Sin}[\lambda_1 + \lambda_2]\eta_1\Lambda_2\xi_1 + 4\mathrm{Cos}[\lambda_1 - \lambda_2]\Lambda_2\xi_1^2 - 3\mathrm{Cos}[3\lambda_1 - \lambda_2]\Lambda_2\xi_1^2 - \mathrm{Cos}[\lambda_1 + \lambda_2]\Lambda_2\xi_1^2 - 6\mathrm{Sin}[\lambda_1 - 3\lambda_2]\eta_2\Lambda_1\xi_2 + 2\mathrm{Sin}[\lambda_1 + \lambda_2]\eta_2\Lambda_1\xi_2 -$$
$$6\mathrm{Sin}[2\lambda_1]\eta_1\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_2 + 2\mathrm{Sin}[2\lambda_1 - 2\lambda_2]\eta_1\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_2 - 6\mathrm{Sin}[2\lambda_2]\eta_1\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_2 - 18\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1\xi_2 + 6\mathrm{Cos}[2\lambda_1]\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1$$
$$\xi_2 - 2\mathrm{Cos}[2\lambda_1 - 2\lambda_2]\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1\xi_2 + 6\mathrm{Cos}[2\lambda_2]\sqrt{\Lambda_1}\sqrt{\Lambda_2}\xi_1\xi_2 - 3\mathrm{Cos}[\lambda_1 - 3\lambda_2]\Lambda_1\xi_2^2 + 4\mathrm{Cos}[\lambda_1 - \lambda_2]\Lambda_1\xi_2^2 - \mathrm{Cos}[\lambda_1 + \lambda_2]\Lambda_1\xi_2^2\big)\Big)$$

Evolution equations

To obtain the evolution equations we need to average the equations of motion written in terms of the Poincare variables over the "fast" variables λ_1, λ_2. Averaging of a function F over λ_1, λ_2 is given by

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} F\,d\lambda_1\,d\lambda_2.$$

Then the secular part of the perturbing function $R_{10}^{(\mathrm{sec})} = -\frac{\ddot\gamma_1}{2\gamma_1}(r_1^2)^{(\mathrm{sec})}$ is determined by the condition that all sine and cosine functions depending on λ_1 are set to zero

```
1/(4 π^2) Integrate[Integrate[#, {λ_2, 0, 2 π}], {λ_1, 0, 2 π}] & /@ r1sqε2
```

$$\frac{a_1^2\gamma_1^2(3\eta_1^2 + 2\Lambda_1 + 3\xi_1^2)}{2\Lambda_1}$$

$$R_{10}^{(\mathrm{sec})} = -\frac{\ddot\gamma_1}{2\gamma_1}r_1^2 = -\frac{\ddot\gamma_1}{2}a_1^2\gamma_1\left(1 + \frac{3}{2\Lambda_1}(\eta_1^2 + \xi_1^2)\right)$$

Then the secular part of the perturbing function $R_{20}^{(\mathrm{sec})} = -\frac{\ddot\gamma_2}{2\gamma_2}r_2^2$ is determined by the condition that all sine and cosine functions depending on λ_1 are set to zero

```
1/(4 π^2) Integrate[Integrate[#, {λ_2, 0, 2 π}], {λ_1, 0, 2 π}] & /@ r2sqε2
```

$$\frac{a_2^2\gamma_2^2(3\eta_2^2 + 2\Lambda_2 + 3\xi_2^2)}{2\Lambda_2}$$

$$R_{20}^{(\mathrm{sec})} = -\frac{\ddot\gamma_2}{2\gamma_2}r_2^2 = -\frac{\ddot\gamma_2}{2}a_2^2\gamma_2\left(1 + \frac{3}{2\Lambda_2}(\eta_2^2 + \xi_2^2)\right)$$

To compute the secular part of $\frac{1}{|\vec r_2 - \vec r_1|}$ we use the following Fourier series

$$\frac{1}{\Delta_0} = \frac{1}{2}\sum_{k=-\infty}^{\infty} A_k\cos(k(\lambda_1 - \lambda_2));\quad \frac{\gamma_1\gamma_2a_1a_2}{\Delta_0^3} = \frac{1}{2}\sum_{k=-\infty}^{\infty} B_k\cos(k(\lambda_1 - \lambda_2));\quad \frac{\gamma_1^2\gamma_2^2a_1^2a_2^2}{\Delta_0^5} = \frac{1}{2}\sum_{k=-\infty}^{\infty} C_k\cos(k(\lambda_1 - \lambda_2));$$

where A_k, B_k, C_k are known as the Laplace coefficients. As we do the calculations up to the second order in a small parameters it is sufficient to take into account only the terms for $k = 0, \pm1, \pm2 \pm3$. Then we obtain

```
R12Sexp =
  ((1/2 Coefficient[R12Ser, Δ0, -1] (A0_0 + 2 Sum[A0_i Cos[i (λ_1 - λ_2)], {i, 3}]) // Expand // TrigReduce //
       Collect[#, {A0_0, A0_1, A0_2, A0_3}, Simplify] &) +
    (1/(2 a_1 a_2 γ_1 γ_2) Coefficient[R12Ser, Δ0, -3] (B0_0 + 2 Sum[B0_i Cos[i (λ_1 - λ_2)], {i, 3}]) // Expand // TrigReduce  //
       Collect[#, {B0_0, B0_1, B0_2, B0_3}, Simplify] &) +
    (1/(2 a_1^2 a_2^2 γ_1^2 γ_2^2) Coefficient[R12Ser, Δ0, -5] (C0_0 + 2 Sum[C0_i Cos[i (λ_1 - λ_2)], {i, 3}]) // Expand // TrigReduce)) // Expand;
```

The corresponding expression is very cumbersome and we do not show it. Averaging the term R12Sexp, we obtain the secular part of the Hamiltonian $\frac{1}{|\vec r_2 - \vec r_1|}$.

```
R12Secular = ((1/(4 π^2) Integrate[Integrate[#, {λ_1, 0, 2 π}], {λ_2, 0, 2 π}] &) /@ R12Sexp) //
   Collect[#, {A0_0, B0_0, B0_1, B0_2, B0_3, C0_0, C0_1, C0_2, C0_3}, Simplify] &
```

$$\frac{A0_0}{2} + \frac{B0_2(\eta_1\eta_2 + \xi_1\xi_2)}{8\sqrt{\Lambda_1}\sqrt{\Lambda_2}} + \frac{3C0_3(\eta_1\eta_2 + \xi_1\xi_2)}{16\sqrt{\Lambda_1}\sqrt{\Lambda_2}} -$$
$$\frac{1}{8\Lambda_1\Lambda_2}B0_1(p_2^2\Lambda_1 + q_2^2\Lambda_1 + 2\eta_2^2\Lambda_1 - 2p_1p_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} - 2q_1q_2\sqrt{\Lambda_1}\sqrt{\Lambda_2} + p_1^2\Lambda_2 + q_1^2\Lambda_2 + 2\eta_1^2\Lambda_2 + 2\Lambda_2\xi_1^2 + 2\Lambda_1\xi_2^2) +$$
$$\frac{3}{16}C0_1\left(-\frac{4a_1\gamma_1(\eta_1^2 + \xi_1^2)}{a_2\gamma_2\Lambda_1} + \frac{7(\eta_1\eta_2 + \xi_1\xi_2)}{\sqrt{\Lambda_1}\sqrt{\Lambda_2}} - \frac{4a_2\gamma_2(\eta_2^2 + \xi_2^2)}{a_1\gamma_1\Lambda_2}\right) + \frac{3}{8}B0_0\left(-\frac{a_1\gamma_1(\eta_1^2 + \xi_1^2)}{a_2\gamma_2\Lambda_1} + \frac{3(\eta_1\eta_2 + \xi_1\xi_2)}{\sqrt{\Lambda_1}\sqrt{\Lambda_2}} - \frac{a_2\gamma_2(\eta_2^2 + \xi_2^2)}{a_1\gamma_1\Lambda_2}\right) +$$
$$\frac{1}{16\Lambda_1\Lambda_2}3C0_2\left(\frac{2a_1\gamma_1\sqrt{\Lambda_1}\sqrt{\Lambda_2}(\eta_1\eta_2 + \xi_1\xi_2)}{a_2\gamma_2} + \frac{2a_2\gamma_2\sqrt{\Lambda_1}\sqrt{\Lambda_2}(\eta_1\eta_2 + \xi_1\xi_2)}{a_1\gamma_1} - 3(\eta_2^2\Lambda_1 + \eta_1^2\Lambda_2 + \Lambda_2\xi_1^2 + \Lambda_1\xi_2^2)\right) +$$
$$\frac{1}{16a_1^2a_2^2\gamma_1^2\gamma_2^2\Lambda_1\Lambda_2}3C0_0\big(2a_1^4\gamma_1^4\Lambda_2(\eta_1^2 + \xi_1^2) - 6a_1^3a_2\gamma_1^3\gamma_2\sqrt{\Lambda_1}\sqrt{\Lambda_2}(\eta_1\eta_2 + \xi_1\xi_2) -$$
$$6a_1a_2^3\gamma_1\gamma_2^3\sqrt{\Lambda_1}\sqrt{\Lambda_2}(\eta_1\eta_2 + \xi_1\xi_2) + 2a_2^4\gamma_2^4\Lambda_1(\eta_2^2 + \xi_2^2) + 5a_1^2a_2^2\gamma_1^2\gamma_2^2(\eta_2^2\Lambda_1 + \eta_1^2\Lambda_2 + \Lambda_2\xi_1^2 + \Lambda_1\xi_2^2)\big)$$

Let us introduce a parameter $\alpha = \frac{a_1\gamma_1}{a_2\gamma_2} < 1$ if $a_1 < a_2$.

```
R12Sec = (R12Secular // Expand) /. γ_1 → α a_2 γ_2/a_1 // Collect[#, {ξ_1, η_1, ξ_2, η_2, p_1, q_1, p_2, q_2}, Simplify] &
```

$$\frac{A0_0}{2} - \frac{B0_1p_1^2}{8\Lambda_1} - \frac{B0_1q_1^2}{8\Lambda_1} - \frac{(6\alpha B0_0 + 4B0_1 - 15C0_0 - 6\alpha^2C0_0 + 12\alpha C0_1 + 9C0_2)\eta_1^2}{16\Lambda_1} - \frac{B0_1p_2^2}{8\Lambda_2} -$$
$$\frac{B0_1q_2^2}{8\Lambda_2} - \frac{(6\alpha B0_0 + 4\alpha^2B0_1 - 6C0_0 - 15\alpha^2C0_0 + 12\alpha C0_1 + 9\alpha^2C0_2)\eta_2^2}{16\alpha^2\Lambda_2} + \frac{B0_1p_1p_2}{4\sqrt{\Lambda_1}\sqrt{\Lambda_2}} + \frac{B0_1q_1q_2}{4\sqrt{\Lambda_1}\sqrt{\Lambda_2}} +$$
$$\frac{(18\alpha B0_0 + 2\alpha B0_2 + 3(-6(1 + \alpha^2)C0_0 + 7\alpha C0_1 + 2C0_2 + 2\alpha^2C0_2 + \alpha C0_3))\eta_1\eta_2}{16\alpha\sqrt{\Lambda_1}\sqrt{\Lambda_2}} - \frac{(6\alpha B0_0 + 4B0_1 - 15C0_0 - 6\alpha^2C0_0 + 12\alpha C0_1 + 9C0_2)\xi_1^2}{16\Lambda_1} +$$
$$\frac{(18\alpha B0_0 + 2\alpha B0_2 + 3(-6(1 + \alpha^2)C0_0 + 7\alpha C0_1 + 2C0_2 + 2\alpha^2C0_2 + \alpha C0_3))\xi_1\xi_2}{16\alpha\sqrt{\Lambda_1}\sqrt{\Lambda_2}} - \frac{(6\alpha B0_0 + 4\alpha^2B0_1 - 6C0_0 - 15\alpha^2C0_0 + 12\alpha C0_1 + 9\alpha^2C0_2)\xi_2^2}{16\alpha^2\Lambda_2}$$

One can readily check that secular parts of the perturbing function $\frac{\vec r_1\vec r_2}{r_1^3}$ and $\frac{\vec r_1\vec r_2}{r_2^3}$ are equal to zero.

```
U1Sec // Simplify
```

```
U1Sec = (r1r2ε2 / (r2aγ)^3 /. rul31 // Series[#, {ε, 0, 2}] & // Normal // Expand // TrigReduce // Expand) /. ε → 1;
```

```
((1/(4 π^2) Integrate[Integrate[#, {λ_1, 0, 2 π}], {λ_2, 0, 2 π}] &) /@ U1Sec)
```

$$0$$

```
U2Sec = (r1r2ε2 / (r1aγ)^3 /. rul31 // Series[#, {ε, 0, 2}] & // Normal // Expand // TrigReduce // Expand) /. ε → 1;
```

```
((1/(4 π^2) Integrate[Integrate[#, {λ_1, 0, 2 π}], {λ_2, 0, 2 π}] &) /@ U2Sec)
```

$$0$$

Let us introduce the following coefficients

```
Π11 = 2 Coefficient[R12Sec, ξ_1, 2]
```

$$-\frac{6\alpha B0_0 + 4B0_1 - 15C0_0 - 6\alpha^2C0_0 + 12\alpha C0_1 + 9C0_2}{8\Lambda_1}$$

```
Π22 = 2 Coefficient[R12Sec, ξ_2, 2] // Collect[#, {α}] & // Simplify
```

$$-\frac{6\alpha B0_0 + 4\alpha^2B0_1 - 6C0_0 - 15\alpha^2C0_0 + 12\alpha C0_1 + 9\alpha^2C0_2}{8\alpha^2\Lambda_2}$$

```
Π12  = 2 Coefficient[R12Sec, ξ_1 ξ_2] // Collect[#, {α}] & // Simplify
```

$$\frac{18\alpha B0_0 + 2\alpha B0_2 + 3(-6(1 + \alpha^2)C0_0 + 7\alpha C0_1 + 2C0_2 + 2\alpha^2C0_2 + \alpha C0_3)}{8\alpha\sqrt{\Lambda_1}\sqrt{\Lambda_2}}$$

```
Coefficient[R12Sec, p_1, 2]
```

$$-\frac{B0_1}{8\Lambda_1}$$

```
Coefficient[R12Sec, q_1, 2]
```

$$-\frac{B0_1}{8\Lambda_1}$$

```
Coefficient[R12Sec, p_2, 2]
```

$$-\frac{B0_1}{8\Lambda_2}$$

```
Coefficient[R12Sec, q_2, 2]
```

$$-\frac{B0_1}{8\Lambda_2}$$

```
Coefficient[R12Sec, p_1 p_2]
```

$$\frac{B0_1}{4\sqrt{\Lambda_1}\sqrt{\Lambda_2}}$$

```
Coefficient[R12Sec, q_1 q_2]
```

$$\frac{B0_1}{4\sqrt{\Lambda_1}\sqrt{\Lambda_2}}$$

Finally, we obtain the secular parts of the perturbing functions in the form

$$W_1^{(\mathrm{sec})} = \frac{f\,m_2}{2}\left(A0_0 + \Pi_{11}(\eta_1^2 + \xi_1^2) + \Pi_{22}(\eta_2^2 + \xi_2^2) + \Pi_{12}(\eta_1\eta_2 + \xi_1\xi_2) - \frac{B0_1}{2}\left(\frac{p_1^2 + q_1^2}{2\Lambda_1} + \frac{p_2^2 + q_2^2}{2\Lambda_2} - \frac{(p_1p_2 + q_1q_2)}{\sqrt{\Lambda_1\Lambda_2}}\right)\right)$$

$$W_2^{(\mathrm{sec})} = \frac{f\,m_1}{2}\left(A0_0 + \Pi_{11}(\eta_1^2 + \xi_1^2) + \Pi_{22}(\eta_2^2 + \xi_2^2) + \Pi_{12}(\eta_1\eta_2 + \xi_1\xi_2) - \frac{B0_1}{2}\left(\frac{p_1^2 + q_1^2}{2\Lambda_1} + \frac{p_2^2 + q_2^2}{2\Lambda_2} - \frac{(p_1p_2 + q_1q_2)}{\sqrt{\Lambda_1\Lambda_2}}\right)\right)$$

Laplace coefficients A1

Laplace coefficients $A1_0$ and $A0_0$ are given by

```
2/(π a_2 γ_2) Integrate[1/Sqrt[1 + α^2 - 2 α Cos[x]], {x, 0, π}, Assumptions → {0 < α < 1 && α ∈ Reals}] // Simplify // TraditionalForm
```

$$\frac{4K\left(\frac{4\alpha}{(\alpha+1)^2}\right)}{\pi a_2(\alpha + 1)\gamma_2}$$

Mathematica denotes the elliptic functions as

```
{EllipticK[x], EllipticE[x]} // TraditionalForm
```

$$\{K(x), E(x)\}$$

```
2/(π a_2 γ_2) Integrate[Cos[x]/Sqrt[1 + α^2 - 2 α Cos[x]], {x, 0, π}, Assumptions → {0 < α < 1 && α ∈ Reals}] // TraditionalForm
```

$$\frac{2\left((\alpha^2 + 1)K\left(\frac{4\alpha}{(\alpha+1)^2}\right) - (\alpha + 1)^2E\left(\frac{4\alpha}{(\alpha+1)^2}\right)\right)}{\pi a_2\alpha(\alpha + 1)\gamma_2}$$

Finally, the secular parts of the Hamiltonians may be written in the form

$$H_1^{(\mathrm{sec})} = -\frac{K_1}{2\gamma_1^2\Lambda_1^2} + \frac{1}{2}\gamma_1\ddot\gamma_1a_1^2\left(1 + \frac{3}{2\Lambda_1}(\xi_1^2 + \eta_1^2)\right) -$$
$$\frac{f\,m_2}{2}\left(A0_0 + \Pi_{11}(\eta_1^2 + \xi_1^2) + \Pi_{22}(\eta_2^2 + \xi_2^2) + \Pi_{12}(\eta_1\eta_2 + \xi_1\xi_2) - \frac{B0_1}{2}\left(\frac{p_1^2 + q_1^2}{2\Lambda_1} + \frac{p_2^2 + q_2^2}{2\Lambda_2} - \frac{(p_1p_2 + q_1q_2)}{\sqrt{\Lambda_1\Lambda_2}}\right)\right)$$

$$H_2^{(\mathrm{sec})} = -\frac{K_2}{2\gamma_2^2\Lambda_2^2} + \frac{1}{2}\gamma_2\ddot\gamma_2a_2^2\left(1 + \frac{3}{2\Lambda_2}(\xi_2^2 + \eta_2^2)\right) -$$
$$\frac{f\,m_1}{2}\left(A0_0 + \Pi_{11}(\eta_1^2 + \xi_1^2) + \Pi_{22}(\eta_2^2 + \xi_2^2) + \Pi_{12}(\eta_1\eta_2 + \xi_1\xi_2) - \frac{B0_1}{2}\left(\frac{p_1^2 + q_1^2}{2\Lambda_1} + \frac{p_2^2 + q_2^2}{2\Lambda_2} - \frac{(p_1p_2 + q_1q_2)}{\sqrt{\Lambda_1\Lambda_2}}\right)\right)$$

where

$$K_1 = f(m_{00} + m_{10}),\ K_2 = f(m_{00} + m_{20}),\ \Lambda_1 = \sqrt{K_1a_1},\ \Lambda_2 = \sqrt{K_2a_2},$$
$$\gamma_1(t) = \frac{m_{00} + m_{10}}{m_0(t) + m_1(t)},\ \gamma_2(t) = \frac{m_{00} + m_{20}}{m_0(t) + m_2(t)},$$

The Hamiltonian equations are given by

$$\dot\eta_j = \frac{\partial H_1^{(\mathrm{sec})}}{\partial\xi_j},\ \dot\xi_j = -\frac{\partial H_1^{(\mathrm{sec})}}{\partial\eta_j},\ \dot q_j = \frac{\partial H_1^{(\mathrm{sec})}}{\partial p_j},\ \dot p_j = -\frac{\partial H_1^{(\mathrm{sec})}}{\partial q_j},\ j = 1, 2.$$

As the Hamiltonians $H_j^{(\mathrm{sec})}$ do not depend on coordinates λ_j then the corresponding momenta $\Lambda_j = \mathrm{const}$, and the semi-major axes $a_j = \mathrm{const}$.
Evolution equations are given by

$$\dot\eta_1 = \frac{\partial H_1^{(\mathrm{sec})}}{\partial\xi_1} = \frac{3}{2\Lambda_1}\gamma_1\ddot\gamma_1a_1^2\xi_1 - \frac{f\,m_2}{2}(2\Pi_{11}\xi_1 + \Pi_{12}\xi_2),$$
$$\dot\xi_1 = -\frac{\partial H_1^{(\mathrm{sec})}}{\partial\eta_1} = -\frac{3}{2\Lambda_1}\gamma_1\ddot\gamma_1a_1^2\eta_1 + \frac{f\,m_2}{2}(2\Pi_{11}\eta_1 + \Pi_{12}\eta_2),$$
$$\dot\eta_2 = \frac{\partial H_2^{(\mathrm{sec})}}{\partial\xi_2} = \frac{3}{2\Lambda_2}\gamma_2\ddot\gamma_2a_2^2\xi_2 - \frac{f\,m_1}{2}(2\Pi_{22}\xi_2 + \Pi_{12}\xi_1),$$
$$\dot\xi_2 = -\frac{\partial H_2^{(\mathrm{sec})}}{\partial\eta_2} = -\frac{3}{2\Lambda_2}\gamma_2\ddot\gamma_2a_2^2\eta_2 + \frac{f\,m_1}{2}(2\Pi_{22}\eta_2 + \Pi_{12}\eta_1),$$
$$\dot q_1 = \frac{\partial H_1^{(\mathrm{sec})}}{\partial p_1} = \frac{f\,m_2}{2}\frac{B0_1}{2}\left(\frac{p_1}{\Lambda_1} - \frac{p_2}{\sqrt{\Lambda_1\Lambda_2}}\right),$$
$$\dot p_1 = -\frac{\partial H_1^{(\mathrm{sec})}}{\partial q_1} = -\frac{f\,m_2}{2}\frac{B0_1}{2}\left(\frac{q_1}{\Lambda_1} - \frac{q_2}{\sqrt{\Lambda_1\Lambda_2}}\right),$$
$$\dot q_2 = \frac{\partial H_2^{(\mathrm{sec})}}{\partial p_2} = \frac{f\,m_1}{2}\frac{B0_1}{2}\left(\frac{p_2}{\Lambda_2} - \frac{p_1}{\sqrt{\Lambda_1\Lambda_2}}\right),$$
$$\dot p_2 = -\frac{\partial H_2^{(\mathrm{sec})}}{\partial q_2} = -\frac{f\,m_1}{2}\frac{B0_1}{2}\left(\frac{q_2}{\Lambda_2} - \frac{q_1}{\sqrt{\Lambda_1\Lambda_2}}\right).$$

Let us introduce the following dimensionless variables: time t will be measured in the dimensionless units $t^* = \omega_0 t$, where $\omega_0 = \sqrt{f\,m_{00}/a_1^3}$. The distances will be measured in the dimensional units $r^* = r/a_1$. Masses - in the dimensionless mass $m^* = m/m_{00}$: $m_1 \to m_{00}m_1^*$, $m_2 \to m_{00}m_2^*$. Canonical variables ξ, η, p, q will get a multiplier $(G\,m_{00}\,a_1)^{1/4}$: $\xi \to (G\,m_{00}\,a_1)^{1/4}\xi^*$, $a_1^* \to 1$, $a_2 \to a_1a_2^*$. Then $\Lambda_1 \to \sqrt{G\,m_{00}\,a_1}\sqrt{1 + m_{10}^*}$, $\Lambda_2 \to \sqrt{G\,m_{00}\,a_1}\sqrt{(1 + m_{20}^*)a_2^*}$. We'll use the same notations without the sign "*"

As a result we obtain the equations of motion in the form

$$\dot\eta_1 = \frac{3}{2\sqrt{1 + m_{10}}}\gamma_1\ddot\gamma_1\xi_1 - \frac{m_2}{2\sqrt{1 + m_{10}}}(2\Pi_{11}\xi_1 + \Pi_{12}\xi_2),$$
$$\dot\xi_1 = -\frac{3}{2\sqrt{1 + m_{10}}}\gamma_1\ddot\gamma_1\eta_1 + \frac{m_2}{2\sqrt{1 + m_{10}}}(2\Pi_{11}\eta_1 + \Pi_{12}\eta_2),$$
$$\dot\eta_2 = \frac{3}{2\sqrt{(1 + m_{20})}}\gamma_2\ddot\gamma_2a_2^{3/2}\xi_2 - \frac{m_1}{2\sqrt{a_2}(1 + m_{20})}(2\Pi_{22}\xi_2 + \Pi_{12}\xi_1),$$
$$\dot\xi_2 = -\frac{3}{2\sqrt{(1 + m_{20})}}\gamma_2\ddot\gamma_2a_2^{3/2}\eta_2 + \frac{m_1}{2\sqrt{a_2}(1 + m_{20})}(2\Pi_{22}\eta_2 + \Pi_{12}\eta_1),$$
$$\dot q_1 = \frac{B0_1m_2}{4\sqrt{1 + m_{10}}}(p_1 - p_2),$$
$$\dot p_1 = -\frac{B0_1m_2}{4\sqrt{1 + m_{10}}}(q_1 - q_2),$$
$$\dot q_2 = \frac{B0_1m_1}{4\sqrt{a_2}(1 + m_{20})}(p_2 - p_1),$$
$$\dot p_2 = -\frac{B0_1m_1}{4\sqrt{a_2}(1 + m_{20})}(q_2 - q_1).$$

All variables here are dimensionless.

Conclusion

Summarizing all the results, one can make the following conclusion.

We have shown all kinds of symbolic computation to be performed to derive evolutionary equations determining the motion of nonstationary bodies in the central gravitational field.

We have obtained the evolution equations describing the secular perturbations of the bodies motion.

Next step: we'll analyze the evolution equations numerically for some chosen functions $m_1(t), m_2(t), \ldots, m_n(t)$.

Thank you very much for your attention!!!