Secular Perturbations in the Three-Body Problem with Variable Masses

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Abstract. *Secular perturbations in the three-body problem with variable masses changing at the same specific rate are considered. The analytical expressions of secular perturbations in the protoplanetary three-body problem with variable masses are obtained with the computer algebra system Mathematica.*

1 Introduction

Real space bodies are non-stationary ones. Their mass, size, shape and structure of the mass distribution within the body change in time [1]–[3]. These processes take place intensively in the binary and multiple systems [4]. In this connection we investigate the three-body problem with masses varying isotropic at the same specific rate. Bodies are assumed to be material points. Secular perturbations in the corresponding three-body problem with variable masses are investigated on the basis of perturbation theory based on aperiodic motion on quasiconic section [5].

2 Problem statement

Let's consider a system of three reciprocally gravitating bodies with masses m_0, m_1, m_2, changing isotropic at the same specific rate as

$$m_0 = \frac{m_{00}}{\varphi(t)}, \quad m_1 = \frac{m_{10}}{\varphi(t)}, \quad m_2 = \frac{m_{20}}{\varphi(t)}. \qquad (1)$$

Using Jacobi coordinates [5], [6], one can write equations of motion of the system in the form

$$\mu_1 \ddot{\vec{r}}_1 = \frac{1}{\varphi(t)} grad_{\vec{r}_1} U, \quad \mu_2 \ddot{\vec{r}}_2 = \frac{1}{\varphi(t)} grad_{\vec{r}_2} U. \qquad (2)$$

where

$$\mu_1 = \frac{m_{10} m_{00}}{m_{00} + m_{10}} = const, \quad \mu_2 = \frac{m_{20}(m_{10} + m_{00})}{m_{00} + m_{10} + m_{20}} = const. \qquad (3)$$

$$U = f\left(\frac{m_{00}m_{10}}{r_{01}} + \frac{m_{00}m_{20}}{r_{02}} + \frac{m_{10}m_{20}}{r_{12}}\right), \tag{4}$$

$$r_{01}^2 = x_1^2 + y_1^2 + z_1^2 = r_1^2, \tag{5}$$

$$r_{02}^2 = (x_2 + \nu_1 x_1)^2 + (y_2 + \nu_1 y_1)^2 + (z_2 + \nu_1 z_1)^2, \tag{6}$$

$$r_{12}^2 = (x_2 - \nu_0 x_1)^2 + (y_2 - \nu_0 y_1)^2 + (z_2 - \nu_0 z_1)^2, \tag{7}$$

$$\nu_1 = \frac{m_{10}}{m_{00} + m_{10}} = const, \quad \nu_0 = \frac{m_{00}}{m_{00} + m_{10}} = const, \tag{8}$$

and f is the gravitational constant.

Equations of motion (2) can be analyzed in the framework of the perturbation theory based on aperiodic motion on quasiconic section (see [5]). Introducing the analogue of the canonical Delaunay elements

$$L, \quad G, \quad H, \quad l, \quad g, \quad h, \tag{9}$$

we rewrite the equations of motion (2) in the form

$$\left.\begin{aligned} \dot{L}_i &= \frac{\partial R^*}{\partial l_i}, \quad \dot{G}_i = \frac{\partial R^*}{\partial g_i}, \quad \dot{H}_i = \frac{\partial R^*}{\partial h_i}, \\ \dot{l}_i &= -\frac{\partial R^*}{\partial L_i}, \quad \dot{g}_i = -\frac{\partial R^*}{\partial G_i}, \quad \dot{h}_i = -\frac{\partial R^*}{\partial H_i}, \quad (i = 1, 2) \end{aligned}\right\} \tag{10}$$

where the disturbing function is given by

$$R^* = \sum_{i=1}^{2} \frac{1}{2\varphi^2(t)}\left(\frac{\beta_i^4}{\mu_i L_i^2}\right) + R, \tag{11}$$

$$R = \frac{f}{\varphi(t)}\left[\frac{m_{00}m_{20}}{r_{02}} + \frac{m_{10}m_{20}}{r_{12}} - \frac{m_{20}(m_{00} + m_{10})}{r_2}\right] - \frac{1}{2}\frac{\ddot{\varphi}}{\varphi}[\mu_1 r_{01}^2 + \mu_2 r_{2}^2]. \tag{12}$$

Secular perturbations are determined by the equations (10), if the disturbing function (11) is averaged with respect to mean anomalies l_1, l_2 and is given by

$$R_{sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} R^* dl_1 dl_2. \tag{13}$$

Then we have

$$\dot{L}_1 = 0, \quad \dot{L}_2 = 0, \tag{14}$$

and, therefore, secular perturbations are determined by the following system of eight differential equations

$$\left.\begin{aligned} \dot{G}_i &= \frac{\partial R_{sec}}{\partial g_i}, \quad \dot{g}_i = -\frac{\partial R_{sec}}{\partial G_i}, \\ \dot{H}_i &= \frac{\partial R_{sec}}{\partial h_i}, \quad \dot{h}_i = -\frac{\partial R_{sec}}{\partial H_i}, \quad (i = 1, 2). \end{aligned}\right\} \tag{15}$$

Note that, as in the classical case, analogs of the second system of the Poincare elements [6]

$$\Lambda, \quad \lambda, \quad \xi, \quad \eta, \quad p, \quad q \tag{16}$$

are preferable in our problem [5]. They are defined as follows

$$\left.\begin{array}{ll} \Lambda = L, & \lambda = l + g + h, \\ \xi = \sqrt{2(L-G)}\cos(g+h), & \eta = -\sqrt{2(L-G)}\sin(g+h), \\ p = \sqrt{2(G-H)}\cos h, & q = -\sqrt{2(G-H)}\sin h. \end{array}\right\} \tag{17}$$

Then the secular perturbations are determined by the equations

$$\left.\begin{array}{ll} \dot{\xi}_i = \dfrac{\partial R_{\rm sec}}{\partial \eta_i}, & \dot{p}_i = \dfrac{\partial R_{\rm sec}}{\partial q_i}, \\ \dot{\eta}_i = -\dfrac{\partial R_{\rm sec}}{\partial \xi_i}, & \dot{q}_i = -\dfrac{\partial R_{\rm sec}}{\partial p_i}, \quad (i = 1, 2) \end{array}\right\} \tag{18}$$

where

$$R_{\rm sec} = \frac{1}{2\varphi^2(t)}\sum_{i=1}^{2}\left(\frac{\beta_i^4}{\mu_i L_i^2}\right) + \frac{f}{\varphi^2(t)}[F_{\rm sec}] - \frac{1}{2}\ddot{\varphi}\varphi[F_{\rho\,\rm sec}], \tag{19}$$

$$[F_{\rm sec}] = \left[\frac{m_{00}m_{20}}{\rho_{02}} + \frac{m_{10}m_{20}}{\rho_{12}} - \frac{m_{20}(m_{00}+m_{10})}{\rho_2}\right]_{\rm sec}, \tag{20}$$

$$[F_{\rho\,\rm sec}] = [\mu_1\rho_{01}^2 + \mu_2\rho_2^2]_{\rm sec}. \tag{21}$$

Thus, the problem is to calculate the secular perturbations in accordance with the equations (18)–(21).

3 Expansion of the disturbing function

As in the classical case, the problem of expanding the main part of the disturbing function (20)

$$[F_{\rm sec}] = [F_{main.part}] = \frac{m_{10}m_{20}}{\rho_{12}} \tag{22}$$

remains in our case, as well. In general, it is quite difficult and time-consuming work but it can be done successfully with computer algebra methods, as it was done in [7], for example. One can also expect successful application of computer algebra in our problem [8], that is supposed hereafter.

Note that the secular part of expression (21) has a simple form (see [5])

$$[F_{\rho\,\rm sec}] = \frac{\Lambda_1^4}{\mu_1}\left[1 + \frac{3}{2}\left(\frac{1}{\Lambda_1}(\xi_1^2+\eta_1^2)\right)\right] + \frac{\Lambda_2^4}{\mu_2}\left[1 + \frac{3}{2}\left(\frac{1}{\Lambda_2}(\xi_2^2+\eta_2^2)\right)\right]. \tag{23}$$

4 Secular perturbations in two-protoplanetary three-body problem with variable masses

In this paper we focus on considering two-protoplanetary three-body problem with variable masses, assuming that

$$m_{10} << m_{00}, \qquad m_{20} << m_{00}. \tag{24}$$

Let's assume also that the elements e_i, i_i are sufficiently small. Therefore, we can expand the disturbing function in power series in terms of small parameters $\xi/\sqrt{\Lambda}$, $\eta/\sqrt{\Lambda}$, m_{10}, m_{20} and truncate this series after the second-order terms. As a result we obtain a well-known classical formula (see [6])

$$[F_{sec}] = \frac{1}{2}k^2 m_{20} m_{10} A_0 + k^2 m_{20} m_{10} \left\{ \frac{1}{8} B_1 \left(\frac{\xi_1^2 + \eta_1^2}{\Lambda_1} + \frac{\xi_2^2 + \eta_2^2}{\Lambda_2} \right) - \right.$$

$$\left. -\frac{1}{4} B_2 \left(\frac{\xi_1 \xi_2}{\sqrt{\Lambda_1 \Lambda_2}} + \frac{\eta_1 \eta_2}{\sqrt{\Lambda_1 \Lambda_2}} \right) - \frac{1}{8} B_1 \left(\frac{p_1^2 + q_1^2}{\Lambda_1} + \frac{p_2^2 + q_2^2}{\Lambda_2} - \frac{2(p_1 p_2 + q_1 q_2)}{\sqrt{\Lambda_1 \Lambda_2}} \right) \right\}, \tag{25}$$

where $k^2 = f$ and all remaining symbols correspond to the notation of [6].

Introducing new variables according to the following relationships

$$\left.\begin{aligned} \xi_i &= \sqrt{\Lambda_i} e_i \cos \pi_i = \sqrt{\Lambda_i} r_i, \\ \eta_i &= -\sqrt{\Lambda_i} e_i \sin \pi_i = -\sqrt{\Lambda_i} s_i, \\ p_i &= \sqrt{\Lambda_i} \sin i_i \cos \Omega_i = \sqrt{\Lambda_i} u_i, \\ q_i &= -\sqrt{\Lambda_i} \sin i_i \sin \Omega_i = -\sqrt{\Lambda_i} v_i, \quad (i = 1, 2) \end{aligned}\right\} \tag{26}$$

we obtain

$$\left.\begin{aligned} \frac{dr_i}{dt} &= -\frac{1}{\Lambda_i} \frac{\partial [R_{sec}]}{\partial s_i}, \quad \frac{ds_i}{dt} = \frac{1}{\Lambda_i} \frac{\partial [R_{sec}]}{\partial r_i}, \\ \frac{du_i}{dt} &= -\frac{1}{\Lambda_i} \frac{\partial [R_{sec}]}{\partial v_i}, \quad \frac{dv_i}{dt} = \frac{1}{\Lambda_i} \frac{\partial [R_{sec}]}{\partial u_i}, \quad (i = 1, 2) \end{aligned}\right\} \tag{27}$$

$$[R_{sec}] = \frac{1}{2\varphi^2(t)} \sum_{i=1}^{2} \left(\frac{\beta_i^4}{\mu_i \Lambda_i} \right) + \frac{1}{\varphi^2(t)} [F_{sec}] - \frac{1}{2} \ddot{\varphi} \varphi [F_{\rho\, sec}], \tag{28}$$

$$[F_{sec}] = \frac{1}{2} k^2 m_{20} m_{00} A_0 + k^2 m_{20} m_{00} \left\{ \frac{1}{8} B_1 (r_1^2 + s_1^2 + r_2^2 + s_2^2) - \right.$$

$$\left. -\frac{1}{4} B_2 (r_1 r_2 + s_1 s_2) - \frac{1}{8} B_1 (u_1^2 + v_1^2 + u_2^2 + v_2^2) + \frac{1}{4} B_1 (u_1 u_2 + v_1 v_2) \right\}, \tag{29}$$

$$[F_{\rho\, sec}] = \frac{\Lambda_1^4}{\mu_1} \left[1 + \frac{3}{2} (r_1^2 + s_1^2) \right] + \frac{\Lambda_2^4}{\mu_2} \left[1 + \frac{3}{2} (r_2^2 + s_2^2) \right].$$

Using the following notations

$$\left.\begin{aligned} \aleph_1 &= \frac{1}{\varphi^2(t)} \frac{k^2 m_{20} m_{10}}{4\Lambda_1} B_1 - \frac{3}{2} \frac{\Lambda_1^3}{\mu_1} \varphi \ddot{\varphi}, \quad \aleph_2 = \frac{1}{\varphi^2(t)} \frac{k^2 m_{20} m_{10}}{4\Lambda_1} B_2, \\ \aleph_1' &= \frac{1}{\varphi^2(t)} \frac{k^2 m_{20} m_{10}}{4\Lambda_2} B_1 - \frac{3}{2} \frac{\Lambda_2^3}{\mu_2} \varphi \ddot{\varphi}, \quad \aleph_2' = \frac{1}{\varphi^2(t)} \frac{k^2 m_{20} m_{10}}{4\Lambda_2} B_2, \\ E_1 &= \frac{1}{\varphi^2(t)} \frac{k^2 m_{20} m_{10} B_1}{4\Lambda_1}, \qquad E_1' = \frac{1}{\varphi^2(t)} \frac{k^2 m_{20} m_{10} B_1}{4\Lambda_2}, \end{aligned}\right\} \tag{30}$$

we obtain two systems of differential equations following from the relations (27)–(30)

$$\left.\begin{aligned} \frac{dr_1}{dt} &= -\aleph_1 s_1 + \aleph_2 s_2, \\ \frac{ds_1}{dt} &= \aleph_1 r_1 - \aleph_2 r_2, \\ \frac{dr_2}{dt} &= -\aleph_1' s_2 + \aleph_2' s_1, \\ \frac{ds_2}{dt} &= \aleph_1' r_2 - \aleph_2' r_1, \end{aligned}\right\} \tag{31}$$

and

$$\left.\begin{aligned} \frac{du_1}{dt} &= E_1(v_1 - v_2), \\ \frac{dv_1}{dt} &= E_1(-u_1 + u_2), \\ \frac{du_2}{dt} &= E_1'(v_2 - v_1), \\ \frac{dv_2}{dt} &= E_1'(-u_2 + u_1). \end{aligned}\right\} \tag{32}$$

One can use various methods for solving the systems of equations (31)–(32), including approximate methods. Here we consider only one special case when the systems of differential equations (31)–(32) become autonomous and, hence, we can obtain exact solutions.

Let the following condition be satisfied

$$\ddot{\varphi}\varphi^3 = \alpha = const. \tag{33}$$

Differential equation (33) can be easily solved and its general solution is given by

$$\varphi = \varphi(t) = \sqrt{C_1 t^2 + 2C_2 t + C_3}, \quad C_1 C_3 - C_2^2 = \alpha \tag{34}$$

Introducing new independent variable τ in accordance with the relationship

$$\frac{d\tau}{dt} = \frac{1}{\varphi^2(t)}, \tag{35}$$

we can rewrite the equations (31)–(32) in the form

$$\left.\begin{aligned} \frac{dr_1}{d\tau} &= -\widetilde{\aleph}_1 s_1 + \widetilde{\aleph}_2 s_2, \\ \frac{ds_1}{d\tau} &= \widetilde{\aleph}_1 r_1 - \widetilde{\aleph}_2 r_2, \\ \frac{dr_2}{d\tau} &= -\widetilde{\aleph}_1' s_2 + \widetilde{\aleph}_2' s_1, \\ \frac{ds_2}{d\tau} &= \widetilde{\aleph}_1' r_2 - \widetilde{\aleph}_2' r_1, \end{aligned}\right\} \tag{36}$$

and

$$
\left.
\begin{aligned}
\frac{du_1}{d\tau} &= \widetilde{E}_1(v_1 - v_2), \\
\frac{dv_1}{d\tau} &= \widetilde{E}_1(-u_1 + u_2), \\
\frac{du_2}{d\tau} &= \widetilde{E}_1'(v_2 - v_1), \\
\frac{dv_2}{d\tau} &= \widetilde{E}_1'(-u_2 + u_1).
\end{aligned}
\right\}
\tag{37}
$$

Here the following notations are used

$$
\begin{gathered}
\widetilde{\aleph}_1 = \frac{k^2 m_{20} m_{10}}{4\Lambda_1} B_1 - \frac{3}{2}\frac{\Lambda_1^3}{\mu_1}\alpha, \quad \widetilde{\aleph}_2 = \frac{k^2 m_{20} m_{10}}{4\Lambda_1} B_2, \\
\widetilde{\aleph}_1' = \frac{k^2 m_{20} m_{10}}{4\Lambda_2} B_1 - \frac{3}{2}\frac{\Lambda_2^3}{\mu_2}\alpha, \quad \widetilde{\aleph}_2' = \frac{k^2 m_{20} m_{10}}{4\Lambda_2} B_2, \\
\widetilde{E}_1 = \frac{k^2 m_{20} m_{10} B_1}{4\Lambda_1}, \quad \widetilde{E}_1' = \frac{k^2 m_{20} m_{10} B_1}{4\Lambda_2}.
\end{gathered}
\tag{38}
$$

Note that solutions of differential equations (36)–(37) are well-known (see [6], for example) and are given by

$$
\left.
\begin{aligned}
r_1 &= N_1 \cos(\widetilde{g}_1\tau + \widetilde{\beta}_1) + N_2 \cos(\widetilde{g}_2\tau + \widetilde{\beta}_2), \\
s_1 &= N_1 \sin(\widetilde{g}_1\tau + \widetilde{\beta}_1) + N_2 \sin(\widetilde{g}_2\tau + \widetilde{\beta}_2), \\
r_2 &= N_1' \cos(\widetilde{g}_1\tau + \widetilde{\beta}_1) + N_2' \cos(\widetilde{g}_2\tau + \widetilde{\beta}_2), \\
s_2 &= N_1' \sin(\widetilde{g}_1\tau + \widetilde{\beta}_1) + N_2' \sin(\widetilde{g}_2\tau + \widetilde{\beta}_2),
\end{aligned}
\right\}
\tag{39}
$$

and

$$
\left.
\begin{aligned}
&\widetilde{E}_1' u_1 + \widetilde{E}_1 u_2 = b_1, \\
&\widetilde{E}_1' v_1 + \widetilde{E}_1 v_2 = b_2, \\
&\widetilde{E}_1'(u_1^2 + v_1^2) + \widetilde{E}_1(u_2^2 + v_2^2) = b_3,
\end{aligned}
\right\}
\tag{40}
$$

where N_i, N_i', $\widetilde{\beta}_i$, b_i are constants of integration, $\widetilde{g}_i$ are positive real roots of the characteristic equation of the system (36), and new independent variable τ is given by

$$
1)\ \tau = \frac{1}{\sqrt{C_1C_3 - C_2^2}} \operatorname{arctg} \frac{C_1 t + C_2}{\sqrt{C_1C_3 - C_2^2}}, \quad C_1C_3 > C_2^2, \tag{41}
$$

$$
2)\ \tau = \frac{1}{2\sqrt{C_2^2 - C_1C_3}} \ln \left| \frac{C_1 t + C_2 - \sqrt{C_2^2 - C_1C_3}}{C_1 t + C_2 + \sqrt{C_2^2 - C_1C_3}} \right|, \quad C_2^2 > C_1C_3, \tag{42}
$$

$$
3)\ \tau = -\frac{1}{(C_1 t + C_2)C_1}, \quad C_2^2 = C_1C_3. \tag{43}
$$

Solutions (39) make possible to analyze the motion of pericentre longitude π and change of eccentricity e depending on the law of mass variation (1), (34). In particular, we obtain

$$
e_1^2 = N_1^2 + N_2^2 + 2N_1N_2 \cos[(\widetilde{g}_1 - \widetilde{g}_2)\tau(t) + \widetilde{\beta}_1 - \widetilde{\beta}_2]. \tag{44}
$$

Well-known relations follow from the integrals (40) [5], [6]:

$$\Omega_1 = \Omega_2 + 180^0, \tag{45}$$

$$\frac{d\Omega_1}{d\tau} = -(\tilde{E}_1 + \tilde{E}_1'). \tag{46}$$

Taking into account the system (35), the last equation can be rewritten as

$$\frac{d\Omega_1}{dt} = -\frac{(\tilde{E}_1 + \tilde{E}_1')}{\varphi^2(t)}. \tag{47}$$

Thus, in our case a motion of the line of nodes has a variable speed in contrast to the classical problem [6]. System (37) is solved separately from the equations (36). Therefore, in (47) $\varphi^2(t)$ can be treated as arbitrary function characterizing law of mass variation (1).

5 Conclusion

Analysis of the secular perturbations in the considered problem shows that the effects of mass changes influence significantly on the change of orbital elements. In this regard, it is interesting to study the secular perturbations in general case, when $m_0(t)$, $m_1(t)$ and $m_2(t)$ are comparable in magnitude and vary in different tempos. Note that symbolic calculations are very bulky in this case and can be effectively done only with some computer algebra system, for example, the system Mathematica [8].

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