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Mathematics in Computer Science

# Computing Perturbations in the Two-Planetary Three-Body Problem with Masses Varying Non-isotropically at Different Rates

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**Abstract** The classical problem of three bodies of variable masses is considered in the case when two of the bodies are protoplanets and all the masses vary non-isotropically at different rates. The problem is analyzed in the framework of the planetary perturbation theory in terms of the osculating elements of aperiodic motion on quasi-conic sections. An algorithm for symbolic computation of the disturbing function and its expansion into power series in terms of the eccentricities and inclinations is discussed in detail. Differential equations describing the long-term evolution of the orbital parameters are derived in the form of Lagrange's planetary equations. All the relevant calculations are done with the computer algebra system Wolfram Mathematica.

**Keywords** Three-body problem · Variable masses · Protoplanets · Perturbations · Wolfram Mathematica

**Mathematics Subject Classification** Primary 70F07; Secondary 68W30

## 1 Introduction

The three-body problem is one of the most important models of celestial mechanics, and it has numerous applications (e.g., see [1,2]). Recall that it describes the dynamical behaviour of three bodies $P_0$, $P_1$, $P_2$ of masses $m_0, m_1, m_2$, respectively, under their mutual gravitational attraction. Such a model provides good approximation for the motion of two planets around the Sun, a satellite motion in the Sun-planet and double star systems (e.g., see [3–5]). Using Newton's second law, one can easily write out the differential equations of motion but their general solution cannot be obtained in symbolic form even in the simplest case when the bodies are assumed to be point particles of constant

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mass. This stimulates development of different approximate analytic and analytic-numerical methods for studying the system motion (see [3,6–11]). Such methods can be also applied to investigation of more complicated models when some parameters of the system change with time (see [12–14]).

Discovery of the exoplanetary systems [15] revived an interest to the two-planetary three-body problem. Since parameters of real-life celestial bodies such as mass, size, shape, and internal constitution vary with time (see [16–20]), it is of interest to investigate the effect of these changes on the dynamical evolution of a system. In the case of isotropical mass variation the three-body problem was investigated in [21–23]. It was shown in particular that such changes of mass can modify substantially the long-term evolution of the orbital parameters in comparison to the case of constant masses but they do not practically affect the spatial orientation of the orbital planes of the bodies.

In the present paper we study a general case of the two-planetary system when the bodies are assumed to be spherical with a spherically symmetric mass distribution but their masses vary non-isotropically with different rates and due to this the reactive forces arise. Performing quite cumbersome symbolic calculations, we obtain the disturbing functions in the form of power series in terms of the eccentricities and inclinations up to the second order and derive the differential equations determining behaviour of the orbital parameters in the form of Lagrange's planetary equations (see [2,3]). Averaging the equations of motion over the mean anomalies of the bodies in the absence of a mean-motion resonances, we obtain the differential equations describing the evolution of orbital parameters over long periods of time. These equations are rather complicated and their analysis requires to use modern computer algebra systems. Here all the relevant computations are performed with the computer algebra system Wolfram Mathematica (see [24]).

## 2 Equations of Motion

Let us consider an exoplanetary system consisting of a central body $P_0$ of mass $m_0(t)$ and two planets $P_1$ and $P_2$ of masses $m_1(t)$ and $m_2(t)$, respectively, mutually attracting each other according to Newton's universal law of gravitation. Masses of the bodies are assumed to vary non-isotropically at different rates, and

$$\frac{\dot{m}_0}{m_0} \neq \frac{\dot{m}_1}{m_1} \neq \frac{\dot{m}_2}{m_2},$$

where the dot over a symbol denotes a total derivative of the corresponding function with respect to time.

Using a relative coordinate system with the origin located at the center of the body $P_0$, we can write out the equations of motion of the bodies $P_1$ and $P_2$ in the form [19,25,26]

$$\ddot{\vec{r}}_1 + G\,(m_0 + m_1)\,\frac{\vec{r}_1}{r_1^3} - \frac{\ddot{\gamma}_1}{\gamma_1}\vec{r}_1 = grad_{\vec{r}_1} W_1, \tag{2.1}$$

$$\ddot{\vec{r}}_2 + G\,(m_0 + m_2)\,\frac{\vec{r}_2}{r_2^3} - \frac{\ddot{\gamma}_2}{\gamma_2}\vec{r}_2 = grad_{\vec{r}_2} W_2, \tag{2.2}$$

where $\vec{r}_j = (x_j, y_j, z_j)$, $j = 1, 2$ are the position vectors of the bodies $P_1$ and $P_2$ with respect to the body $P_0$, $G$ is the constant of gravitation. The functions $W_1$, $W_2$ in the right-hand side of (2.1), (2.2) may be represented in the form

$$W_j = U_j + \vec{F}_j \cdot \vec{r}_j - \frac{\ddot{\gamma}_j}{2\gamma_j} r_j^2, \tag{2.3}$$

and are called the disturbing functions. The force functions $U_1$, $U_2$ in (2.3) are determined by the mutual attraction of the bodies $P_1$, $P_2$ and are given by

$$U_1 = \mu_2 \left( \frac{1}{r_{12}} - \frac{\vec{r}_1 \cdot \vec{r}_2}{r_2^3} \right), \qquad U_2 = \mu_1 \left( \frac{1}{r_{12}} - \frac{\vec{r}_1 \cdot \vec{r}_2}{r_1^3} \right). \tag{2.4}$$

Here the following notations are used

$$r_j = |\vec{r}_j| = \left(x_j^2 + y_j^2 + z_j^2\right)^{1/2}, \quad r_{12} = |\vec{r}_2 - \vec{r}_1|,$$
$$\gamma_j = \frac{m_0(t_0) + m_j(t_0)}{m_0(t) + m_j(t)} = \gamma_j(t), \quad \gamma_j(t_0) = 1, \quad \mu_j = Gm_j,$$

where $t_0$ is the initial point in time. The reactive forces $\vec{F}_j(t) = \left(F_{jx}, F_{jy}, F_{jz}\right)$ arising due to non-isotropic variation of masses of the bodies may be represented in the form

$$\vec{F}_1 = \vec{F}_1(t) = \frac{\dot{m}_1}{m_1}\vec{V}_1 - \frac{\dot{m}_0}{m_0}\vec{V}_0, \qquad \vec{F}_2 = \vec{F}_2(t) = \frac{\dot{m}_2}{m_2}\vec{V}_2 - \frac{\dot{m}_0}{m_0}\vec{V}_0,$$

where $\vec{V}_j$, $j = 0, 1, 2$, denotes relative velocity of the particles leaving the body $P_j$. The masses $m_j(t)$ and velocities $\vec{V}_j$ are obtained from the observation of motion of celestial bodies and so the forces $\vec{F}_j$ are assumed to be given functions of time.

Note that Eqs. (2.1) and (2.2) are not integrable even in the case of constant masses of the bodies. So we apply the perturbation theory to investigate the dynamics of the system. One can readily check that in case of $W_1 = 0$, $W_2 = 0$ Eqs. (2.1) and (2.2) become independent and each of them has an exact solution for an arbitrary twice differentiable continuous function $\gamma_j(t) > 0$ (see [19]). These solutions may be represented in the form

$$x_j = \gamma_j\rho_j(\cos(f_j + \omega_j)\cos\Omega_j - \sin(f_j + \omega_j)\sin\Omega_j\cos i_j),$$
$$y_j = \gamma_j\rho_j(\cos(f_j + \omega_j)\sin\Omega_j + \sin(f_j + \omega_j)\cos\Omega_j\cos i_j),$$
$$z_j = \gamma_j\rho_j\sin(f_j + \omega_j)\sin i_j, \tag{2.5}$$

where parameters $i_j, \omega_j, \Omega_j$ are some constants determined from the initial conditions of the motion. The variables $\rho_j$ and $f_j$ define a conic section that is given by the polar equation

$$\rho_j = \frac{a_j(1 - e_j^2)}{1 + e_j\cos f_j}, \quad (j = 1, 2). \tag{2.6}$$

It is an ellipse of eccentricity $e_j$ and semi-major axis $a_j$ in case of $0 < e_j < 1$, and the true anomaly $f_j$ is defined by the equation

$$\int_0^{f_j} \frac{df_j}{(1 + e_j\cos f_j)^2} = \frac{1}{(1 - e_j^2)^{3/2}}(E_j - e_j\sin E_j)$$
$$= \frac{M_j}{(1 - e_j^2)^{3/2}} = \frac{\sqrt{K_{j0}}}{a_j^{3/2}(1 - e_j^2)^{3/2}}(\Phi_j(t) - \Phi_j(\tau_j)). \tag{2.7}$$

Here parameter $\tau_j$ corresponds to the time of perihelion passage,

$$\Phi_j(t) = \int_0^t \frac{dt}{\gamma_j^2(t)}, \quad K_{j0} = G(m_0(t_0) + m_j(t_0)), \quad (j = 1, 2), \tag{2.8}$$

$M_j = E_j - e_j\sin E_j$ is the mean anomaly, and the eccentric anomaly $E_j$ is determined from the equation

$$\tan\frac{f_j}{2} = \sqrt{\frac{1 + e_j}{1 - e_j}}\,\tan\frac{E_j}{2}. \tag{2.9}$$

One can readily see that in case of constant masses of the bodies when $\gamma_j(t) \equiv 1$ Eqs. (2.5)–(2.9) define a well-known solution to the two-body problem which describes a motion of the bodies $P_1$, $P_2$ around the body

$P_0$ on conic sections (see [2,3]). Consequently, parameters $a_j, e_j, i_j, \omega_j, \Omega_j$ correspond to the Keplerian orbital elements such as the semi-major axis, the eccentricity, the inclination, the longitude of pericenter and the longitude of the ascending node. The presence of a time-dependent factor $\gamma_j(t)$ in the right-hand side of the expressions (2.5) for the Cartesian coordinates $x_j, y_j, z_j$ results in deviation of the bodies trajectories from the conic sections while the orbital parameters $a_j, e_j, i_j, \omega_j, \Omega_j$ remain constant. Besides, the motion of the bodies is not periodic and the mean anomaly $M_j$ becomes non-linear function of time depending on $\gamma_j(t)$ [see (2.7), (2.8)]. For that reason, the solution (2.5) is said to describe aperiodic motion of the bodies on quasi-conic sections and the constants $a_j, e_j, i_j, \omega_j, \Omega_j, \tau_j$ are called analogs of the Keplerian orbital elements (see [19]).

In the case of $W_1 \neq 0$, $W_2 \neq 0$ we can also seek the solutions to Eqs. (2.1) and (2.2) in the form (2.5) but the orbital parameters $a_j, e_j, i_j, \omega_j, \Omega_j, \tau_j$ are now considered to be variables. Such approach to solving differential equations is often used in celestial mechanics and is known as the method of the variation of parameters (e.g., see [2]). To obtain the differential equations determining behaviour of the orbital parameters one should substitute the solution (2.5) into Eqs. (2.1) and (2.2) and resolve them with respect to the time derivatives of the orbital parameters. However, realization of this approach involves very bulky symbolic computations; so it is much more convenient to rewrite the equations of motion (2.1) and (2.2) in the Hamiltonian form and to change to the new canonical variables known as the Delaunay elements which are defined by (see [2,3,7])

$$\begin{aligned} &l_j = M_j, \quad L_j = \sqrt{K_{j0} a_j}, \quad g_j = \omega_j, \quad G_j = \sqrt{K_{j0} a_j (1 - e_j^2)}, \\ &h_j = \Omega_j, \quad H_j = \sqrt{K_{j0} a_j (1 - e_j^2)} \cos i_j, \quad (j = 1, 2), \end{aligned} \tag{2.10}$$

where $l_j, g_j, h_j$ are the coordinates and $L_j, G_j, H_j$ are the corresponding conjugate momenta. The corresponding Hamiltonian is given by

$$\mathcal{H}_j = -\frac{K_{j0}^2}{2\gamma_j^2 L_j^2} - W_j, \quad (j = 1, 2), \tag{2.11}$$

where the disturbing functions $W_1$, $W_2$ defined in (2.3) are expressed in terms of the Delaunay variables. Using (2.11), one can define the equations of motion in the Hamiltonian form

$$\begin{aligned} &\frac{dl_j}{dt} = \frac{\partial \mathcal{H}_j}{\partial L_j} = \frac{K_{j0}^2}{\gamma_j^2 \Lambda_j^3} - \frac{\partial W_j}{\partial L_j}, \quad \frac{dL_j}{dt} = -\frac{\partial \mathcal{H}_j}{\partial l_j} = \frac{\partial W_j}{\partial l_j}, \\ &\frac{dg_j}{dt} = \frac{\partial \mathcal{H}_j}{\partial G_j} = -\frac{\partial W_j}{\partial G_j}, \quad \frac{dG_j}{dt} = -\frac{\partial \mathcal{H}_j}{\partial g_j} = \frac{\partial W_j}{\partial g_j} \\ &\frac{h_j}{dt} = \frac{\partial \mathcal{H}_j}{\partial H_j} = -\frac{\partial W_j}{\partial H_j}, \quad \frac{dH_j}{dt} = -\frac{\partial \mathcal{H}_j}{\partial h_j} = \frac{\partial W_j}{\partial h_j}. \end{aligned} \tag{2.12}$$

Taking into account the expressions (2.10) and resolving the system (2.12) with respect to the derivatives of the orbital elements, we obtain the following Lagrange's planetary equations:

$$\begin{aligned} &\frac{da_j}{dt} = \frac{2}{n_j a_j} \frac{\partial W_j}{\partial M_j}, \\ &\frac{de_j}{dt} = \frac{1}{n_j a_j^2 e_j} \left( \left(1 - e_j^2\right) \frac{\partial W_j}{\partial M_j} - \sqrt{1 - e_j^2} \frac{\partial W_j}{\partial \omega_j} \right), \\ &\frac{d\omega_j}{dt} = \frac{\sqrt{1 - e_j^2}}{n_j a_j^2 e_j} \frac{\partial W_j}{\partial e_j} - \frac{\cot i_j}{n_j a_j^2 \sqrt{1 - e_j^2}} \frac{\partial W_j}{\partial i_j}, \\ &\frac{d\Omega_j}{dt} = \frac{1}{n_j a_j^2 \sqrt{1 - e_j^2} \sin i_j} \frac{\partial W_j}{\partial i_j}, \end{aligned}$$

$$\frac{di_j}{dt} = \frac{1}{n_j a_j^2 \sqrt{1-e_j^2}} \left( \cot i_j \frac{\partial W_j}{\partial \omega_j} - \frac{1}{\sin i_j} \frac{\partial W_j}{\partial \Omega_j} \right),$$

$$\frac{dM_j}{dt} = \frac{n_j}{\gamma_j^2(t)} - \frac{2}{n_j a_j} \frac{\partial W_j}{\partial a_j} - \frac{1-e_j^2}{n_j a_j^2 e_j} \frac{\partial W_j}{\partial e_j}, \quad (j = 1, 2), \tag{2.13}$$

where $n_j = \sqrt{K_{j0}}/a_j^{3/2}$ is called the mean motion.

## 3 Computation of the Disturbing Functions

To compute the partial derivatives of the disturbing functions $W_j$ and to write out the Eq. (2.13) in the explicit form we need to express the disturbing functions $W_1, W_2$ in terms of the orbital elements $a_j, e_j, i_j, \omega_j, \Omega_j, M_j$. To simplify the calculations we consider here the case of small eccentricities $e_j \ll 1$ and inclinations $i_j \ll 1$ of the orbits which is often met in celestial mechanics. Then the Kepler equation $M_j = E_j - e \sin E_j$ may be resolved with respect to the eccentric anomaly $E_j$ which is represented in the form of power series in terms of $e_j$ that is rapidly converges for small values of $e_j$ (see [2,3]). The corresponding series accurate up to the second order in $e_j$ is

$$E_j = M_j + e_j \sin M_j + \frac{e_j^2}{2} \sin(2M_j) + \cdots . \tag{3.1}$$

Using the series solution (3.1) and the built-in Mathematica function *Series* (see [24]), we can obtain the following expansions:

$$\cos E_j = \cos M_j + \frac{e_j}{2} \left(\cos(2M_j) - 1\right) + \frac{3e_j^2}{8} \left(\cos(3M_j) - \cos M_j\right) + \cdots ,$$

$$\sin E_j = \sin M_j + \frac{e_j}{2} \sin(2M_j) + \frac{e_j^2}{8} \left(3\sin(3M_j) - \sin M_j\right) + \cdots ,$$

$$r_j = \gamma_j a_j \left(1 - e_j \cos M_j + \frac{e_j^2}{2} \left(1 - \cos(2M_j)\right)\right) + \cdots . \tag{3.2}$$

Consequently, the cosine and sine functions of the true anomalies are represented in the form

$$\cos f_j = \frac{\cos E_j - e_j}{1 - e_j \cos E_j} = \cos M_j + e_j \left(\cos(2M_j) - 1\right) + \frac{9e_j^2}{8} \left(\cos(3M_j) - \cos M_j\right) + \cdots ,$$

$$\sin f_j = \frac{\sin E_j \sqrt{1-e_j^2}}{1 - e_j \cos E_j} = \sin M_j + e_j \sin(2M_j) + \frac{e_j^2}{8} \left(9\sin(3M_j) - 7\sin M_j\right) + \cdots . \tag{3.3}$$

On substituting the expansions (3.3) into (2.5) and doing quite standard but cumbersome symbolic calculations, we derive the following expressions for the Cartesian coordinates of the bodies $P_1$ and $P_2$:

$$\begin{aligned}\frac{x_j}{r_j} &= \cos(M_j + \omega_j + \Omega_j) + e_j \left(\cos(2M_j + \omega_j + \Omega_j) - \cos(\omega_j + \Omega_j)\right) \\ &\quad - \frac{e_j^2}{8} \left(\cos(M_j - \omega_j - \Omega_j) + 8\cos(M_j + \omega_j + \Omega_j) - 9\cos(3M_j + \omega_j + \Omega_j)\right) \\ &\quad + s_j^2 \left(\cos(M_j + \omega_j - \Omega_j) - \cos(M_j + \omega_j + \Omega_j)\right),\end{aligned}$$

$$\frac{y_j}{r_j} = \sin(M_j + \omega_j + \Omega_j) + e_j \left(\sin(2M_j + \omega_j + \Omega_j) - \sin(\omega_j + \Omega_j)\right)$$

$$+\frac{e_j^2}{8}\left(\sin(M_j-\omega_j-\Omega_j)-8\sin(M_j+\omega_j+\Omega_j)+9\sin(3M_j+\omega_j+\Omega_j)\right)$$
$$-s_j^2\left(\sin(M_j+\omega_j-\Omega_j)+\sin(M_j+\omega_j+\Omega_j)\right),$$
$$\frac{z_j}{r_j}=2s_j\sin(M_j+\omega_j)-2e_js_j\left(\sin(\omega_j)-\sin(2M_j+\omega_j)\right), \tag{3.4}$$

where $r_j=\gamma_j\rho_j$, and $s_j=\sin(i_j/2)\ll 1$ is a small parameter because of the assumption $i_j\ll 1$. The obtained expansions (3.4) are accurate up to the second order in small parameters $e_j$ and $s_j$.

Note that the force functions (2.4) contains the scalar product

$$\vec{r}_1\cdot\vec{r}_2=x_1x_2+y_1y_2+z_1z_2=r_1r_2\cos\psi,$$

where $\psi$ is the angle between the position vectors $\vec{r}_1$ and $\vec{r}_2$. Using the expansions (3.4) and doing the corresponding symbolic calculations, to second order in small parameters $e_j, s_j$, we obtain

$$\cos\psi=\frac{x_1x_2+y_1y_2+z_1z_2}{r_1r_2}=\cos(\nu_1-\nu_2)+\Psi, \tag{3.5}$$

where $\nu_j=f_j+\omega_j+\Omega_j$ is the true longitude of the body $P_j$, $(j=1,2)$, the term $\cos(\nu_1-\nu_2)$ depends only on the eccentricities $e_j$ and is given by

$$\begin{aligned}\cos(\nu_1-\nu_2)=\cos(\lambda_1-\lambda_2)&\\ +e_1(\cos(2\lambda_1-\lambda_2-\omega_1-\Omega_1)-\cos(\lambda_2-\omega_1-\Omega_1))&\\ +e_2(\cos(\lambda_1-2\lambda_2+\omega_2+\Omega_2)-\cos(\lambda_1-\omega_2-\Omega_2))&\\ +\frac{e_1^2}{8}(-8\cos(\lambda_1-\lambda_2)+9\cos(3\lambda_1-\lambda_2-2\omega_1-2\Omega_1)-\cos(\lambda_1+\lambda_2-2\omega_1-2\Omega_1))&\\ +\frac{e_2^2}{8}(-8\cos(\lambda_1-\lambda_2)+9\cos(\lambda_1-3\lambda_2+2\omega_2+2\Omega_2)-\cos(\lambda_1+\lambda_2-2\omega_2-2\Omega_2))&\\ +e_1e_2(\cos(2\lambda_1-2\lambda_2-\omega_1+\omega_2-\Omega_1+\Omega_2)+\cos(\omega_1-\omega_2+\Omega_1-\Omega_2)&\\ -\cos(2\lambda_1-\omega_1-\omega_2-\Omega_1-\Omega_2)&\\ -\cos(2\lambda_2-\omega_1-\omega_2-\Omega_1-\Omega_2)),&\end{aligned} \tag{3.6}$$

where $\lambda_j=M_j+\omega_j+\Omega_j$ is the mean longitude of the body $P_j$, $(j=1,2)$. The second term in the right-hand side of (3.5) may be written in the form

$$\begin{aligned}\Psi=&s_1^2(-\cos(\lambda_1-\lambda_2)+\cos(\lambda_1+\lambda_2-2\Omega_1))\\ &+s_2^2(-\cos(\lambda_1-\lambda_2)+\cos(\lambda_1+\lambda_2-2\Omega_2))\\ &-2s_1s_2(\cos(\lambda_1+\lambda_2-\Omega_1-\Omega_2)-\cos(\lambda_1-\lambda_2-\Omega_1+\Omega_2)).\end{aligned} \tag{3.7}$$

Thus, to second order in small parameters $e_j$, $s_j$, we obtain that the term $\Psi$ in (3.5) is of second order and is independent of $e_j$. Since the distances $r_1, r_2$ depend only on eccentricities [see (3.2)], to second order in $e_j, s_j$, we can write

$$r_{12}=\left(r_1^2+r_2^2-2r_1r_2\cos\psi\right)^{1/2}=\left(\Delta_0^2-2r_1r_2\Psi\right)^{1/2}=\Delta_0\left(1-\frac{r_1r_2}{\Delta_0^2}\Psi\right), \tag{3.8}$$

where

$$\Delta_0=\left(r_1^2+r_2^2-2r_1r_2\cos(\nu_1-\nu_2)\right)^{1/2}. \tag{3.9}$$

Taking into account the expansions (3.2) and (3.6), to the second order in eccentricities $e_j$, we obtain

$$\begin{aligned}
\frac{1}{\Delta_0} &= \frac{1}{\rho_0} + \frac{1}{\rho_0^3}\Big(e_1(\gamma_1^2 a_1^2 \cos(\lambda_1-\omega_1-\Omega_1) \\
&+ \frac{\gamma_1\gamma_2 a_1 a_2}{2}(\cos(2\lambda_1-\lambda_2-\omega_1-\Omega_1) \\
&- \ 3\cos(\lambda_2-\omega_1-\Omega_1))) + e_2(\gamma_2^2 a_2^2\cos(\lambda_2-\omega_2-\Omega_2) \\
&+ \frac{\gamma_1\gamma_2 a_1 a_2}{2}(\cos(\lambda_1-2\lambda_2+\omega_2+\Omega_2) \\
&- 3\cos(\lambda_1-\omega_2-\Omega_2))) + \frac{e_1^2}{8}(2\gamma_1^2 a_1^2(\cos(2\lambda_1-2\omega_1-2\Omega_1)-3) \\
&+ \gamma_1\gamma_2 a_1 a_2(\cos(\lambda_1+\lambda_2-2\omega_1-2\Omega_1) \\
&+ 3\cos(3\lambda_1-\lambda_2-2\omega_1-2\Omega_1) - 4\cos(\lambda_1-\lambda_2))) + \frac{e_2^2}{8}(2\gamma_2^2 a_2^2(\cos(2\lambda_2-2\omega_2-2\Omega_2)-3) \\
&+ \gamma_1\gamma_2 a_1 a_2(\cos(\lambda_1+\lambda_2-2\omega_2-2\Omega_2) + 3\cos(\lambda_1-3\lambda_2+2\omega_2+2\Omega_2) - 4\cos(\lambda_1-\lambda_2))) \\
&- \frac{e_1 e_2}{4}\gamma_1\gamma_2 a_1 a_2(3\cos(2\lambda_1-\omega_1-\omega_2-\Omega_1-\Omega_2) + 3\cos(2\lambda_2-\omega_1-\omega_2-\Omega_1-\Omega_2) \\
&- \ 9\cos(\omega_1-\omega_2+\Omega_1-\Omega_2) - \cos(2\lambda_1-2\lambda_2-\omega_1+\omega_2-\Omega_1+\Omega_2))) \\
&+ \frac{3}{8\rho_0^5}(2e_1\gamma_1^2 a_1^2\cos(\lambda_1-\omega_1-\Omega_1) + \gamma_1\gamma_2 a_1 a_2(e_1(\cos(2\lambda_1-\lambda_2-\omega_1-\Omega_1) \\
&- 3\cos(\lambda_2-\omega_1-\Omega_1)) + e_2(\cos(\lambda_1-2\lambda_2+\omega_2+\Omega_2) - 3\cos(\lambda_1-\omega_2-\Omega_2))) \\
&+ 2e_2\gamma_2^2 a_2^2\cos(\lambda_2-\omega_2-\Omega_2))^2,
\end{aligned} \tag{3.10}$$

where

$$\rho_0 = \left(\gamma_1^2 a_1^2 + \gamma_2^2 a_2^2 - 2\gamma_1\gamma_2 a_1 a_2\cos(\lambda_1-\lambda_2)\right)^{1/2}. \tag{3.11}$$

Therefore, to second order in small parameters, we can write

$$\frac{1}{r_{12}} = \frac{1}{\Delta_0} + \frac{\gamma_1\gamma_2 a_1 a_2}{\rho_0^3}\Psi, \tag{3.12}$$

where $\Psi$ and $1/\Delta_0$ are determined in (3.7) and (3.10), respectively.

Performing similar calculations, we obtain

$$\begin{aligned}
\frac{\vec{r}_1\cdot\vec{r}_2}{r_1^3} &= \frac{r_2\cos\psi}{r_1^2} = \frac{\gamma_2 a_2}{\gamma_1^2 a_1^2}\,(\cos(\lambda_1-\lambda_2) + 2e_1\cos(2\lambda_1-\lambda_2-\omega_1-\Omega_1) \\
&- \frac{e_2}{2}(3\cos(\lambda_1-\omega_2-\Omega_2) - \cos(\lambda_1-2\lambda_2+\omega_2+\Omega_2)) \\
&- \frac{e_1^2}{8}(4\cos(\lambda_1-\lambda_2) - 27\cos(3\lambda_1-\lambda_2-2\omega_1-2\Omega_1) - \cos(\lambda_1+\lambda_2-2\omega_1-2\Omega_1)) \\
&\frac{e_2^2}{8}(4\cos(\lambda_1-\lambda_2) - 3\cos(\lambda_1-3\lambda_2+2\omega_2+2\Omega_2) - \cos(\lambda_1+\lambda_2-2\omega_2-2\Omega_2)) \\
&- e_1 e_2(3\cos(2\lambda_1-\omega_1-\omega_2-\Omega_1-\Omega_2) - \cos(2\lambda_1-2\lambda_2-\omega_1+\omega_2-\Omega_1+\Omega_2)) \\
&- s_1^2(\cos(\lambda_1-\lambda_2) - \cos(\lambda_1+\lambda_2-2\Omega_1)) - s_2^2(\cos(\lambda_1-\lambda_2) - \cos(\lambda_1+\lambda_2-2\Omega_2)) \\
&- \ 2s_1 s_2(\cos(\lambda_1+\lambda_2-\Omega_1-\Omega_2) - \cos(\lambda_1-\lambda_2-\Omega_1+\Omega_2)))\,,
\end{aligned} \tag{3.13}$$

$$\frac{\vec{r}_1\cdot\vec{r}_2}{r_2^3} = \frac{r_1\cos\psi}{r_2^2} = \frac{\gamma_1 a_1}{\gamma_2^2 a_2^2}\,(\cos(\lambda_1-\lambda_2) + 2e_2\cos(\lambda_1-2\lambda_2+\omega_2+\Omega_2)$$

$$
\begin{aligned}
&-\frac{e_1}{2}(3\cos(\lambda_2-\omega_1-\Omega_1)-\cos(2\lambda_1-\lambda_2-\omega_1-\Omega_1))\\
&-\frac{e_1^2}{8}(4\cos(\lambda_1-\lambda_2)-3\cos(3\lambda_1-\lambda_2-2\omega_1-2\Omega_1)-\cos(\lambda_1+\lambda_2-2\omega_1-2\Omega_1))\\
&\frac{e_2^2}{8}(4\cos(\lambda_1-\lambda_2)-27\cos(\lambda_1-3\lambda_2+2\omega_2+2\Omega_2)-\cos(\lambda_1+\lambda_2-2\omega_2-2\Omega_2))\\
&-e_1e_2(3\cos(2\lambda_2-\omega_1-\omega_2-\Omega_1-\Omega_2)-\cos(2\lambda_1-2\lambda_2-\omega_1+\omega_2-\Omega_1+\Omega_2))\\
&-s_1^2(\cos(\lambda_1-\lambda_2)-\cos(\lambda_1+\lambda_2-2\Omega_1))-s_2^2(\cos(\lambda_1-\lambda_2)-\cos(\lambda_1+\lambda_2-2\Omega_2))\\
&-\ 2s_1s_2(\cos(\lambda_1+\lambda_2-\Omega_1-\Omega_2)-\cos(\lambda_1-\lambda_2-\Omega_1+\Omega_2)))\,. \qquad (3.14)
\end{aligned}
$$

Note that expansions (3.12)–(3.14) enable to obtain the force functions (2.4) in the form of power series in terms of eccentricities $e_j$ and inclinations $s_j$ up to second order. However, these expressions are quite cumbersome and we do not write out them here.

The force functions corresponding to the reactive forces in (2.3) are derived in a similar way and, to second order in small parameters, are given by

$$
\begin{aligned}
\vec{F}_j\cdot\vec{r}_j=\gamma_j a_j\Big(&F_{jx}\Big(\cos\lambda_j-\frac{e_j}{2}(3\cos(\omega_j+\Omega_j)-\cos(2\lambda_j-\omega_j-\Omega_j))\\
&-\frac{e_j^2}{8}(4\cos\lambda_j-\cos(\lambda_j-2\omega_j-2\Omega_j)-3\cos(3\lambda_j-2\omega_j-2\Omega_j))\\
&-s_j^2(\cos\lambda_j-\cos(\lambda_j-2\Omega_j))\Big)\\
&+F_{jy}(\sin\lambda_j-\frac{e_j}{2}(3\sin(\omega_j+\Omega_j)-\sin(2\lambda_j-\omega_j-\Omega_j))\\
&-\frac{e_j^2}{8}(4\sin\lambda_j+\sin(\lambda_j-2\omega_j-2\Omega_j)-3\sin(3\lambda_j-2\omega_j-2\Omega_j))\\
&-s_j^2(\sin\lambda_j+\sin(\lambda_j-2\Omega_j)))\\
&+F_{jz}(2s_j\sin(\lambda_j-\Omega_j)-e_js_j(3\sin\omega_j-\sin(2\lambda_j-\omega_j-\Omega_j)))\Big)\,, \qquad (3.15)
\end{aligned}
$$

where components of reactive forces $F_{jx}$, $F_{jy}$, $F_{jz}$ are considered as given functions of time.

At last, to second order in small parameters, we can write the third term of the disturbing functions (2.3) in the form

$$
\frac{\ddot{\gamma}_j}{2\gamma_j}r_j^2=\frac{1}{2}\ddot{\gamma}_j\gamma_j a_j^2\left(1-2e_j\cos(\lambda_j-\omega_j-\Omega_j)+\frac{e_j^2}{2}\left(3-\cos\left(2\lambda_j-2\omega_j-2\Omega_j\right)\right)\right). \qquad (3.16)
$$

Thus, to second order in small parameters $e_j$ and $s_j$, expressions (3.12)–(3.16) define the disturbing functions (2.3) in terms of the orbital elements of the bodies $P_1$, $P_2$.

## 4 Secular Perturbations of the Orbital Elements

Differential equations determining the secular perturbations of the orbital elements are obtained if the disturbing functions $W_j$ in Lagrange's planetary equations (2.13) are replaced by their quantities averaged over the mean anomalies $M_j$ of the bodies $P_1$, $P_2$. Note that due to relationship $M_j = \lambda_j - \omega_j - \Omega_j$, one can perform the averaging scheme over the mean longitudes $\lambda_j$, as well.

To second order in eccentricities $e_j$ and inclinations $s_j$, the disturbing functions $W_j$ are the polynomials with coefficients being periodic functions of the mean longitudes $\lambda_j$. To realize the averaging scheme, one needs to replace the rational expressions $1/\rho_0$, $1/\rho_0^3$, $1/\rho_0^5$ in (3.10) and (3.12) by the corresponding Fourier series given by (see [3])

$$\frac{1}{\rho_0} = \frac{1}{2}\sum_{k=-\infty}^{+\infty} A_k \cos(k(\lambda_1 - \lambda_2)),$$
$$\frac{\gamma_1\gamma_2 a_1 a_2}{\rho_0^3} = \frac{1}{2}\sum_{k=-\infty}^{+\infty} B_k \cos(k(\lambda_1 - \lambda_2)),$$
$$\frac{\gamma_1^2\gamma_2^2 a_1^2 a_2^2}{\rho_0^5} = \frac{1}{2}\sum_{k=-\infty}^{+\infty} C_k \cos(k(\lambda_1 - \lambda_2)), \tag{4.1}$$

where coefficients $A_k$, $B_k$, $C_k$ are known as the Laplace coefficients and satisfy the following recurrence relations:

$$A_k = \frac{2(k-1)}{2k-1}\left(\alpha + \frac{1}{\alpha}\right) A_{k-1} - \frac{2k-3}{2k-1} A_{k-2}, \quad k \geq 2,$$
$$B_k = \frac{(2k+1)\alpha(1+\alpha^2)}{(1-\alpha^2)^2} A_k - \frac{2\alpha^2(2k+1)}{(1-\alpha^2)^2} A_{k+1}, \quad k \geq 0,$$
$$C_k = \frac{(2k+3)\alpha(1+\alpha^2)}{3(1-\alpha^2)^2} B_k - \frac{2\alpha^2(2k-1)}{3(1-\alpha^2)^2} B_{k+1}, \quad k \geq 0.$$

All the Laplace coefficients can be expressed in terms of two coefficients $A_0$ and $A_1$ given by

$$A_0 = \frac{2}{\pi a_2 \gamma_2}\int_0^{\pi} \frac{d\lambda}{(1+\alpha^2 - 2\alpha\cos\lambda)^{1/2}} = \frac{4}{\pi a_2\gamma_2(1+\alpha)} K\left(\frac{4\alpha}{(1+\alpha)^2}\right),$$
$$A_1 = \frac{2}{\pi a_2 \gamma_2}\int_0^{\pi} \frac{\cos\lambda\, d\lambda}{(1+\alpha^2 - 2\alpha\cos\lambda)^{1/2}} = \frac{2}{\pi a_2\gamma_2\alpha(1+\alpha)}\left((1+\alpha^2)K\left(\frac{4\alpha}{(1+\alpha)^2}\right) - (1+\alpha)^2 E\left(\frac{4\alpha}{(1+\alpha)^2}\right)\right),$$

where the functions $K(4\alpha/(1+\alpha)^2)$ and $E(4\alpha/(1+\alpha)^2)$ denote the corresponding complete elliptic integrals of the first and the second kind, respectively, and parameter

$$\alpha = \frac{\gamma_1 a_1}{\gamma_2 a_2} < 1.$$

It is assumed here that the trajectory of body $P_1$ is located inside the trajectory of body $P_2$, and the condition $r_1 < r_2$ is satisfied for any instant of time.

Using the series (4.1) in the expressions $1/\rho_0$, $1/\rho_0^3$, $1/\rho_0^5$ and averaging the disturbing functions $W_j$ according to the rule

$$W_j^{(sec)} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} W_j\, d\lambda_1 d\lambda_2,$$

we obtain the secular parts of the disturbing functions in the form

$$W_j^{(sec)} = U_j^{(sec)} - \frac{3}{2}\gamma_j a_j e_j \left(F_{jx}\cos(\omega_j + \Omega_j) + F_{jy}\sin(\omega_j + \Omega_j) + 2 s_j F_{jz}\sin\omega_j\right) - \frac{1}{2}\ddot{\gamma}_j \gamma_j a_j^2 \left(1 + \frac{3}{2}e_j^2\right), \tag{4.2}$$

where

$$\frac{2U_1^{(sec)}}{\mu_2} = \frac{2U_2^{(sec)}}{\mu_1} = A_0 + K_{11}e_1^2 + K_{22}e_2^2 + K_{12}e_1e_2\cos(\omega_1 - \omega_2 + \Omega_1 - \Omega_2)$$
$$- \left(B_1 + \frac{9}{16}B_3\right)(s_1^2 + s_2^2) + \frac{1}{8}(16B_1 + 9B_3)s_1s_2\cos(\Omega_1 - \Omega_2),$$
$$K_{11} = \frac{3\alpha^2}{4}C_0 - \frac{3\alpha}{32}(8B_0 + 16C_1 + 9C_3) - \frac{1}{128}(64B_1 + 36B_3 - 240C_0 + 225C_2),$$
$$K_{22} = \frac{3}{4\alpha^2}C_0 - \frac{3}{32\alpha}(8B_0 + 16C_1 + 9C_3) - \frac{1}{128}(64B_1 + 36B_3 - 240C_0 + 225C_2),$$
$$K_{12} = -\frac{3(1+\alpha^2)}{64\alpha}(48C_0 - 25C_2) + \frac{1}{128}(288B_0 + 50B_2 + 363C_1 + 237C_3). \quad (4.3)$$

Note that the secular part (4.2) of the disturbing functions $W_j$ is obtained under assumption that the considered two-planetary system is not affected by commensurability between the two mean motions $n_1$, $n_2$. In this case integration over the mean longitudes $\lambda_1$, $\lambda_2$ enables to eliminate rapidly oscillating terms in the expansion of $W_j$ and to retain slowly varying secular terms up to the second order in the eccentricities and inclinations.

Obviously, the secular part (4.2) of the disturbing function $W_j$ does not depend on the mean anomaly $M_j$ and so the semi-major axes $a_j$ do not change with time. Using Lagrange's planetary equations (2.13), to second order in eccentricities and inclinations, we can write the following differential equations for the slowly varying variables:

$$\frac{de_j}{dt} = -\frac{1}{n_j a_j^2 e_j}\frac{\partial W_j^{(sec)}}{\partial \omega_j},$$
$$\frac{d\omega_j}{dt} = \frac{1}{n_j a_j^2 e_j}\frac{\partial W_j^{(sec)}}{\partial e_j} - \frac{1}{n_j a_j^2 i_j}\frac{\partial W_j^{(sec)}}{\partial i_j},$$
$$\frac{d\Omega_j}{dt} = \frac{1}{n_j a_j^2 i_j}\frac{\partial W_j^{(sec)}}{\partial i_j},$$
$$\frac{di_j}{dt} = \frac{1}{n_j a_j^2 i_j}\left(\frac{\partial W_j^{(sec)}}{\partial \omega_j} - \frac{\partial W_j^{(sec)}}{\partial \Omega_j}\right), \quad (j = 1, 2). \quad (4.4)$$

Note that the variable $s_j$ should be replaced in (4.2) by $i_j/2$.

Computing partial derivatives of $W_j^{(sec)}$ in (4.4) with respect to $e_j, i_j, \omega, \Omega_j$, one can easily obtain the system of differential equations determining secular perturbations of these orbital parameters. Due to dependence of masses on time and reactive forces this system cannot be solved analytically. However, choosing some realistic model of masses variation, one can get its numerical solution and investigate the dynamics of the system.

## 5 Conclusion

In the present paper we have considered a general case of the two-planetary system when masses of the all three bodies vary non-isotropically with different rates and due to this the reactive forces arise. As the differential equations of motion are not integrable we use the perturbation theory and consider an exact solution to the two-body problem with variable masses describing an aperiodic motion on quasi-conic sections as initial approximation. Performing quite cumbersome symbolic calculations, we obtain the disturbing functions in the form of power series in terms of the eccentricities and inclinations up to the second order and derive the differential equations determining behaviour of the orbital parameters in the form of Lagrange's planetary equations. Averaging the equations of motion over the mean longitudes of the bodies in the absence of a mean-motion resonances, we obtain the differential equations describing the evolution of orbital parameters over long periods of time. These equations are rather complicated and their solution requires application of numerical methods. All the relevant computations are performed with the computer algebra system Wolfram Mathematica.

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