$f(R)$ Hořava-Lifshitz cosmologies via Noether's symmetries

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We investigate the general approach to finding exact cosmological solutions in $f(R)$ Hořava-Lifshitz gravity, based on Noethers theorem. A feature of this approach is that it uses the behavior of an effective Lagrangian under infinitesimal transformations of the desired symmetry, explicitly determining the form $f(R)$ for which such symmetries exist. It is shown that the dynamics of the scale factor changes according to either a exponential function of time.

Keywords: Hořava-Lifshitz gravity; $f(R)$ gravity; Noether symmetry.

1. Introduction

At the end of the past millennium, it became clear from the observation of type Ia supernovae and the cosmic microwave background that our universe expands with acceleration [1, 2]. The explanation of this phenomenon within the framework of the general theory of relativity led to the formulation of a large number of models, one of which consists in introducing a mysterious substance, the so-called dark energy (see for example Ref. [3] and references therein for some reviews). The nature of dark energy is still not clear, but mathematically it fits well with the ΛCDM model a wide range of data [4]. Nevertheless, this model has strong theoretical flaws [5], which motivated the search for alternative models [6, 7]. One of the alternatives is the modification of the Einstein-Hilbert term, replacing the Ricci scalar R in the action with some common functions $f(R)$ of the Ricci scalar (see for example Ref. [8]).

In parallel, some models for quantum gravity were developed. In 2009, Hořava [9], based on an idea proposed by Lifshitz [10], formulated a model for a theory of quantum gravity, which takes into account ultraviolet mode renormalizability, due to an anisotropic scaling between space and time, so that Lorentz invariance is violated on ultraviolet scales. However, when choosing the parameter $\lambda = 1$, the infrared limit of the theory reproduces the general theory of relativity. This modification of the general theory of relativity consists in introducing high-order terms into the Einstein-Hilbert action, which lead to different scalings and divide the coordinates into space and time. In theory, there are no ghosts, since there are only second-order time derivatives in the action, but a clear violation of general covariance introduces a new scalar degree of freedom, due to which pathologies appear [11]. This model developed further and is known as the Hořava-Lifshitz theory (see for example Ref. [12]). From the above considerations, it is clear that the quantum theory of Hořava-Lifshitz, combined with the alternative theory $f(R)$, is a promising candidate for completing the general theory of relativity in the ultraviolet range [13, 14].

In this paper, we consider the Friedman-Lemaître-Robertson-Walker (FLRW) flat spacetime in the framework of the metric formalism $f(R)$ of gravity. We consider a general approach to constructing modified gravity, which is invariant with respect to diffeomorphisms and preserve foliation. The approach was proposed in [14], where special attention was paid to the formulation of modified $f(R)$ HořavaLifshitz gravity and its Hamiltonian structure. Following [15], we calculated an effective Lagrangian in which the scale factor a and the Ricci scalar R play the role of independent dynamic variables. This Lagrangian is constructed in such a way that its variation with respect to a and R gives the equations of motion of the HořavaLifshitz theory. The form of the function $f(R)$, appearing in the modified action, is then determined by the requirement that the Lagrangian admits the required Noether symmetry [16]. Under Noether symmetry of this cosmological model, we understand that there is a vector field X, which is an infinitesimal generator of a symmetry in the tangent space of the configuration space, such that the derivative of the Lagrangian along this vector field vanishes. We will see that, by requiring the Noether symmetry as a feature of the Lagrangian of the model under consideration, we can obtain the explicit form of the function $f(R)$. Since the existence of symmetry leads to constants of motion, we can integrate the field equations, which then lead to a exponential expansion for the universe.

2. Modified $f(R)$ Hořava-Lifshitz Gravity

In this work, we consider a more general model of the Hořava-Lifshitz gravity proposed in [14]. The action of such a model has the form

$$S_{f(R_{GHL})} = \int \mathrm{d}^4x\sqrt{g^{(3)}}Nf(R_{GHL}). \tag{1}$$

Here $g^{(3)}$ is determinant of the three-dimensional metric tensor $g^{(3)}_{ij}$ for the ADM metric given in the following form

$$ds^2 = -N^2\mathrm{d}t^2 + g^{(3)}_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right), \tag{2}$$

where $i,j = 1,2,3$, N is the so-called lapse variable and N^i is the shift 3-vector. In the action, we use the function $f(R_{GHL})$, which denotes the generalized curvature of the Hořava-Lifshitz gravity R_{GHL} and is defined as

$$R_{GHL} \equiv K^{ij}K_{ij} - \lambda K^2 + 2\mu\nabla_\mu\left(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu\right) - E^{ij}\mathcal{G}_{ijkl}E^{kl}\,, \tag{3}$$

where K_{ij} is the extrinsic curvature

$$K_{ij} = \frac{1}{2N}\left(\dot{g}^{(3)}_{ij} - \nabla^{(3)}_i N_j - \nabla^{(3)}_j N_i\right)\,, \quad K = K^i{}_i\,, \tag{4}$$

n^μ is a unit vector perpendicular to the three-dimensional hypersurface Σ_t defined by $t = \text{constant}$, and $\nabla^{(3)}_i$ expresses the covariant derivative on the hypersurface Σ_t. In the last term of Eq. (3), $\mathcal{G}_{ijkl}$ is the inverse of the generalized De Witt metric

$$\mathcal{G}^{ijkl} = \frac{1}{2}(g^{(3)ik}g^{(3)jl} + g^{(3)il}g^{(3)jk}) - \lambda g^{(3)ij}g^{(3)kl}. \tag{5}$$

Here it is important to note that $\mathcal{G}^{ijkl}$ is singular for $\lambda = 1/3$ and G_{ijkl} exist if $\lambda \neq 1/3$. The expression for E_{ij} is constructed to satisfy the "detailed balance principle" [17] and is defined as

$$\sqrt{g^{(3)}}E^{ij} = \frac{\delta W[g^{(3)}_{kl}]}{\delta g^{(3)}_{ij}}, \tag{6}$$

where the form of $W[g^{(3)}_{kl}]$ is given in [18] for $z = 2$ and $z = 3$.

Consider a spatially flat FLRW universe

$$ds^2 = -N^2\mathrm{d}t^2 + a(t)^2 \sum_{i=1,2,3}\left(\mathrm{d}x^i\right)^2\,, \tag{7}$$

where N can be considered as time-independent and we will fix as $N = 1$. The scalar curvature (3) can be written as

$$R_{GHL} = 3(1 - 3\lambda + 4\mu)\frac{\dot{a}^2}{a^2} + 6\mu\frac{\ddot{a}}{a}. \tag{8}$$

Following [16], in order to investigate Noether's symmetry of the model, it is necessary to determine the effective Lagrangian of the action (1) of the minisuperspace under consideration, in which the scale factor a and the scalar curvature R_{GHL} play the role of independent dynamical variables

$$\begin{aligned} S = &\int dt\mathcal{L}(a, \dot{a}, R_{GHL}, \dot{R}_{GHL}) = \\ &\int dt\Big[a^3 f(R_{GHL}) - \nu\Big\{R_{GHL} - \Big(3(1 - 3\lambda + 4\mu)\frac{\dot{a}^2}{a^2} + 6\mu\frac{\ddot{a}}{a}\Big)\Big\}\Big], \end{aligned} \tag{9}$$

where $\nu = a^3 df(R_{GHL})/dR_{GHL}$ is a Lagrange multiplier. Then, the effective Lagrangian will have the form

$$\mathcal{L}(a, \dot{a}, R_{GHL}, \dot{R}_{GHL}) = (9\lambda - 3)\dot{a}^2 af' + 6\mu\dot{a}\dot{R}_{GHL}a^2 f'' + a^3(f'R_{GHL} - f). \tag{10}$$

The equations of motion will have then the following form

$$\begin{aligned} 3H^2 + 2\dot{H} = -\frac{2}{3\lambda - 1}\frac{1}{f'}\Big[&\mu f'''\dot{R}^2_{GHL} + \mu f''\ddot{R}_{GHL} + \\ &+(3\lambda - 1)f''H\dot{R}_{GHL} + \frac{1}{2}\left(f - R_{GHL}f'\right)\Big]. \end{aligned} \tag{11}$$

Also, we have the zero energy condition associated with the above Lagrangian as

$$H^2 = \frac{1}{3(3\lambda - 1)} \frac{1}{f'} \left[f' R_{GHL} - f - 6\mu \dot{R}_{GHL} H f'' \right]. \quad (12)$$

Knowing the Lagrangian of the generalized Hořava-Lifshitz gravity, it is possible to determine the Noether symmetry.

3. The Noether symmetries in $f(R_{GHL})$ theory of gravity

Here, our aim is to find the function $f(R_{GHL})$ such that the corresponding Lagrangian exhibits the desired symmetry. Following [16], we define the Noether symmetry induced on the model by a vector field X on the tangent space $TQ = \left(a, \dot{a}, R, \dot{R}_{GHL}\right)$ of the configuration space $Q = (a, R_{GHL})$ of the Lagrangian (10) through

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R_{GHL}} + \frac{d\alpha}{dt} \frac{\partial}{\partial \dot{a}} + \frac{d\beta}{dt} \frac{\partial}{\partial \dot{R}_{GHL}}, \quad (13)$$

such that the Lie derivative of the Lagrangian with respect to this vector field vanishes

$$L_X \mathcal{L} = 0. \quad (14)$$

In Eq.(13), α and β are functions of a and R_{GHL} and $\frac{d}{dt}$ represents the Lie derivative along the dynamical vector field, that is,

$$\frac{d}{dt} = \dot{a} \frac{\partial}{\partial a} + \dot{R}_{GHL} \frac{\partial}{\partial R_{GHL}}. \quad (15)$$

Here we substitute the expressions for the effective Lagrangian (10) and combine the coefficients in front of $\dot{a}^2, \dot{R}^2_{GHL}, \dot{a}\dot{R}_{GHL}$. Equating to zero the resulting expression, we obtain the following equations

$$3(3\lambda - 1)\left(\alpha + 2a\alpha_a\right) f' + \left[3(3\lambda - 1)\beta a + 6\mu a^2 \beta_a\right] f'' = 0, \quad (16)$$

$$6\mu a^2 \alpha_{R_{GHL}} f'' = 0, \quad (17)$$

$$6\mu\left(2a\alpha + a^2\alpha_a\right) f'' + 6(3\lambda - 1)a\alpha_{R_{GHL}} f' + 6\mu a^2 \left(\beta f''' + \beta_{R_{GHL}} f''\right) = 0. \quad (18)$$

Then, we collect the remaining free member

$$3\alpha a^2 (f'R - f) + \beta a^3 R f'' = 0. \quad (19)$$

Now, our task is to solve the system of equations (16) - (19) in order to find the cosmological parameters that describe the dynamics of the universe in the framework of the $f(R_{GHL})$ Hořava-Lifshitz gravity. From Eq.(17), it is clear that two cases must be considered: $f'' = 0$ and $\frac{d\alpha}{dR_{GHL}} = 0$, but the solution for $f'' = 0$ has no physical meaning; therefore, we consider only $\frac{d\alpha}{dR_{GHL}} = 0$, then

$$\alpha(a) = \alpha_0 a^{\frac{\beta_0}{\alpha_0}+1}, \beta(a, R) = \beta_0 a^{\frac{\beta_0}{\alpha_0}} R, \quad (20)$$

and

$$f = f_0 R^{-3\frac{\alpha_0}{\beta_0}}, \tag{21}$$

where $\alpha_0 = \frac{2\mu}{3\lambda-6\mu-1}\beta_0, f_0, \beta_0$ are integral constants.

4. Cosmological solutions

In this section, to describe the dynamics of the universe, we solve analytically the field equations (11) - (12). To this end, we need to find the explicit dependence of the scale factor a in terms of the time t. We rewrite Eqs.(11) - (12) as follows

$$3H^2 + 2\dot{H} = -\frac{4\mu(3\lambda-3\mu-1)}{(3\lambda-6\mu-1)^2}\frac{\dot{R}^2_{GHL}}{R^2_{GHL}} + \frac{2\mu}{3\lambda-6\mu-1}\frac{\ddot{R}_{GHL}}{R_{GHL}} + \frac{2(3\lambda-1)}{3\lambda-6\mu-1}\frac{\dot{R}_{GHL}}{R_{GHL}}H + \frac{1}{6\mu}R_{GHL}, \tag{22}$$

$$H^2 = \frac{R_{GHL}}{18\mu} - \frac{2\mu}{1-3\lambda+6\mu}\frac{\dot{R}_{GHL}}{R_{GHL}}H\ . \tag{23}$$

To solve this system we obtain

$$H = \sqrt{\frac{C_1}{C_2}}\tanh(\sqrt{C_1 C_2}t), \tag{24}$$

where $C_1 = \frac{Z}{3\mu+3\lambda-1}$, $C_2 = \frac{(3\lambda-1)(3\mu-3\lambda+1)}{2\mu(3\mu+3\lambda-1)}$, or, equivalently,

$$a = a_0\left(e^{\sqrt{C_1C_2}t} + e^{-\sqrt{C_1C_2}t}\right)^{\frac{1}{C_2}}\ . \tag{25}$$

Thus, we have found a general solution to the modified $f(R)$ Hořava-Lifshitz gravity theory. In general, it represents an expanding cosmological model with a scale factor with grows exponentially.

5. Conclusion

In this work, we analyzed the $f(R)$ Hořava-Lifhitz gravity model, which is a modification of the original Hořava-Lifhitz model for quantum gravity. We consider the spatially flat FLRW line element to find out what kind of cosmological scenarios are allowed in this modified gravity. Since usually it is very difficult to find analytic solutions to the field equations, we use here the alternative method, which is based on the analysis of the Noether symmetries of a particular effective Lagrangian. In this work, we derive the effective Lagrangian, from the corresponding field equations for the $f(R)$ Hořava-Lifhitz theory in the presence of spatially flat, isotropic and homogeneous line element.

It turns out that the field equations can be represented as a set of two second-order ordinary differential equations, which can be expressed as one differential equation for different variables so that it can be solved by using the standard method of separation of variables. This simple representation allows us to integrate explicitly the scale factor as well as the Ricci scalar. Both quantities are then explicit functions of time.

The results presented in this work show that the method of Noether symmetries can be applied also in the case of the modified $f(R)$ Hořava-Lifhitz gravity model to derive cosmological solutions. The resulting functions for the scale factor show that the corresponding universe expands either exponentially. This scenario is possible in relativistic cosmology so that, in principle, we could compare our results with observational data from different epochs of the universe evolution. This could be used to set limits on the values of the parameters that enter the Hořava-Lifhitz action.

Acknowledgments

This work was partially supported by UNAM-DGAPA-PAPIIT, Grant No. 111617, and by the Ministry of Education and Science of RK, Grant No. BR05236322 and AP05133630.

References

1. A. G. Riess, *et. al.*, *Astron. J.* **116** 1009 (1998);
   S. Perlmutter, *et. al.*, *Astrophys. J.* **517** 565 (1999).
2. C. L. Bennett, *et. al.*, *Astrophys. J. Suppl.* **148** 1 (2003).
3. S. Nojiri and S. D. Odintsov, *Int. J. Geom. Meth. Mod. Phys.* **4** 115 (2007).
4. U. Seljak, *et.al.*, *Phys. Rev. D* **71** 103515 (2005).
5. S.M. Carroll , W.H. Press , E.L. Turner, *J.Ann. Rev. Astron. Astroph.* **30** 499 (1992).
6. P.J.E.Peebles, B.Rathra , *Rev. Mod. Phys.* **75**, 559 (2003).
7. E.J. Copeland, M. Sami, S. Tsujikawa, *Int. J. Mod. Phys.* **D 15**, 1753 (2006).
8. M. Calz, M.Rinaldi, L.Sebastiani *L. Eur. Phys. J. C* **78**178 (2018).
9. P. Horava, *Phys. Rev. D* **79**, 084008 (2009);
10. P. Horava, *JHEP* **0903**, 020 (2009);
    P. Horava, *Phys. Lett. B* **694**, 172-176 (2010).
11. R. G. Cai, B. Hu and H. B. Zhang, *Phys. Rev. D* **80**, 041501 (2009).
12. G.E. Volovik,*JETP Lett.* **89**, 525-528 (2009).
13. J. Kluson, *Phys. Rev. D* **81**, 064028 (2010).
14. M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen and A. Tureanu, *Class. Quant. Grav.* **27**, 185021 (2010).
15. S. Capozziello and A. De Felice, *JCAP* **0808**, 016 (2008).
16. B. Vakili, N.Khosravi and H. R. Sepangi, *Class. Quantum Grav.* **24** 931 (2007).
17. A. De Felice and S. Tsujikawa, *Living Rev.Rel.* **13**, 3 (2010).
18. G.J. Olmo, *Phys. Rev. D* **75**, 023511 (2007).
19. Fourteenth Marcel Grossmann Meeting (2015), `http://www.icra.it/MG/mg14/`.