Numerical Algorithm for Solving the Inverse Problem for the Helmholtz Equation

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Abstract. In this paper we consider acoustic equation. The equation by separation of variables is reduced to a boundary value problem for the Helmholtz equation. We consider problem for the Helmholtz equation. We reduce the solution of the operator equation to the problem of minimizing the functional. And we build numerical algorithm for solving the inverse problem. At the end of the article is given the numerical calculations of this problem.

Keywords: continuation problem, regularization problem, comparative analysis, numerical methods, Landweber's method

1 Introduction

For mathematical modelling of physical processes and the phenomena occurring in nature, it is necessary to face ill-posed problems, including with the Cauchy problem for the Helmholtz equation. The Helmholtz equation is used in many physical processes associated with the propagation of waves and has numerous applications. If the law of oscillations of the physical medium harmonically depends on time, then the wave equation can be transformed to the Helmholtz equation. In particular, the Cauchy problem for the Helmholtz equation describes the propagation of electromagnetic or acoustic waves[1]. The aim of the paper is that an effective numerical solution for investigating inverse elliptic-type problems by the Landweber method. A significant theoretical and applied contribution to this topic has been accumulated in monographs by A.N. Tikhonova, M.M. Lavrentyeva, V.K. Ivanova, A.V. Goncharsky. The Cauchy problem for elliptic equations is of fundamental importance in all inverse problems. An important application of the Helmholtz equation is the acoustic wave problem, which is considered in the works of DeLillo, Isakov, Valdivia, Wang (2003)] L. Marin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic and H. Wen. The Landweber method is effective and makes it possible to substantially simplify the investigation of inverse problems [2], [3].

2 Formulation of the Problem

Consider the acoustics equation [4] in domain $Q = \Omega \times (0, +\infty)$, where $\Omega = (0,1) \times (0,1)$:

$$c^{-2}(x,y)U_{tt} = \Delta U - \nabla \ln(\rho(x,y))\nabla U \qquad (x,y,t) \in Q \tag{1}$$

Suppose that a harmonic oscillation regime was established in Ω:

$$U(x,y,t) = u(x,y)e^{i\omega t}, \qquad (x,y,t) \in Q \tag{2}$$

Putting (2) into (1) we obtain Helmholtz equation:

$$-\omega^2 c^{-2} u = \Delta u - \nabla \ln(\rho(x,y))\nabla u, \qquad (x,y) \in \Omega$$

We consider the initial-boundary value problem:

$$-\omega^2 c^{-2} u = \Delta u - \nabla \ln(\rho(x,y))\nabla u, \qquad (x,y) \in \Omega, \tag{3}$$
$$u(0,y) = h_1(y), \qquad y \in [0,1], \tag{4}$$
$$u(x,0) = h_2(x), \qquad x \in [0,1], \tag{5}$$
$$u_x(0,y) = f_1(y), \qquad y \in [0,1], \tag{6}$$
$$u_y(x,0) = f_2(x), \qquad x \in [0,1]. \tag{7}$$

Problem (3) — (7) appears ill-posed. For a numerical solution of the problem, we first reduce it to the inverse problem $Aq = f$ with respect to some direct (well-posed) problem. Further, we reduce the solution of the operator equation $Aq = f$ to the problem of minimizing the objective functional $J(q) = \langle Aq - f, Aq - f\rangle$. After calculating the gradient $J'q$ of the objective functional, we apply the method of Landweber to minimize it [5], [6].

3 The Conditional Stability Theorem

Let us consider the initial-boundary value problem:

$$\Delta u = 0, \qquad (x,y) \in \Omega, \tag{8}$$
$$u(0,y) = f_1(y), \quad u_x(0,y) = h_1(y), \qquad y \in [0,1], \tag{9}$$
$$u(x,0) = f_2(x), \quad u_y(x,0) = h_2(x), \qquad x \in [0,1], \tag{10}$$
$$\tag{11}$$

Let us divide the problem into two parts:

Problem 1

$$\Delta u = 0,$$
$$u(0,y) = f_1(y),$$
$$u(x,0) = 0,$$
$$u_x(0,y) = h_1(y),$$
$$u_y(x,0) = 0.$$

Problem 2

$$\Delta u = 0,$$
$$u(0,y) = 0,$$
$$u(x,0) = f_2(x),$$
$$u_x(0,y) = 0,$$
$$u_y(x,0) = h_2(x).$$

Problem 1, we continue the field along the axis x, then at $y = 1$ we can admit the boundary at zero. And also, problem 2, we continue the field along the axis y, then at $x = 1$ we can admit the boundary at zero. Suppose $h_2(x) = 0, h_1(y) = 0$.

Problem 1

$$\Delta u = 0, \quad (x, y) \in \Omega, \quad (12)$$
$$u(0, y) = f_1(y), \quad y \in [0, 1], \quad (13)$$
$$u(x, 0) = 0, \quad x \in [0, 1], \quad (14)$$
$$u_x(0, y) = 0, \quad y \in [0, 1], \quad (15)$$
$$u(x, 1) = 0, \quad x \in [0, 1]. \quad (16)$$

Problem 2

$$\Delta u = 0, \quad (x, y) \in \Omega, \quad (17)$$
$$u(0, y) = 0, \quad y \in [0, 1], \quad (18)$$
$$u(x, 0) = f_2(x), \quad x \in [0, 1], \quad (19)$$
$$u(1, y) = 0, \quad y \in [0, 1], \quad (20)$$
$$u_y(x, 0) = 0, \quad x \in [0, 1]. \quad (21)$$

Theorem 1 (of the conditional stability). *Let us suppose that for $f_1 \in L_2(0, 1)$ and there is a solution $u \in L_2(\Omega)$ of the problem* (12) — (16)*. Then the following estimate of conditional stability is right*

$$\int_0^1 u^2(x, y)dy \le \left(\int_0^1 f_1^2(y)dy \right)^{1-x} \left(\int_0^1 u^2(1, y)dy \right)^{x}. \quad (22)$$

Theorem 2 (of the conditional stability). *Let us suppose that for $f_2 \in L_2(0, 1)$ and there is a solution $u \in L_2(\Omega)$ of the problem* (17) — (21)*. Then the following estimate of conditional stability is right*

$$\int_0^1 u^2(x, y)dx \le \left(\int_0^1 f_2^2(x)dx \right)^{1-y} \left(\int_0^1 u^2(x, 1)dx \right)^{y}. \quad (23)$$

More details proof such estimates are shown in works [7], [8].

4 Reduction of the Initial Problem to the Inverse Problem

Let us show that the solution of the problem (3) — (7) is possible to reduce to the solution of the inverse problem with respect to some direct (well-posed) problem [9], [10].

As a direct problem, we consider the following one

$$-\omega^2 c^{-2} u = \Delta u - \nabla \ln(\rho(x, y)) \nabla u, \quad (x, y) \in \Omega, \quad (24)$$
$$u(0, y) = h_1(y), \quad y \in [0, 1], \quad (25)$$
$$u(x, 0) = h_2(x), \quad x \in [0, 1], \quad (26)$$
$$u(1, y) = q_1(y), \quad y \in [0, 1], \quad (27)$$
$$u(x, 1) = q_2(x), \quad x \in [0, 1]. \quad (28)$$

The inverse problem to problem (24) — (28) consist in defining the function $q_1(x), q_2(y)$ by the additional information on the solution of direct problem.

$$u_x(0,y) = f_1(y), \qquad y \in [0,1], \tag{29}$$

$$u_y(x,0) = f_2(x), \qquad x \in [0,1]. \tag{30}$$

We introduce the operator

$$A\colon (q_1, q_2) \mapsto (u_x(0,y), u_y(x,0)). \tag{31}$$

Then the inverse problem can be written in operator form

$$Aq = f.$$

We introduce the objective functional

$$J(q_1,q_2) = \int\limits_0^1 \big[u_x(0,y;q_1,q_2) - f_1(y)\big]^2 dy + \int\limits_0^1 \big[u_y(x,0;q_1,q_2) - f_2(x)\big]^2 dx. \tag{32}$$

We shall minimize the quadratic functional(32) by Landweber's method. Let the approximation be known q^n. The subsequent approximation is determined from:

$$q^{n+1} = q^n - \alpha J'(q^n) \tag{33}$$

here $\alpha \in (0, ||A||^{-2})$ [4], [9],[10].

Algorithm for solving the inverse problem

1. We choose the initial approximation $q^0 = (q_1^0, q_2^0)$;
2. Let us assume that q_n is known, then we solve the direct problem numerically

$$u_{xx} + u_{yy} - \Big(\frac{\rho_x}{\rho}u_x + \frac{\rho_y}{\rho}u_y\Big) + \Big(\frac{\omega}{c}\Big)^2 u = 0, \qquad (x,y) \in \Omega,$$
$$u(0,y) = h_1(y), \quad u(1,y) = q_1^n(y), \qquad y \in [0,1],$$
$$u(x,0) = h_2(x), \quad u(x,1) = q_2^n(x), \qquad x \in [0,1].$$

3. We calculate the value of the functional $J(q_{n+1}) = \int\limits_0^1 \big[u_x(0,y;q_1^{n+1}) - f_1(y)\big]^2 dy + \int\limits_0^1 \big[u_y(x,0;q_2^{n+1}) - f_2(x)\big]^2 dx;$
4. If the value of the functional is not sufficiently small, then go to next step;
5. We solve the conjugate problem

$$\psi_{xx} + \psi_{yy} + \Big(\frac{\rho_x}{\rho}\psi\Big)_x + \Big(\frac{\rho_y}{\rho}\psi\Big)_y + \Big(\frac{\omega}{c}\Big)^2 \psi = 0, \qquad (x,y) \in \Omega,$$
$$\psi(0,y) = 2\big(u_x(0,y;q_1) - f_1(y)\big), \psi(1,y) = 0, \qquad y \in [0,1],$$
$$\psi(x,0) = 2\big(u_y(x,0;q_2) - f_2(x)\big), \psi(x,1) = 0, \qquad x \in [0,1].$$

6. Calculate the gradient of the functional $J'(q^n) = \big(-\psi_x(1,y), -\psi_y(x,1)\big)$;
7. Calculate the following approximation $q^{n+1} = q^n - \alpha J'(q^n)$, then turn to step 2;;

5 Numerical Solution of the Inverse Problem

First we consider the initial problem in a discrete statement. We carry out a numerical study of the stability of the problem in a discrete statement [11].

Discretization of the original problem

The corresponding difference problem for the original problem (3) — (7) has the following

$$
\begin{aligned}
&\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}\\
&-\frac{\rho_{i+1,j}-\rho_{i-1,j}}{2h\rho_{i,j}}\cdot\frac{u_{i+1,j}-u_{i-1,j}}{2h}\\
&-\frac{\rho_{i,j+1}-\rho_{i,j-1}}{2h\rho_{i,j}}\cdot\frac{u_{i,j+1}-u_{i,j-1}}{2h}+\left(\frac{\omega}{c}\right)^2 u_{i,j}=0, \qquad i,j=\overline{1,N-1},\\
&u_{0,j}=h_1^j, \qquad j=\overline{0,N},\\
&u_{i,0}=h_2^i, \qquad i=\overline{0,N},\\
&u_{1,j}=h_1^j+h\cdot f_1^j, \qquad j=\overline{0,N},\\
&u_{i,1}=h_2^i+h\cdot f_2^i, \qquad i=\overline{0,N}.
\end{aligned}
$$

For convenience, we introduce the new denotations $a_{i,j}=1+\dfrac{\rho_{i+1,j}-\rho_{i-1,j}}{4\rho_{i,j}}$,
$b_{i,j}=1+\dfrac{\rho_{i,j+1}-\rho_{i,j-1}}{4\rho_{i,j}}$, $\quad c=-4+\left(\dfrac{\omega\cdot h}{c}\right)^2$,
$d_{i,j}=1-\dfrac{\rho_{i+1,j}-\rho_{i-1,j}}{4\rho_{i,j}}$, $\quad e_{i,j}=1-\dfrac{\rho_{i,j+1}-\rho_{i,j-1}}{4\rho_{i,j}}$.

$$a_{i,j}u_{i-1,j}+b_{i,j}u_{i,j-1}+cu_{i,j}+d_{i,j}u_{i,j+1}+e_{i,j}u_{i+1,j}=0, \quad i,j=\overline{1,N-1}, \tag{34}$$

$$u_{0,j}=h_1^j, \qquad j=\overline{0,N}, \tag{35}$$

$$u_{i,0}=h_2^i, \qquad i=\overline{0,N}, \tag{36}$$

$$u_{1,j}=h_1^j+h\cdot f_1^j, \tag{37}$$

$$u_{i,1}=h_2^i+h\cdot f_2^i, \qquad i=\overline{0,N}. \tag{38}$$

Let us construct a system of difference equations [12, p.379]

$$A\cdot X=B. \tag{39}$$

Here A — of matrix $(N+1)^2$ size , X — unknown vector of the form

$X=\left(u_{0,0},u_{0,1},u_{0,2}\ldots u_{0,N},u_{1,0},u_{1,1},u_{1,2}\ldots u_{1,N},\ldots u_{N,0},u_{N,1},u_{N,2},\ldots u_{N,N}\right)$,

B — data vector (boundary and additional conditions).

Analysis of the stability of the matrix of the initial problem

Description of the numerical experiment $c = 1, \quad \omega = 0.5$

$$h_1(y) = \frac{1-\cos(8\pi y)}{4}, \quad h_2(x) = \frac{1-\cos(8\pi x)}{4},$$
$$q_1(y) = \frac{1-\cos(8\pi y)}{4}, \quad q_2(x) = \frac{1-\cos(8\pi x)}{4},$$
$$\rho(x,y) = e^{-\frac{(x-0.5)^2+(y-0.5)^2}{2b^2}}, \quad b = 0.1.$$

Table 1 presents the results of a singular decomposition of the matrix of the initial problem A and a direct problem A_T for the values $N = 50$

Matrices	$\sigma_{max}(A)$	$\sigma_{min}(A)$	$\mu(A)$
A_T	743.404	0.015	47056.2
A	743.404	$9.07 \cdot 10^{-19}$	$8.19 \cdot 10^{20}$

Table 1. Singular decomposition of matrices with size $(N+1)^2$

The matrix of the original problem has a poor conditionality [13].

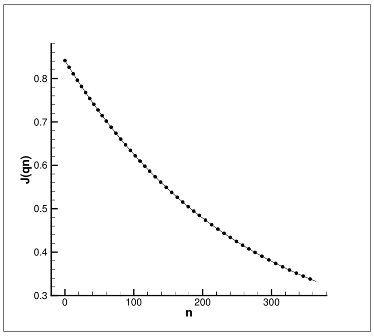
Numerical Results of the Inverse Problem by the Landweber Method

In this section, to solve the two-dimensional direct problem for the Helmholtz equation, the finite element method is used. Triangulation with the number of triangles — N_t; vertices — N_v; and the number of points at the border — N. The problem is solved using the computational package FreeFEM++.
Description of the numerical experiment $c = 1, \quad \omega = 0.5$

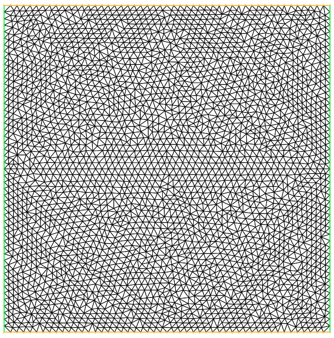
$$h_1(y) = \frac{1-\cos(8\pi y)}{4}, \quad h_2(x) = \frac{1-\cos(8\pi x)}{4},$$
$$q_1(y) = \frac{1-\cos(8\pi y)}{4}, \quad q_2(x) = \frac{1-\cos(8\pi x)}{4},$$
$$\rho(x,y) = e^{-\frac{(x-0.5)^2+(y-0.5)^2}{2b^2}}, \quad b = 0.1.$$

Number of iterations, n	$J(q)$	$\|u_T - \tilde{u}\|$
10	0.8158	0.1491
100	0.6254	0.1013
300	0.3788	0.0553
365	0.3323	0.0538

Table 2. Solution results by the Landweber iteration method without noise



a) $J(q_n)$



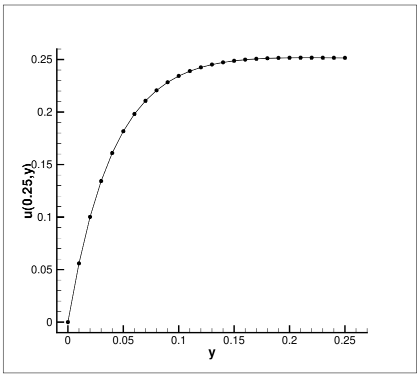
b) $N = 50$, $N_t = 5862$ and $N_v = 3032$

Fig. 1. a) The value of the functional by iteration, b) Ω area grid with N number of points on the border

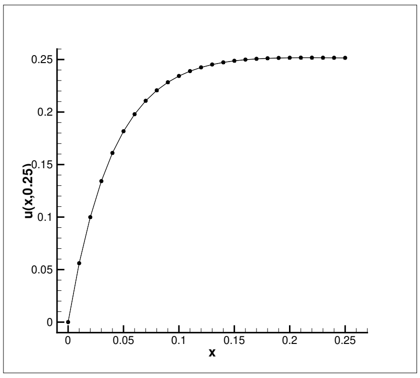
6 Conclusion

The paper is devoted to the investigation of an ill-posed problem by initial-boundary value problems for the Helmholtz equation, the construction of numerical optimization methods for solving problems, the construction of corresponding algorithms and the computational experiments of this problem.

The numerical results of the solution of the initial-boundary value problem for the Helmholtz equation, in which, together with the data on the surface, the



a) $u(0.25, y)$

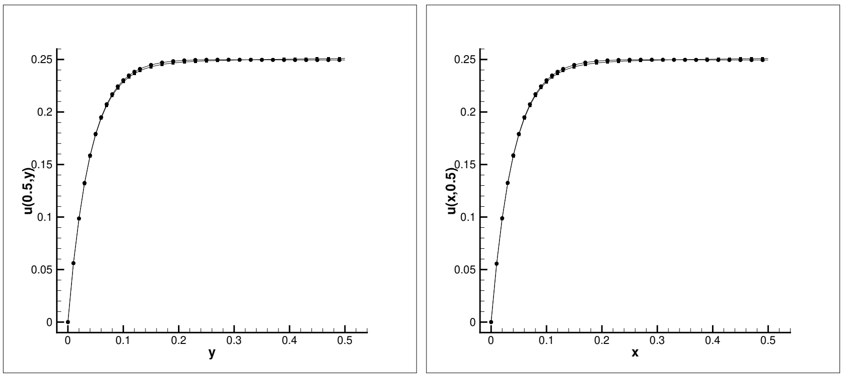


b) $u(x, 0.25)$

Denotation: (symbol ■) — Landweber solution, (symbol •) — exact solution

Fig. 2. The figure a) comparison of boundaries $u(x, y)$ at $x = 0.25$, the figure b) comparison of boundaries $u(x, y)$ at $y = 0.25$

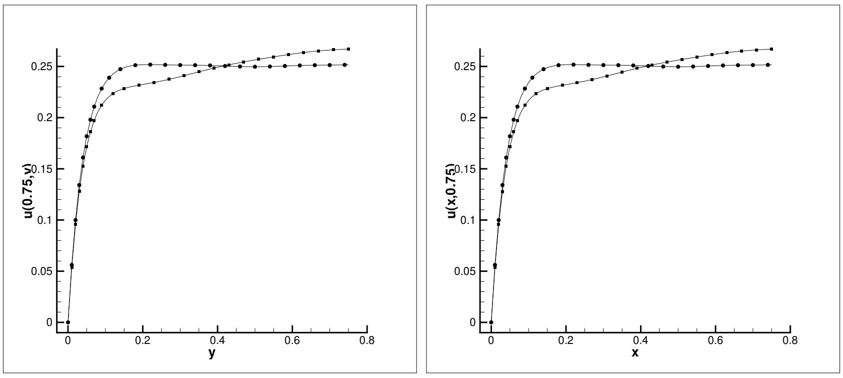
data in depth are used, show that if we want to calculate the squaring problem, it is better to measure the data larger and deeper and start solving the problem in a large square . This gives a more stable solution.



a) $u(0.5, y)$ b) $u(x, 0.5)$

Denotation: (symbol ■) — Landweber solution, (symbol •) — exact solution

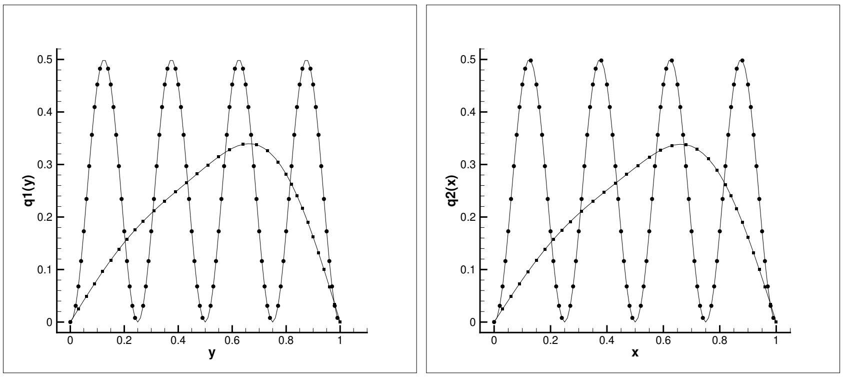
Fig. 3. The figure a) comparison of boundaries $u(x, y)$ at $x = 0.5$, the figure b) comparison of boundaries $u(x, y)$ at $y = 0.5$



a) $u(0.75, y)$ b) $u(x, 0.75)$

Denotation: (symbol ■) — Landweber solution, (symbol •) — exact solution

Fig. 4. The figure a) comparison of boundaries $u(x, y)$ at $x = 0.75$, the figure b) comparison of boundaries $u(x, y)$ at $y = 0.75$



a) $u(1, y)$ b) $u(x, 1)$

Denotation: (symbol ■) — Landweber solution, (symbol •) — exact solution

Fig. 5. The figure a) comparison of boundaries $u(x, y)$ at $x = 1$, the figure b) comparison of boundaries $u(x, y)$ at $y = 1$

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