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On linear and nonlinear heat equations in degenerating domains

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Abstract. Earlier we studied the homogeneous boundary value problem for the heat equation in degenerating domains. For this problem in the weight class of essentially bounded functions it was established the existence of a nontrivial solution up to a constant multiplier. In this paper, on the basis of the above result, we study the issues of the existence of nontrivial solutions of homogeneous nonlinear heat equations, including the homogeneous Burgers equation in degenerating domains. The nonhomogeneous boundary value problems for the Burgers equation are studied separately.

INTRODUCTION

Earlier in several papers [1], [2], [3], [4], [5], [6] it was shown the existence of non-trivial solutions for the homogeneous boundary value problem for linear heat equation in degenerating (in the power-law order) domains. In this paper, based on our previous results established for a linear problem, we study the existence of nontrivial solutions of homogeneous boundary value problems for two nonlinear heat equation, given in degenerate domains. Classes of essentially bounded functions with weight in which there exists or does not exist a nontrivial solution are found. Nontrivial solutions of the nonlinear equations under consideration are found in explicit form. We show in the degenerating domain the existence of nontrivial solutions for a nonlinear heat equation (1st part of the work) and for Burgers equation (2nd part of the work).

We will note the paper [7] where for linear and semilinear equations of Tricomi type, existence, uniqueness and qualitative properties of weak solutions to the degenerate hyperbolic Goursat problem on characteristic triangles were established. For the linear problem, a robust L_2-based theory was developed, including well-posedness, elements of a spectral theory, partial regularity results and maximum and comparison principles. For the nonlinear problem, existence of weak solutions with nonlinearities of unlimited polynomial growth at infinity was proved by combining standard topological methods of nonlinear analysis with the linear theory developed there.

In the work [8] for semilinear partial differential equations of mixed elliptic-hyperbolic type with various boundary conditions, the nonexistence of nontrivial solutions was showed for domains which are suitably star-shaped and for nonlinearities with supercritical growth in a suitable sense.

NONLINEAR HEAT EQUATION

Let $G = \{x, t | \, 0 < x < t, \ t > 0\}$. In the infinite corner domain G we study the existence of nontrivial solutions for the following boundary value problem [9]:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + w_x^2(x,t), \quad \{x,t\} \in G, \\ w(x,t)|_{x=0} = w(x,t)|_{x=t} = 0. \end{cases} \tag{1}$$

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Using the transform

$$u(x,t) = \exp\{w(x,t)\} - 1 \tag{2}$$

the boundary value problem (1) is reduced to a linear homogeneous boundary value problem for the heat equation:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & \{x,t\} \in G, \\ u(x,t)|_{x=0} = u(x,t)|_{x=t} = 0. \end{cases} \tag{3}$$

The reduction of the boundary value problem (3) to an integral equation

Search for a solution to the boundary value problem (3) as the sum of double-layer heat potentials [1], [2], [3], [4], [5], [6], [10]:

$$u(x,t) = \frac{1}{4\sqrt{\pi}} \int_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4(t-\tau)}\right\} \nu(\tau)\, d\tau + \frac{1}{4\sqrt{\pi}} \int_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\} \varphi(\tau)\, d\tau. \tag{4}$$

It is known [10] that the function (4) satisfies the equation (3) for any $\nu(t)$ and $\varphi(t)$. Using boundary conditions from (3) and the properties of heat potentials, we obtain the following integral equation with respect to the unknown function $\varphi(t)$:

$$[I - \mathbf{K}]\varphi \equiv \varphi(t) - \int_0^t K(t,\tau)\varphi(\tau) d\tau = 0, \ \ t > 0, \tag{5}$$

where

$$K(t,\tau) = \frac{1}{2\sqrt{\pi}} \left[\frac{t+\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(t+\tau)^2}{4(t-\tau)}\right\} + \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{t-\tau}{4}\right\} \right].$$

$$\nu(t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} \varphi(\tau)\, d\tau. \tag{6}$$

We have shown earlier [3], [4], [5], [6] that the integral equation (5) along with the trivial solution has a non-trivial solution up to a constant multiplier.

Theorem 1. *The general solution of equation (5) has the form*

$$\varphi(t) = C \cdot \varphi_0(t), \ \ C = const, \tag{7}$$

$$\varphi_0(t) = t^{-1/2} \exp\{-t/4\} + \sqrt{\pi}/2 \left[1 + \operatorname{erf}\left(t^{1/2}/2\right)\right]. \tag{8}$$

In the following paragraph we clarify the class of solutions (8) of the integral equation (5), which we have set earlier in works [3], [4], [5], [6].

The class of solutions of the integral equation (5)

First set the property of the integral operator $\mathbf{K}$ in equation (5). Let us introduce the weight class of essentially bounded functions

$$L_\infty(\mathbb{R}_+ ;\, \theta_1(t,T)) = \{\varphi |\, \theta_1(t,T)\varphi(t) \in L_\infty(\mathbb{R}_+)\}, \tag{9}$$

$$\theta_1(t,T) = \begin{cases} t^{1/2}, & \text{if} \quad 0 < t \le T, \\ T^{1/2}, & \text{if} \quad T < t < +\infty, \end{cases} \tag{10}$$

and T is an arbitrary positive finite number. Let us introduce the weight class of essentially bounded functions (uniqueness class)

$$L_\infty(\mathbb{R}_+ ;\, \theta_2(t,\varepsilon,T)) = \{\varphi |\, \theta_2(t,\varepsilon,T)\varphi(t) \in L_\infty(\mathbb{R}_+)\}, \tag{11}$$

$$\theta_2(t,\varepsilon,T) = \begin{cases} t^{1/2-\epsilon}, & \text{if} \quad 0 < t \le T, \\ T^{1/2-\epsilon}, & \text{if} \quad T < t < +\infty, \end{cases} \tag{12}$$

and T is an arbitrary positive finite number, $\varepsilon > 0$.

Lemma 1. *The integral operator* $\mathbf{K}$ *in the equation (5) is bounded in the space* $L_\infty(\mathbb{R}_+;\theta_1(t,T))$ *(9)–(10), i.e.*

$$\mathbf{K} \in \mathcal{L}\left(L_\infty(\mathbb{R}_+;\theta_1(t,T))\right). \tag{13}$$

Proof of Lemma 1

Estimate the integral operator $\mathbf{K}$. We show the following inequality

$$\|\mathbf{K}\| \leq I_{(0,T)} + I_{(T,Inf)}, \tag{14}$$

where by $I_{(0,T)}$ and $I_{(T,Inf)}$ are indicated the norms of restriction of the integral operator $\mathbf{K}$ acting respectively in the classes and for which the following estimates true

$$I_{(0,T)} \leq I_{1,(0,T)} + I_{2,(0,T)}, \tag{15}$$

$$I_{(T,Inf)} \leq I_{1,(T,Inf)} + I_{2,(T,Inf)}, \tag{16}$$

where

$$I_{1,(0,T)} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{t^{3/2}}{\tau^{1/2}(t-\tau)^{3/2}} \exp\left\{-\frac{t\tau}{t-\tau}\right\} \exp\left\{-\frac{t-\tau}{4}\right\} d\tau,$$

$$I_{2,(0,T)} = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{t^{1/2}}{\tau^{1/2}(t-\tau)^{1/2}} \left[1 - \exp\left\{-\frac{t\tau}{t-\tau}\right\}\right] \exp\left\{-\frac{t-\tau}{4}\right\} d\tau,$$

$$I_{1,(T,Inf)} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{t}{(t-\tau)^{3/2}} \exp\left\{-\frac{(t+\tau)^2}{4(t-\tau)}\right\} d\tau,$$

$$I_{2,(T,Inf)} = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} \left[1 - \exp\left\{-\frac{t\tau}{t-\tau}\right\}\right] \exp\left\{-\frac{t-\tau}{4}\right\} d\tau.$$

Now for the proof of Lemma 1 is sufficient to show the boundedness of integrals in the right-hand sides of inequalities (15)–(16). Perform the change $\tau = t\sin^2\alpha$ in the integrals $I_{1,(0,T)}$ and $I_{2,(0,T)}$ from (15), we have:

$$I_{1,(0,T)} \leq \frac{2}{\sqrt{\pi}} \int_0^{\pi/2} \exp\left\{-\left(\sqrt{t}\cdot tg\alpha\right)^2\right\} d\left(\sqrt{t}\cdot tg\alpha\right) = 1, \tag{17}$$

$$I_{2,(0,T)} \leq \frac{\sqrt{t}}{2\sqrt{\pi}} \int_0^{\pi/2} \frac{2t\sin\alpha\cdot\cos\alpha}{t^{1/2}\sin\alpha\cdot t^{1/2}\cos\alpha} d\alpha = \frac{\sqrt{\pi t}}{4}. \tag{18}$$

Making the change $\tau = t\sin^2\alpha$ in the integral $I_{1,(T,Inf)}$ and estimating the integral $I_{2,(T,Inf)}$ from (16), we obtain:

$$I_{1,(T,Inf)} \leq \frac{2}{\sqrt{\pi}} \int_0^{\pi/2} \exp\left\{-\left(\sqrt{t}\cdot tg\alpha\right)^2\right\} d\left(\sqrt{t}\cdot tg\alpha\right) = 1, \tag{19}$$

$$I_{2,(T,Inf)} \leq -\frac{2}{\sqrt{\pi}} \int_0^t \exp\left\{-\frac{t-\tau}{2}\right\} d\sqrt{\frac{t-\tau}{2}} = \operatorname{erf}\left(\frac{\sqrt{t}}{\sqrt{2}}\right). \tag{20}$$

From (17)–(20) follows the inequality (14), i.e. the assertion (13) is valid. Lemma 1 is proved.
From the assertion of Lemma 1 follows the validity of the following Lemmas.
Lemma 2. *The solution of the integral equation (5) belongs to the class $L_\infty(\mathbb{R}_+;\theta_1(t,T))$ (9)–(10), i.e.*

$$\theta_1(t,T)\varphi_0(t) \in L_\infty(\mathbb{R}_+). \tag{21}$$

Lemma 3 (uniqueness class). *For the integral equation (5) the space $L_\infty(\mathbb{R}_+;\theta_2(t,\varepsilon,T))$ is a class of uniqueness, i.e., in the class (11)–(12) the equation (5) has only a trivial solution.*

The class of solutions of the boundary value problem for the heat equation (3)

For further we transform the representation (4) of the solution of the boundary value problem (3). For this purpose, substituting the representation of the function $v(t)$ (6) in (4), we get:

$$u(x,t) = \frac{1}{4\sqrt{\pi}}\int_0^t M(x,t,\tau)\varphi(\tau)\,d\tau, \tag{22}$$

$$M(x,t,\tau) = \frac{x+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(x+\tau)^2}{4(t-\tau)}\right\} + \frac{x-\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\}. \tag{23}$$

$$\theta_3(t,\varepsilon,T) = \begin{cases} t^{1/2}, & \text{if} \quad 0 < t \le T, \\ T^{1/2}\exp\left\{-\left(\frac{1}{4}+\varepsilon\right)(t-T)\right\}, & \text{if} \quad T < t < +\infty, \end{cases} \tag{24}$$

and T is an arbitrary positive finite number, $\varepsilon > 0$.

Lemma 4. *Let $G_{0,T} = \{0 < x < t,\ 0 < t \le T\}$. Then the solution $u(x,t)$ of the boundary value problem (3) belongs to the class*

$$L_\infty(G_{0,T};\theta_3(t,\varepsilon,T)).$$

Proof of Lemma 4

Since $\{x,t\} \in G_{0,T}$, then by definition (24) we get $\theta_3(t,\varepsilon,T) = t^{1/2}$. Estimate the solution (22)–(23) on the set $G_{0,T}$. Let us show that

$$\|u(x,t)\|_{L_\infty(G_{0,T};\theta_3(t,\varepsilon,T))} \le \sum_{k=1}^{4} I_k(x,t), \tag{25}$$

where

$$I_1(x,t) = \frac{t^{1/2}}{4\sqrt{\pi}}\exp\left\{\frac{x+t}{2}\right\}\int_0^t \frac{x+t}{\tau^{1/2}(t-\tau)^{3/2}}\exp\left\{-\frac{(x+t)^2}{4(t-\tau)}\right\}d\tau, \tag{26}$$

$$I_2(x,t) = \frac{t^{1/2}}{4\sqrt{\pi}}\exp\left\{\frac{x+t}{2}\right\}\int_0^t \frac{1}{\tau^{1/2}(t-\tau)^{1/2}}\exp\left\{-\frac{(x+t)^2}{4(t-\tau)}\right\}d\tau, \tag{27}$$

$$I_3(x,t) = \frac{t^{1/2}}{4\sqrt{\pi}}\exp\left\{\frac{t-x}{2}\right\}\int_0^t \frac{t-x}{\tau^{1/2}(t-\tau)^{3/2}}\exp\left\{-\frac{(t-x)^2}{4(t-\tau)}\right\}d\tau, \tag{28}$$

$$I_4(x,t) = \frac{t^{1/2}}{4\sqrt{\pi}}\exp\left\{\frac{t-x}{2}\right\}\int_0^t \frac{1}{\tau^{1/2}(t-\tau)^{1/2}}\exp\left\{-\frac{(t-x)^2}{4(t-\tau)}\right\}d\tau \tag{29}$$

are valid. To obtain the expressions (26)–(29) in representation of the solution $u(x,t)$ (22)–(23) the following transformations are applied

$$x+\tau = x+t-(t-\tau), \;\; x-\tau = x-t+(t-\tau).$$

It remains to show the boundedness of the integrals (26)–(29) on the set $G_{0,T}$. Making the change $\tau = t\sin^2\alpha$ in the integrals (26)–(29), we obtain:

$$I_1(x,t) = \frac{1}{2\sqrt{\pi}}\exp\left\{\frac{t^2-x^2}{4t}\right\}\int\limits_0^{\pi/2}\exp\left\{-\left(\frac{x+t}{2\sqrt{t}}tg\alpha\right)^2\right\}d\left(\frac{x+t}{2\sqrt{t}}tg\alpha\right) = \frac{1}{4}\exp\left\{\frac{t^2-x^2}{4t}\right\}. \tag{30}$$

$$I_2(x,t) \le \frac{t^{1/2}}{4\sqrt{\pi}}\exp\left\{\frac{x+t}{2}\right\}\int\limits_0^t \frac{1}{\tau^{1/2}(t-\tau)^{1/2}}d\tau = \frac{\sqrt{\pi}\,t^{1/2}}{4}\exp\left\{\frac{x+t}{2}\right\}, \tag{31}$$

$$I_3(x,t) = \frac{1}{2\sqrt{\pi}}\exp\left\{-\frac{t^2-x^2}{4t}\right\}\int\limits_0^{\pi/2}\exp\left\{-\left(\frac{t-x}{2\sqrt{t}}tg\alpha\right)^2\right\}d\left(\frac{t-x}{2\sqrt{t}}tg\alpha\right) = \frac{1}{4}\exp\left\{-\frac{t^2-x^2}{4t}\right\}. \tag{32}$$

$$I_4(x,t) \le \frac{t^{1/2}}{4\sqrt{\pi}}\exp\left\{\frac{t-x}{2}\right\}\int\limits_0^t \frac{1}{\tau^{1/2}(t-\tau)^{1/2}}d\tau = \frac{\sqrt{\pi}\,t^{1/2}}{4}\exp\left\{\frac{t-x}{2}\right\}. \tag{33}$$

From the boundedness on the set $G_{0,T}$ of the right-hand sides of the relations (30)–(33) the assertion of Lemma 4 follows.

Lemma 5. *Let $G_{T,Inf} = \{0 < x < t,\ T < t < \infty\}$. Then the solution $u(x,t)$ of the boundary value problem (3) belongs to the class*

$$L_\infty(G_{T,Inf}\,;\,\theta_3(t,\varepsilon,T)).$$

Proof of Lemma 5

Since $\{x,t\} \in G_{T,Inf}$, then by definition (24) we have

$$\theta_3(t,\varepsilon,T) = T^{1/2}\exp\left\{-\left(\frac{1}{4}+\varepsilon\right)(t-T)\right\}.$$

Estimate the solution (22)–(23) on the set $G_{T,Inf}$. Let us show that

$$\|u(x,t)\|_{L_\infty(G_{T,Inf}\,;\,\theta_3(t,\varepsilon,T))} \le \sum_{k=1}^{4} T^{1/2}J_k(x,t), \tag{34}$$

where

$$J_1(x,t) = \frac{1}{4\sqrt{\pi}}\exp\left\{\frac{2x+t}{4}-\varepsilon\,t\right\}\int\limits_0^t \frac{x+t}{(t-\tau)^{3/2}}\exp\left\{-\frac{(x+t)^2}{4(t-\tau)}\right\}d\tau, \tag{35}$$

$$J_2(x,t) = \frac{1}{4\sqrt{\pi}}\exp\left\{\frac{2x+t}{4}-\varepsilon\,t\right\}\int\limits_0^t \frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{(x+t)^2}{4(t-\tau)}\right\}d\tau, \tag{36}$$

$$J_3(x,t) = \frac{1}{4\sqrt{\pi}}\exp\left\{-\frac{2x-t}{4}-\varepsilon\,t\right\}\int\limits_0^t \frac{t-x}{(t-\tau)^{3/2}}\exp\left\{-\frac{(x-t)^2}{4(t-\tau)}\right\}d\tau, \tag{37}$$

$$J_4(x,t) = \frac{1}{4\sqrt{\pi}} \exp\left\{-\frac{2x-t}{4} - \varepsilon t\right\} \int\limits_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{(x-t)^2}{4(t-\tau)}\right\} d\tau \tag{38}$$

are valid. To obtain the expressions (35)–(38) in representation of the solution $u(x,t)$ (22)–(23) the following transformations are applied

$$x+\tau = x+t-(t-\tau), \;\; x-\tau = x-t+(t-\tau).$$

It remains to show the boundedness of the integrals (35)–(38) on the set $G_{T,Inf}$. Making the change $\tau = t\sin^2\alpha$ in the integrals (35)–(38), we get:

$$J_1(x,t) \le \frac{1}{2\sqrt{\pi}} \exp\left\{-\frac{x^2}{4t} - \varepsilon t\right\} \int\limits_0^{\pi/2} \exp\left\{-\left(\frac{x+t}{2\sqrt{t}} tg\alpha\right)^2\right\} d\left(\frac{x+t}{2\sqrt{t}} tg\alpha\right) = \frac{1}{4} \exp\left\{-\frac{x^2}{4t} - \varepsilon t\right\}, \tag{39}$$

$$J_2(x,t) \le \frac{t^{1/2}}{2\sqrt{\pi}} \exp\left\{-\frac{x^2}{4t} - \varepsilon t\right\} \int\limits_0^{\pi/2} \exp\left\{-\left(\frac{x+t}{2\sqrt{t}}\, tg\alpha\right)^2\right\} d\alpha \le \frac{\sqrt{\pi}\, t^{1/2}}{4} \exp\left\{-\frac{x^2}{4t} - \varepsilon t\right\}, \tag{40}$$

$$J_3(x,t) \le \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{x^2}{4t} - \varepsilon t\right\} \int\limits_0^{\pi/2} \exp\left\{-\left(\frac{t-x}{2\sqrt{t}}\, tg\alpha\right)^2\right\} d\left(\frac{t-x}{2\sqrt{t}} tg\alpha\right) = \frac{1}{2} \exp\left\{-\frac{x^2}{4t} - \varepsilon t\right\}, \tag{41}$$

$$J_4(x,t) \le \frac{t^{1/2}}{2\sqrt{\pi}} \exp\left\{-\frac{x^2}{4t} - \varepsilon t\right\} \int\limits_0^{\pi/2} \exp\left\{-\left(\frac{t-x}{2\sqrt{t}}\, tg\alpha\right)^2\right\} d\alpha \le \frac{\sqrt{\pi}\, t^{1/2}}{4} \exp\left\{-\frac{x^2}{4t} - \varepsilon t\right\}. \tag{42}$$

From the boundedness on the set $G_{T,Inf}$ of the right-hand sides of the relations (39)–(42) the assertion of Lemma 5 follows. From the assertions of Lemmas 4 and 5 follows the validity of the following theorem.

Theorem 2. *Along with a trivial solution, the boundary value problem (3) has a family of nontrivial solutions*

$$\{C \cdot u(x,t),\, C = const \neq 0\},$$

where

$$u(x,t) \in L_\infty(G\,;\, \theta_3(t,\varepsilon,T)),$$

defined by the relations (22)–(23).

Theorem 3 (uniqueness class). *The boundary value problem (3) has only a trivial solution in the class* $L_\infty(G\,;\, \theta_4(t,\varepsilon_1,\varepsilon_2,T))$, *where*

$$\theta_4(t,\varepsilon_1,\varepsilon_2,T) = \begin{cases} t^{1/2-\varepsilon_1-\varepsilon_2}, & \text{if} \quad 0 < t \le T, \\ T^{1/2-\varepsilon_1-\varepsilon_2} \exp\left\{-\left(\frac{1}{4}+\varepsilon_2\right)(t-T)\right\}, & \text{if} \quad T < t < +\infty, \end{cases} \tag{43}$$

and T *is an arbitrary positive finite number,* $\varepsilon_1 \ge 0,\;\, \varepsilon_2 \ge 0$.

The main result for the nonlinear heat equation

From the theorems and lemmas which were set above it follows

Theorem 4. *Along with a trivial solution, the boundary value problem (1) has a family of nontrivial solutions*

$$\{w(x,t) = \ln[1 + C \cdot u(x,t)],\, C = const \neq 0\},$$

which is determined by transformation (2), where

$$u(x,t) \in L_\infty(G\,;\, \theta_3(t,\varepsilon,T)).$$

To solve the nonlinear equation (1), we have:

$$\exp\{w(x,t)\} - 1 \in L_\infty(G\,;\, \theta_3(t,\varepsilon,T)).$$

THE BURGERS EQUATION

We consider the following boundary value problem

$$\begin{cases} w_t + ww_x - w_{xx} = 0, & 0 < x < t,\ t > 0, \\ w\big|_{x=0} = 0, \ \ w\big|_{x=t} = 0. \end{cases} \tag{44}$$

Using the Hopf-Cole transformation

$$w(x,t) = -2 \cdot \frac{u_x(x,t)}{u(x,t)}, \tag{45}$$

the boundary value problem (44) reduces to the following auxiliary boundary value problem

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < t,\ t > 0, \\ u_x\big|_{x=0} = 0, \ \ u_x\big|_{x=t} = 0. \end{cases} \tag{46}$$

The literature on Burgers equations is quite numerous. We give only the following [11], [12], [13], [14].

Reduction to the integral equation

Search for a solution to the boundary value problem (46) as the sum of simple-layer heat potentials [1], [2], [3], [4], [5], [6], [10]:

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{x^2}{4(t-\tau)}\right\} \nu(\tau) + \frac{1}{2\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\} \varphi(\tau) d\tau. \tag{47}$$

It is known [10] that the function (47) satisfies the equation (46) for any $\nu(t)$ and $\varphi(t)$. Using the boundary conditions from (46) and the properties of the heat potentials [3], [4], [5], [6], we obtain the following integral equation with respect to the unknown function $\varphi(t)$:

$$[I - \mathbf{K}]\varphi \equiv \varphi(t) - \int_0^t K(t,\tau)\varphi(\tau) d\tau = 0, \ \ t > 0, \tag{48}$$

where

$$K(t,\tau) = \frac{1}{2\sqrt{\pi}} \left[\frac{t+\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(t+\tau)^2}{4(t-\tau)}\right\} + \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{t-\tau}{4}\right\} \right],$$

$$\nu(t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{\tau^2}{4(t-\tau)}\right\} \varphi(\tau)\, d\tau, \tag{49}$$

Remark. *Note that this equation coincides with the integral equation (5) for the nonlinear boundary value problem (1) which was considered above. And for it Theorem 1 and Lemma 1-3 are true.*

Taking into account the formulas (47), (49) for solving the boundary value problem for the heat equation (46) we obtain the representation

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} \left[\exp\left\{-\frac{(x+\tau)^2}{4(t-\tau)}\right\} + \exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\} \right] \varphi(\tau) d\tau. \tag{50}$$

The class of solutions $u(x,t)$ (50) for the equation (46)

Let us introduce the weight class of essentially bounded functions

$$L_\infty(G\,;\,\theta_3(t,\varepsilon,T)) = \{u|\ \theta_3(t,\varepsilon,T)u(x,t) \in L_\infty(G)\}, \tag{51}$$

where weight function $\theta_3(t,\varepsilon,T)$ is defined in (24):

$$\theta_3(t,\varepsilon,T) = \begin{cases} t^{1/2}, & \text{if} \quad 0 < t \le T, \\ T^{1/2}\exp\left\{-\left(\frac{1}{4}+\varepsilon\right)(t-T)\right\}, & \text{if} \quad T < t < +\infty, \end{cases}$$

and T is an arbitrary positive finite number, $\varepsilon > 0$.

Lemma 6. *Let* $G_{0,T} = \{0 < x < t,\ 0 < t \le T\}$*. Then the solution* $u(x,t)$ *(50) of the boundary value problem (46) belongs to the class*

$$L_\infty(G_{0,T};\theta_3(t,\varepsilon,T)).$$

Proof of Lemma 6

Since $\{x,t\} \in G_{0,T}$, then by definition (24) we have

$$\theta_3(t,\varepsilon,T) = t^{1/2}.$$

Estimate the solution (50) on the set $G_{0,T}$. Let us show, that

$$||u(x,t)||_{L_\infty(G_{0,T};\theta_3(t,\varepsilon,T))} \le I_{B1}(x,t) + I_{B2}(x,t). \tag{52}$$

Indeed, the following relations

$$I_{B1}(x,t) \le \frac{t^{1/2}}{2\sqrt{\pi}}\int_0^t \frac{1}{\tau^{1/2}(t-\tau)^{1/2}}\exp\left\{-\frac{(x+\tau)^2}{4(t-\tau)}\right\}d\tau \le \frac{\sqrt{\pi}\,t}{2}, \tag{53}$$

$$I_{B2}(x,t) \le \frac{t^{1/2}}{2\sqrt{\pi}}\int_0^t \frac{1}{\tau^{1/2}(t-\tau)^{1/2}}\exp\left\{-\frac{(x-\tau)^2}{4(t-\tau)}\right\}d\tau \le \frac{\sqrt{\pi}\,t}{2} \tag{54}$$

are true. This completes the proof of Lemma 6.

Lemma 7. *Let* $G_{T,Inf} = \{0 < x < t,\ T < t < \infty\}$*. Then the solution* $u(x,t)$ *(50) of the boundary value problem (46) belongs to the class*

$$L_\infty(G_{T,Inf}\,;\,\theta_3(t,\varepsilon,T)).$$

Proof of Lemma 7

Since $\{x,t\} \in G_{T,Inf}$, then by definition (24) we have

$$\theta_3(t,\varepsilon,T) = T^{1/2}\exp\left\{-\left(\frac{1}{4}+\varepsilon\right)(t-T)\right\}.$$

Estimate the solution (50) on the set $G_{T,Inf}$. Let us show, that

$$||u(x,t)||_{L_\infty(G_{T,Inf}\,;\,\theta_3(t,\varepsilon,T))} \le J_{B1}(x,t) + J_{B2}(x,t). \tag{55}$$

Indeed, the following relations

$$J_{B1}(x,t) \le \frac{1}{2\sqrt{\pi}} \exp\{-\varepsilon t\} \int_0^t \frac{1}{(t-\tau)^{1/2}} d\tau = \frac{\sqrt{t}}{\sqrt{\pi}} \exp\{-\varepsilon t\}, \tag{56}$$

$$J_{B2}(x,t) \le \frac{1}{2\sqrt{\pi}} \exp\{-\varepsilon t\} \int_0^t \frac{1}{(t-\tau)^{1/2}} d\tau = \frac{\sqrt{t}}{\sqrt{\pi}} \exp\{-\varepsilon t\} \tag{57}$$

are true. This completes the proof of Lemma 7.

From the assertions of Lemmas 6 and 7 follows the validity of the following theorem.

Theorem 5. *Along with a trivial solution, the boundary value problem (46) has a family of nontrivial solutions (50)*

$$\{C \cdot u(x,t),\ C = const \ne 0\},$$

where

$$u(x,t) \in L_\infty(G\,;\ \theta_3(t,\varepsilon,T)),$$

and $\theta_3(t,\varepsilon,T)$ defined by the relations (24).

The main result for the Burgers equation

From the theorems and lemmas which were set above it follows

Theorem 6. *Along with a trivial solution the boundary value problem (44) has a unique nontrivial solution*

$$w(x,t) = -2u_x(x,t)/u(x,t)$$

which is determined by transformation (2), where

$$u(x,t) \in L_\infty(G\,;\ \theta_3(t,\varepsilon,T)).$$

To solve the nonlinear Burgers equation (44) we have:

$$\exp\left\{-\frac{1}{2}\int_0^x w(\xi,t)d\xi\right\} \in L_\infty(G\,;\ \theta_3(t,\varepsilon,T)).$$

CONCLUSION

From the assertions of Theorems 4 and 6 it follows that some functionals of non-trivial solutions of both the nonlinear equation and Burgers equation allow growth both at the top of the corner and at infinity.

This orders of growth are determined by one weight function $\theta_3(t,\varepsilon,T)$ for each of the considered equations.

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